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Algebro-differential embeddings of compact almost complex structures

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Symplectic geometry and almost complex geometry

An important part of Helmut Hofer's work deals with symplectic geometry, and the geometry of pseudoholomorphic curves.
The subject was given a strong impetus by Mikhail Gromov in his famous Inventiones paper from 1985.
Since then, many interconnections between symplectic geometry, topology and algebraic geometry have been developed, e.g. through the study of Gromov-Witten invariants.

Basic question

Let (M^{2n}, ω) be a compact symplectic manifold and J a compatible almost complex structure. Assume that

$$\int_M c_1(M, J) \wedge \omega^{n-1} > 0.$$

Is it true that there exists a differentiable family of mobile pseudoholomorphic curves $(f_t)_{t \in S} : \mathbb{P}^1 \to M$,
i.e. generically injective and covering an open set in M ?

Existence of rational curves

Related question

Let (X^n, ω) be a compact Kähler manifold. Assume that that $c_1(K_X) \cdot \omega^{n-1} < 0$ or more generally that K_X is not pseudoeffective (this means that the class $c_1(K_X)$ does not contain any closed $(1,1)$-current $T \geq 0$.

Can one conclude that X is covered by rational curves ?

This would be crucial for the theory of compact Kähler manifolds.

Theorem (BDPP = Boucksom - D - Peternell - Paun, 2002)

The answer is positive when X is a complex projective manifold.

The proof uses intersection theory of currents and characteristic p techniques due to Mori.

It would be nice to have a "symplectic proof", especially in the Kähler case.

A question raised by Fedor Bogomolov

Rough question

Can one produce an arbitrary compact complex manifold X / an arbitrary compact Kähler manifold X by means of a "purely algebraic construction" ?

Let Z be a projective algebraic manifold, $\dim_{\mathbb{C}} Z = N$, equipped with a subbundle (or rather subsheaf) $\mathcal{D} \subset \mathcal{O}_Z(T_Z)$.

Assume that X^{2n} is a compact C^∞ real even dimensional manifold that is embedded in Z, as follows:

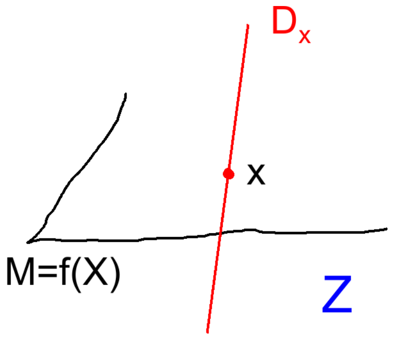
(i) $f : X \hookrightarrow Z$ is a smooth (say C^∞) embedding

(ii) $\forall x \in X, \quad f_* T_{X,x} \oplus \mathcal{D}_{f(x)} = T_{Z,f(x)}$.

(iii) $f(X) \cap \mathcal{D}_{\text{sing}} = \emptyset$.

We say that $X \hookrightarrow (Z, \mathcal{D})$ is a transverse embedding.

A conjecture of Bogomolov

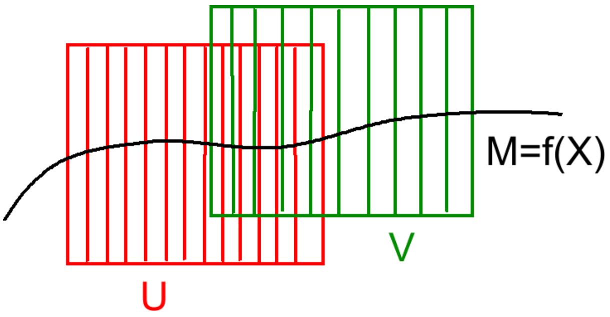


$$f_* T_{X,x} = T_{M,f(x)} \simeq T_{Z,f(x)}/\mathcal{D}_{f(x)}$$

Observation 1

If $\mathcal{D} \subset T_Z$ is an algebraic foliation, i.e. $[\mathcal{D},\mathcal{D}] \subset \mathcal{D}$, then the almost complex structure J_f on X induced by $(Z,\mathcal{D})$ is integrable.

Proof:

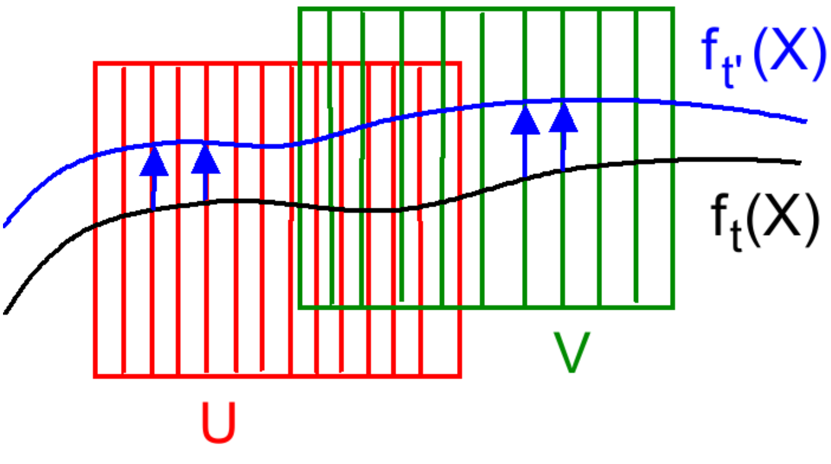


A conjecture of Bogomolov (2)

Observation 2

If $\mathcal{D} \subset T_Z$ is an algebraic foliation and $f_t : X \hookrightarrow (Z,\mathcal{D})$ is an isotopy of transverse embeddings, $t \in [0,1]$, then all complex structures (X, J_{f_t}) are biholomorphic.

Proof:



A conjecture of Bogomolov (3)

To each triple $(Z, \mathcal{D}, \alpha)$ where

- Z is a complex projective manifold
- $\mathcal{D} \subset T_Z$ is an algebraic foliation
- α is an isotopy class of transverse embeddings $f : X \hookrightarrow (Z, \mathcal{D})$

one can thus associate a biholomorphism class (X, J_f).

Conjecture (Bogomolov, 1995)

One can construct in this way every compact complex manifold X.

Additional question 1

What if (X, ω) is Kähler ? Can one embed in such a way that ω is the pull-back of a transversal Kähler structure on $(Z, \mathcal{D})$?

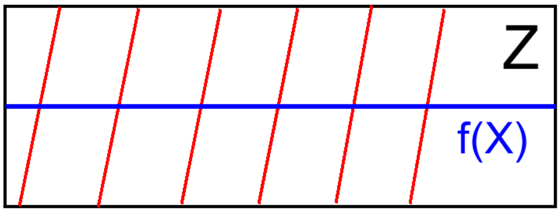
Additional question 2

Can one define moduli spaces of such embeddings, describing the non injectivity of the "Bogomolov fonctor" ?

There exist large classes of examples !

Example 1 : tori

If Z is an Abelian variety and $N \geq 2n$, every n-dimensional compact complex torus $X = \mathbb{C}^n/\Lambda$ can be embedded transversally to a linear codimension n foliation $\mathcal{D}$ on Z.



Example 2 : LVMB manifolds

One obtains a rich class, named after Lopez de Medrano, Verjovsky, Meersseman, Bosio, by considering foliations on $\mathbb{P}^N$ given by a commutative Lie subalgebra of the Lie algebra of $\mathrm{PGL}(N+1, \mathbb{C})$. The corresponding transverse varieties produced include e.g. Hopf surfaces and the Calabi-Eckmann manifolds $S^{2p+1} \times S^{2q+1}$.

What about the almost complex case ?

Easier question : drop the integrability assumption

Can one realize every compact almost complex manifold (X, J) by a transverse embedding into a projective algebraic pair $(Z, \mathcal{D})$, $\mathcal{D} \subset T_Z$, so that $J = J_f$?

Not surprisingly, there are constraints, and Z cannot be "too small". But how large exactly ?

Let $\Gamma^\infty(X, Z, \mathcal{D})$ the Fréchet manifold of transverse embeddings $f : X \hookrightarrow (Z, \mathcal{D})$ and $\mathcal{J}^\infty(X)$ the space of smooth almost complex structures on X.

Further question

When is $f \mapsto J_f$, $\quad \Gamma^\infty(X, Z, \mathcal{D}) \to \mathcal{J}^\infty(X)$ a submersion ?

Note: technically one has to consider rather Banach spaces of maps of $C^{r+\alpha}$ Hölder regularity.

Variation formula for J_f

First, the tangent space to the Fréchet manifold $\Gamma^\infty(X, Z, \mathcal{D})$ at a point f consists of

$$C^\infty(X, f^* T_Z) = C^\infty(X, f^*\mathcal{D}) \oplus C^\infty(X, T_X)$$

Theorem (D - Gaussier, arxiv:1412.2899, 2014)

The differential of the natural map $f \mapsto J_f$ along any infinitesimal variation $w = u + f_* v : X \to f^* T_Z = f^*\mathcal{D} \oplus f_* TX$ of f is given by

$$dJ_f(w) = 2J_f\big(f_*^{-1}\theta(\overline{\partial}_{J_f} f, u) + \overline{\partial}_{J_f} v\big)$$

where

$$\theta : \mathcal{D} \times \mathcal{D} \to TZ/\mathcal{D}, \quad (\xi, \eta) \mapsto [\xi, \eta] \mod \mathcal{D}$$

is the torsion tensor of the holomorphic distribution $\mathcal{D}$, and $\overline{\partial} f = \overline{\partial}_{J_f} f$, $\overline{\partial} v = \overline{\partial}_{J_f} v$ are computed with respect to the almost complex structure (X, J_f).

Sufficient condition for submersivity

Theorem (D - Gaussier, 2014)

Let $f : X \hookrightarrow (Z, \mathcal{D})$ be a smooth transverse embedding. Assume that f and the torsion tensor θ of $\mathcal{D}$ satisfy the following additional conditions:

(ii) f is a totally real embedding, i.e. $\overline{\partial} f(x) \in \mathrm{End}_{\overline{\mathbb{C}}}(T_{X,x}, T_{Z,f(x)})$ is injective at every point $x \in X$;

(ii) for every $x \in X$ and every $\eta \in \mathrm{End}_{\overline{\mathbb{C}}}(T_X)$, there exists a vector $\lambda \in \mathcal{D}_{f(x)}$ such that $\theta(\overline{\partial} f(x) \cdot \xi, \lambda) = \eta(\xi)$ for all $\xi \in T_X$.

Then there is a neighborhood $\mathcal{U}$ of f in $\Gamma^\infty(X, Z, \mathcal{D})$ and a neighborhood $\mathcal{V}$ of J_f in $\mathcal{J}^\infty(X)$ such that

$$\mathcal{U} \to \mathcal{V},\ f \mapsto J_f \text{ is a submersion.}$$

Remark. A necessary condition for (ii) to be possible is that $\mathrm{rank}\, \mathcal{D} = N - n \geq n^2 = \dim \mathrm{End}(T_X)$, i.e. $N \geq n + n^2$.

Existence of universal embedding spaces

Theorem (D - Gaussier, 2014)

For all integers $n \geq 1$ and $k \geq 4n$, there exists a complex affine algebraic manifold $Z_{n,k}$ of dimension $N = 2k + 2(k^2 + n(k - n))$ possessing a real structure (i.e. an anti-holomorphic algebraic involution) and an algebraic distribution $\mathcal{D}_{n,k} \subset T_{Z_{n,k}}$ of codimension n, with the following property:

for every compact n-dimensional almost complex manifold (X, J) admits an embedding $f : X \hookrightarrow Z^{\mathbb{R}}_{n,k}$ transverse to $\mathcal{D}_{n,k}$ and contained in the real part of $Z_{n,k}$, such that $J = J_f$.

The choice $k = 4n$ yields the explicit embedding dimension $N = 38n^2 + 8n$ (and a quadratic bound $N = O(n^2)$ is optimal by what we have seen previously).

Hint. $Z_{n,k}$ is produced by a fiber space construction mixing Grassmannians and twistor spaces ...

Symplectic embeddings

Consider the case of a compact almost complex symplectic manifold (X, J, ω) where the symplectic form ω is assumed to be J-compatible, i.e. $J^*\omega = \omega$ and $\omega(\xi, J\xi) > 0$.

Definition

We say that a closed semipositive $(1,1)$-form β on Z is a transverse Kähler structure to $\mathcal{D} \subset T_Z$ if the kernel of β is contained in $\mathcal{D}$, i.e., if β induces a Kähler form on germs of complex submanifolds transverse to $\mathcal{D}$.

Theorem (D - Gaussier, 2014)

There also exist universal embedding spaces for compact almost complex symplectic manifolds, i.e. a certain triple $(Z, \mathcal{D}, \beta)$ as above, such that every (X, J, ω), $\dim_{\mathbb{C}} X = n$, $\{\omega\} \in H^2(X, \mathbb{Z})$, embeds transversally by $f : X \hookrightarrow (Z, \mathcal{D}, \beta)$ such that

$$J = J_f \text{ and } \omega = f^*\beta.$$

Integrability condition

Recall that

$$N_J(\zeta, \eta) = 4\,\mathrm{Re}\,[\zeta^{0,1}, \eta^{0,1}]^{1,0} = [\zeta, \eta] - [J\zeta, J\eta] + J[\zeta, J\eta] + J[J\zeta, \eta].$$

Nijenhuis tensor formula

If θ denotes the torsion of $(Z, \mathcal{D})$, the Nijenhuis tensor of the almost complex structure J_f induced by a transverse embedding $f : X \hookrightarrow (Z, \mathcal{D})$ is given by $\forall z \in X$, $\forall \zeta, \eta \in T_zX$

$$N_{J_f}(\zeta, \eta) = 4\,\theta(\overline{\partial}_{J_f} f(z) \cdot \zeta, \overline{\partial}_{J_f} f(z) \cdot \eta).$$

Weak solution to the Bogomolov conjecture

There exist universal embeddings spaces $(Z, \mathcal{D}, \mathcal{S})$ where $\mathcal{S} \subset \mathcal{D} \subset T_Z$ are algebraic subsheaves satisfying the partial integrability condition $[\mathcal{S}, \mathcal{S}] \subset \mathcal{D}$, such that every compact complex manifold (X, J) of given dimension n embeds transversally by $f : X \hookrightarrow (Z, \mathcal{D})$, i.e. $J = J_f$, with the additional constraint $\mathrm{Im}(\overline{\partial} f) \subset \mathcal{S}$. [Note: our construction yields $\dim Z = O(n^4)$].

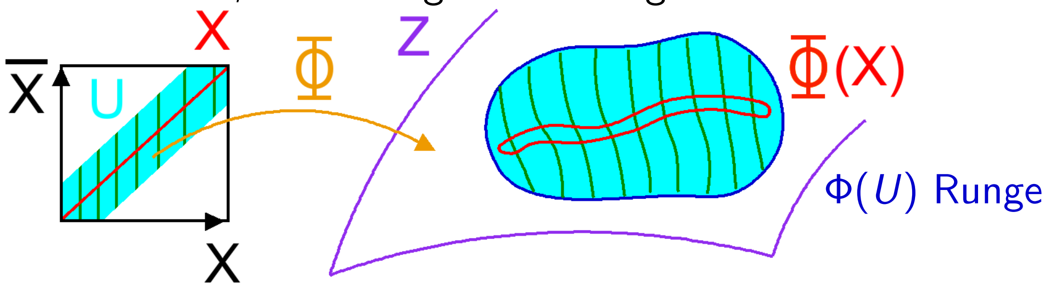
What about Bogomolov's original conjecture ?

Proposition (reduction of the conjecture to another one !)

Assume that holomorphic foliations can be approximated by Nash algebraic foliations uniformly on compact subsets of any polynomially convex open subset of $\mathbb{C}^N$.
Then every compact complex manifold can be approximated by compact complex manifolds that are embeddable in the sense of Bogomolov in foliated projective manifolds.

The proof uses the Grauert technique of embedding X as a totally real submanifold of $X \times \overline{X}$, and taking a Stein neighborhood $U \supset \Delta$.

Proof:



$\exists \Phi : U \to Z$ holomorphic embedding into Z affine algebraic (Stout).

The end

Happy birthday Helmut!