Hermitian-Yang-Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles

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Chern curvature tensor

This is $\Theta_{E,h} = i\nabla^2_{E,h} \in C^\infty(\Lambda^{1,1}T^*_X \otimes \mathrm{Hom}(E,E))$, which can be written

$$\Theta_{E,h} = i \sum_{1\leq j,k\leq n,\, 1\leq\lambda,\mu\leq r} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu$$

in terms of an orthonormal frame $(e_\lambda)_{1\leq\lambda\leq r}$ of E.

Positivity concepts for vector bundles

Griffiths and Nakano positivity

One looks at the associated quadratic form on $S = T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \overline{\xi}) \cdot v, v \rangle_h = \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu.$$

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Then E is said to be

- Griffiths positive (Griffiths 1969) if at any point $z \in X$
$$\widetilde{\Theta}_{E,h}(\xi \otimes v) > 0, \quad \forall 0 \neq \xi \in T_{X,z}, \ \forall 0 \neq v \in E_z$$

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E Nakano positive $\Rightarrow$ E Griffiths positive $\Rightarrow$ E ample.

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Easy and well known facts

E Nakano positive $\Rightarrow$ E Griffiths positive $\Rightarrow$ E ample.

In fact E Griffiths positive $\Rightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ positive.

Dual Nakano positivity – a conjecture

Curvature tensor of the dual bundle E^*

$$\Theta_{E^*,h} = -{}^T\Theta_{E,h} = -\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda}\,dz_j\wedge d\overline{z}_k\otimes(e^*_\lambda)^*\otimes e^*_\mu.$$

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Dual Nakano positivity is clearly stronger than Griffiths positivity. Also, it is better behaved than Nakano positivity, e.g.

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(Very speculative) conjecture

Is it true that E ample $\Rightarrow$ E dual Nakano positive ?

Brief discussion around this positivity conjecture

If true, Griffiths conjecture would follow:

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$$H^{n-1,n-1}(\mathbb{P}^n, \mathbb{C}) = H^{n-1}(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) = 0 \quad !!!$$

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Let us mention here that there are already known subtle relations between ampleness, Griffiths and Nakano positivity are known to hold – for instance, B. Berndtsson has proved that the ampleness of E implies the Nakano positivity of $S^m E \otimes \det E$ for every $m \in \mathbb{N}$.

“Total” determinant of the curvature tensor

If the Chern curvature tensor $\Theta_{E,h}$ is dual Nakano positive, then one can introduce the $(n \times r)$-dimensional determinant of the corresponding Hermitian quadratic form on $T_X \otimes E^*$

$$\det_{T_X \otimes E^*}({}^T\Theta_{E,h})^{1/r} := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

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Basic idea

Assigning a "matrix Monge-Ampère equation"

$$\det{}_{T_X \otimes E^*}({}^T\Theta_{E,h})^{1/r} = f > 0$$

where f is a positive (n,n)-form, may enforce the dual Nakano positivity of $\Theta_{E,h}$ if that assignment is combined with a continuity technique from an initial starting point where positivity is known.

Continuity method (case of rank 1)

For $r = 1$ and $h = h_0 e^{-\varphi}$, we have

$$ {}^T\Theta_{E,h} = \Theta_{E,h} = -i\partial\overline{\partial} \log h = \omega_0 + i\partial\overline{\partial}\varphi, $$

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When the right hand side $f = f_t$ of $(*)$ varies smoothly with respect to some parameter $t \in [0,1]$, one then gets a smoothly varying solution

$$ \Theta_{E,h_t} = \omega_0 + i\partial\overline{\partial}\varphi_t > 0, $$

and the positivity of Θ_{E,h_0} forces the positivity of Θ_{E,h_t} for all t.

Undeterminacy of the equation

Assuming E to be ample of rank $r > 1$, the equation

$$(**) \qquad \det_{T_X \otimes E^*}({}^T\Theta_{E,h})^{1/r} = f > 0$$

becomes underdetermined, as the real rank of the space of hermitian matrices $h = (h_{\lambda\mu})$ on E is equal to r^2, while $(**)$ provides only 1 scalar equation.

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Observation 1 (from the Donaldson-Uhlenbeck-Yau theorem)

Take a Hermitian metric η_0 on $\det E$ so that $\omega_0 := \Theta_{\det E,\eta_0} > 0$. If E is ω_0-polystable, $\exists h$ Hermitian metric h on E such that

$\omega_0^{n-1}\wedge\Theta_{E,h} = \frac{1}{r}\,\omega_0^n\otimes \mathrm{Id}_E$ (Hermite-Einstein equation, slope $\frac{1}{r}$).

Resulting trace free condition

Observation 2

The trace part of the above Hermite-Einstein equation is "automatic", hence the equation is equivalent to the trace free condition

$$\omega_0^{n-1} \wedge \Theta^\circ_{E,h} = 0,$$

when decomposing any endomorphism $u \in \mathrm{Herm}(E,E)$ as

$$u = u^\circ + \frac{1}{r}\,\mathrm{Tr}(u)\,\mathrm{Id}_E \in \mathrm{Herm}^\circ(E,E) \oplus \mathbb{R}\,\mathrm{Id}_E, \quad \mathrm{tr}(u^\circ) = 0.$$

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Remark

In case $\dim X = n = 1$, the trace free condition means that E is projectively flat, and the Umemura proof of the Griffiths conjecture proceeds exactly in that way, using the fact that the graded pieces of the Harder-Narasimhan filtration are projectively flat.

Towards a “cushioned” Hermite-Einstein equation

In general, one cannot expect E to be ω_0-polystable, but Uhlenbeck-Yau have shown that there always exists a smooth solution q_ε to a certain “cushioned” Hermite-Einstein equation.

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To make things more precise, let $\mathrm{Herm}(E)$ be the space of Hermitian (non necessarily positive) forms on E. Given a reference Hermitian metric $H_0 > 0$, let $\mathrm{Herm}_{H_0}(E, E)$ be the space of H_0-Hermitian endomorphisms $u \in \mathrm{Hom}(E, E)$; denote by

$$\mathrm{Herm}(E) \xrightarrow{\simeq} \mathrm{Herm}_{H_0}(E, E), \quad q \mapsto \widetilde{q} \ \text{ s.t. } \ q(v, w) = \langle \widetilde{q}(v), w \rangle_{H_0}$$

the natural isomorphism.

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In the sequel, we fix H_0 on E such that

$$\Theta_{\det E, \det H_0} = \omega_0 > 0.$$

A basic result from Uhlenbeck and Yau

Uhlenbeck-Yau 1986, Theorem 3.1

For every $\varepsilon > 0$, there always exists a (unique) smooth Hermitian metric q_ε on E such that

$$\omega_0^{n-1} \wedge \Theta_{E,q_\varepsilon} = \omega_0^n \otimes \left(\frac{1}{r}\,\mathrm{Id}_E - \varepsilon \log \widetilde{q}_\varepsilon\right),$$

where $\widetilde{q}_\varepsilon$ is computed with respect to H_0, and $\log g$ denotes the logarithm of a positive Hermitian endomorphism g.

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The reason is that the term $-\varepsilon \log \widetilde{q}_\varepsilon$ is a "friction term" that prevents the explosion of the a priori estimates, similarly what happens for Monge-Ampère equations $(\omega_0 + i\partial\overline{\partial}\varphi)^n = e^{\varepsilon\varphi+f}\omega_0^n$.

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The above matrix equation is equivalent to prescribing $\det q_\varepsilon = \det H_0$ and the trace free equation of rank (r^2-1)

$$\omega_0^{n-1} \wedge \Theta^\circ_{E,q_\varepsilon} = -\varepsilon\,\omega_0^n \otimes \log \widetilde{q}_\varepsilon.$$

Search for an appropriate evolution equation

General setup

In this context, given $\alpha > 0$ large enough, it is natural to search for a time dependent family of metrics $h_t(z)$ on the fibers E_z of E, $t \in [0,1]$, satisfying a generalized Monge-Ampère equation

$$(D) \quad \det_{T_X \otimes E^*}\left({}^T\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_{E^*}\right)^{1/r} = f_t\,\omega_0^n, \quad f_t > 0,$$

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and trace free, rank $r^2 - 1$, Hermite-Einstein conditions

$$(T) \quad \omega_t^{n-1} \wedge \Theta^\circ_{E,h_t} = g_t$$

with smoothly varying families of functions $f_t \in C^\infty(X,\mathbb{R})$, Hermitian metrics $\omega_t > 0$ on X and sections

$$g_t \in C^\infty(X, \Lambda^{n,n}_{\mathbb{R}} T^*_X \otimes \mathrm{Herm}^\circ_{h_t}(E,E)), \quad t \in [0,1].$$

Search for an appropriate evolution equation

General setup

In this context, given $\alpha > 0$ large enough, it is natural to search for a time dependent family of metrics $h_t(z)$ on the fibers E_z of E, $t \in [0,1]$, satisfying a generalized Monge-Ampère equation

$$(D) \quad \det_{T_X \otimes E^*}\left({}^T\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_{E^*}\right)^{1/r} = f_t\,\omega_0^n, \quad f_t > 0,$$

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Observe that this is a determined (not overdetermined!) system.

Choice of the initial state ($t = 0$)

We start with the Uhlenbeck-Yau solution $h_0 = q_\varepsilon$ of of the "cushioned" trace free Hermite-Einstein equation, so that $\det h_0 = \det H_0$, and take $\alpha > 0$ so large that

$${}^T\Theta_{E,h_0} + \alpha\,\omega_0 \otimes \mathrm{Id}_{E^*} > 0 \text{ in the sense of Nakano.}$$

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If conditions (D) and (T) can be met for all $t \in [0,1]$, thus without any explosion of the solutions h_t, we infer from (D) that

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Observation

At time $t = 1$, we would then get a Hermitian metric h_1 on E such that Θ_{E,h_1} is dual Nakano positive !!

Possible choices of the right hand side

One still has the freedom of adjusting f_t, ω_t and g_t in the general setup. There are in fact many possibilities:

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Proposition

Let (E, H_0) be a smooth Hermitian holomorphic vector bundle such that E is ample and $\omega_0 = \Theta_{\det E, \det H_0} > 0$. Then the system of determinantal and trace free equations

$$(D)\quad \det_{T_X \otimes E^*}\left({}^T\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_{E^*}\right)^{1/r} = F(t, z, h_t, D_z h_t)$$

$$(T)\quad \omega_0^{n-1} \wedge \Theta^\circ_{E,h_t} = G(t, z, h_t, D_z h_t, D_z^2 h_t) \in \mathrm{Herm}^\circ(E, E)$$

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(where $F > 0$), is a well determined system of PDEs.

It is elliptic whenever the symbol η_h of the linearized operator $u \mapsto DG_{D^2h}(t, z, h, Dh, D^2h) \cdot D^2 u$ has an Hilbert-Schmidt norm

$$\sup_{\xi \in T_X^*, |\xi|_{\omega_0} = 1} \|\eta_{h_t}(\xi)\|_{h_t} \le (r^2+1)^{-1/2}\, n^{-1}$$

for any metric h_t involved, e.g. if G does not depend on $D^2 h$.

Proof of the ellipticity

The (long, computational) proof consists of analyzing the linearized system of equations, starting from the curvature tensor formula

$$\Theta_{E,h} = i\overline{\partial}(h^{-1}\partial h) = i\overline{\partial}(\widetilde{h}^{-1}\partial_{H_0}\widetilde{h}),$$

where $\partial_{H_0} s = H_0^{-1}\partial(H_0 s)$ is the $(1,0)$-component of the Chern connection on $\mathrm{Hom}(E,E)$ associated with H_0 on E.

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Let us recall that the ellipticity of an operator

$$P : C^\infty(V) \to C^\infty(W), \quad f \mapsto P(f) = \textstyle\sum_{|\alpha|\le m} a_\alpha(x) D^\alpha f(x)$$

means the invertibility of the principal symbol

$$\sigma_P(x,\xi) = \textstyle\sum_{|\alpha|\le m} a_\alpha(x)\,\xi^\alpha \in \mathrm{Hom}(V,W)$$

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For instance, on the torus $\mathbb{R}^n/\mathbb{Z}^n$, $f \mapsto P_\lambda(f) = -\Delta f + \lambda f$ has an invertible symbol $\sigma_{P_\lambda}(x,\xi) = -|\xi|^2$, but P_λ is invertible only for $\lambda > 0$.

A more specific choice of the right hand side

Theorem

The elliptic differential system defined by

$$\det{}_{T_X\otimes E^*}\left({}^T\Theta_{E,h}+(1-t)\alpha\,\omega_0\otimes \mathrm{Id}_{E^*}\right)^{1/r}=\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\lambda}a_0(z),$$

$$\omega_0^{n-1}\wedge\Theta_{E^\circ,h}=-\varepsilon\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\mu}(\log\widetilde{h}^\circ)\,\omega_0^n$$

possesses an invertible elliptic linearization for $\varepsilon\geq\varepsilon_0(h_t)$ and $\lambda\geq\lambda_0(h_t)(1+\mu^2)$, with $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ large enough.

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Corollary

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Here, the proof consists of analyzing the total symbol of the linearized operator, and the rest is just linear algebra.

Monge-Ampère volume for vector bundles

If $E \to X$ is an ample vector bundle of rank r that is dual Nakano positive, one can introduce its Monge-Ampère volume to be

$$\mathrm{MAVol}(E) = \sup_h \int_X \det{}_{T_X \otimes E^*} \left((2\pi)^{-1}\, {}^T\Theta_{E,h}\right)^{1/r},$$

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Taking $\omega_0 = \Theta_{\det E}$, the proof is a consequence of the inequality $(\prod \lambda_j)^{1/nr} \leq \frac{1}{nr} \sum \lambda_j$ between geometric and arithmetic means, for the eigenvalues λ_j of $(2\pi)^{-1}\, {}^T\Theta_{E,h}$, after raising to power n.

Concluding remarks

- Siarhei Finski (PostDoc at Institut Fourier right now) has observed that the equality holds iff E is projectively flat.

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- In the split case, it seems natural to conjecture that

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- The Euler-Lagrange equation for the maximizer is 4th order.

Thank you for your attention