

Hermitian-Yang-Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles

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Positive and ample vector bundles

Let X be a projective n -dimensional manifold and $E \rightarrow X$ a holomorphic vector bundle of rank $r \geq 1$.

Ample vector bundles

$E \rightarrow X$ is said to be **ample in the sense of Hartshorne** if the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is ample.

By Kodaira, this is equivalent to the existence of a **smooth hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ with positive curvature** (equivalently, a negatively curved Finsler metric on E^*).

Chern curvature tensor

This is $\Theta_{E,h} = i\nabla_{E,h}^2 \in C^\infty(\Lambda^{1,1} T_X^* \otimes \text{Hom}(E, E))$, which can be written

$$\Theta_{E,h} = i \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

in terms of an orthonormal frame $(e_\lambda)_{1 \leq \lambda \leq r}$ of E .

Positivity concepts for vector bundles

Griffiths and Nakano positivity

One looks at the associated quadratic form on $S = T_X \otimes E$

$$\tilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \bar{\xi}) \cdot v, v \rangle_h = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} \xi_j \bar{\xi}_k v_\lambda \bar{v}_\mu.$$

Then E is said to be

- Griffiths positive (Griffiths 1969) if at any point $z \in X$

$$\tilde{\Theta}_{E,h}(\xi \otimes v) > 0, \quad \forall 0 \neq \xi \in T_{X,z}, \quad \forall 0 \neq v \in E_z$$

- Nakano positive (Nakano 1955) if at any point $z \in X$

$$\tilde{\Theta}_{E,h}(\tau) = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} \tau_{j,\lambda} \bar{\tau}_{k,\mu} > 0, \quad \forall 0 \neq \tau \in T_{X,z} \otimes E_z.$$

Easy and well known facts

$$E \text{ Nakano positive} \Rightarrow E \text{ Griffiths positive} \Rightarrow E \text{ ample.}$$

In fact E Griffiths positive $\Rightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ positive.

Dual Nakano positivity – a conjecture

Curvature tensor of the dual bundle E^*

$$\Theta_{E^*,h} = -{}^T \Theta_{E,h} = - \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{jk\mu\lambda} dz_j \wedge d\bar{z}_k \otimes (e_\lambda^*)^* \otimes e_\mu^*.$$

Dual Nakano positivity

One requires

$$-\tilde{\Theta}_{E^*,h}(\tau) = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{jk\mu\lambda} \tau_{j,\lambda} \bar{\tau}_{k,\mu} > 0, \quad \forall 0 \neq \tau \in T_{X,z} \otimes E_z^*.$$

Dual Nakano positivity is **clearly stronger** than Griffiths positivity.

Also, it is better behaved than Nakano positivity, e.g.

E dual Nakano positive

\Rightarrow any quotient $Q = E/S$ is also dual Nakano positive.

(Very speculative) conjecture

Is it true that E ample $\Rightarrow E$ dual Nakano positive ?

Brief discussion around this positivity conjecture

If true, Griffiths conjecture would follow:

$$E \text{ ample} \Leftrightarrow E \text{ dual Nakano positive} \Leftrightarrow E \text{ Griffiths positive.}$$

Remark

$$E \text{ ample} \not\Rightarrow E \text{ Nakano positive, in fact} \\ E \text{ Griffiths positive} \not\Rightarrow E \text{ Nakano positive.}$$

For instance, $T_{\mathbb{P}^n}$ is easily shown to be ample and Griffiths positive for the Fubini-Study metric, but it is **not Nakano positive**. Otherwise the Nakano vanishing theorem would then yield

$$H^{n-1, n-1}(\mathbb{P}^n, \mathbb{C}) = H^{n-1}(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-1}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) = 0 \quad !!!$$

Let us mention here that there are already known subtle relations between ampleness, Griffiths and Nakano positivity are known to hold – for instance, B. Berndtsson has proved that the ampleness of E implies the Nakano positivity of $S^m E \otimes \det E$ for every $m \in \mathbb{N}$.

“Total” determinant of the curvature tensor

If the Chern curvature tensor $\Theta_{E,h}$ is **dual Nakano positive**, then one can introduce the $(n \times r)$ -dimensional determinant of the corresponding Hermitian quadratic form on $T_X \otimes E^*$

$$\det_{T_X \otimes E^*}({}^T \Theta_{E,h})^{1/r} := \det(c_{jk\mu\lambda})_{(j,\lambda),(k,\mu)}^{1/r} \text{id}z_1 \wedge d\bar{z}_1 \wedge \dots \wedge \text{id}z_n \wedge d\bar{z}_n.$$

This (n, n) -form does not depend on the choice of coordinates (z_j) on X , nor on the choice of the orthonormal frame (e_λ) on E .

Basic idea

Assigning a “matrix Monge-Ampère equation”

$$\det_{T_X \otimes E^*}({}^T \Theta_{E,h})^{1/r} = f > 0$$

where f is a positive (n, n) -form, may enforce the dual Nakano positivity of $\Theta_{E,h}$ if that assignment is combined with a continuity technique from an initial starting point where positivity is known.

Continuity method (case of rank 1)

For $r = 1$ and $h = h_0 e^{-\varphi}$, we have

$${}^T \Theta_{E,h} = \Theta_{E,h} = -i\partial\bar{\partial} \log h = \omega_0 + i\partial\bar{\partial}\varphi,$$

and the equation reduces to a standard Monge-Ampère equation

$$(*) \quad (\Theta_{E,h})^n = (\omega_0 + i\partial\bar{\partial}\varphi)^n = f.$$

If f is given and independent of h , Yau's theorem guarantees the existence of a unique solution $\theta = \Theta_{E,h} > 0$, provided E is an ample line bundle and $\int_X f = c_1(E)^n$.

When the right hand side $f = f_t$ of $(*)$ varies smoothly with respect to some parameter $t \in [0, 1]$, one then gets a smoothly varying solution

$$\Theta_{E,h_t} = \omega_0 + i\partial\bar{\partial}\varphi_t > 0,$$

and the positivity of Θ_{E,h_0} forces the positivity of Θ_{E,h_t} for all t .

Undeterminacy of the equation

Assuming E to be ample of rank $r > 1$, the equation

$$(**) \quad \det_{T_X \otimes E^*} ({}^T \Theta_{E,h})^{1/r} = f > 0$$

becomes underdetermined, as the real rank of the space of hermitian matrices $h = (h_{\lambda\mu})$ on E is equal to r^2 , while $(**)$ provides only 1 scalar equation.

(Solutions might still exist, but lack uniqueness and a priori bounds.)

Conclusion

In order to recover a well determined system of equations, one needs an additional “matrix equation” of rank $(r^2 - 1)$.

Observation 1 (from the Donaldson-Uhlenbeck-Yau theorem)

Take a Hermitian metric η_0 on $\det E$ so that $\omega_0 := \Theta_{\det E, \eta_0} > 0$. If E is ω_0 -polystable, $\exists h$ Hermitian metric h on E such that

$$\omega_0^{n-1} \wedge \Theta_{E,h} = \frac{1}{r} \omega_0^n \otimes \text{Id}_E \quad (\text{Hermite-Einstein equation, slope } \frac{1}{r}).$$

Resulting trace free condition

Observation 2

The trace part of the above Hermite-Einstein equation is “automatic”, hence the equation is equivalent to the trace free condition

$$\omega_0^{n-1} \wedge \Theta_{E,h}^\circ = 0,$$

when decomposing any endomorphism $u \in \text{Herm}(E, E)$ as

$$u = u^\circ + \frac{1}{r} \text{Tr}(u) \text{Id}_E \in \text{Herm}^\circ(E, E) \oplus \mathbb{R} \text{Id}_E, \quad \text{tr}(u^\circ) = 0.$$

Observation 3

The trace free condition is a matrix equation of **rank $(r^2 - 1)$** !!!

Remark

In case $\dim X = n = 1$, the trace free condition means that E is **projectively flat**, and the Umemura proof of the Griffiths conjecture proceeds exactly in that way, using the fact that the graded pieces of the Harder-Narasimhan filtration are projectively flat.

Towards a “cushioned” Hermite-Einstein equation

In general, one cannot expect E to be ω_0 -polystable, but Uhlenbeck-Yau have shown that there always exists a smooth solution q_ε to a certain **“cushioned” Hermite-Einstein equation**.

To make things more precise, let $\text{Herm}(E)$ be the space of Hermitian (non necessarily positive) forms on E . Given a reference Hermitian metric $H_0 > 0$, let $\text{Herm}_{H_0}(E, E)$ be the space of H_0 -Hermitian endomorphisms $u \in \text{Hom}(E, E)$; denote by

$$\text{Herm}(E) \xrightarrow{\cong} \text{Herm}_{H_0}(E, E), \quad q \mapsto \tilde{q} \text{ s.t. } q(v, w) = \langle \tilde{q}(v), w \rangle_{H_0}$$

the natural isomorphism. Let also

$$\text{Herm}_{H_0}^\circ(E, E) = \{q \in \text{Herm}_{H_0}(E, E); \text{tr}(q) = 0\}$$

be the subspace of “trace free” Hermitian endomorphisms.

In the sequel, we fix H_0 on E such that

$$\Theta_{\det E, \det H_0} = \omega_0 > 0.$$

Uhlenbeck-Yau 1986, Theorem 3.1

For every $\varepsilon > 0$, there **always exists** a (unique) smooth Hermitian metric q_ε on E such that

$$\omega_0^{n-1} \wedge \Theta_{E, q_\varepsilon} = \omega_0^n \otimes \left(\frac{1}{r} \text{Id}_E - \varepsilon \log \tilde{q}_\varepsilon \right),$$

where \tilde{q}_ε is computed with respect to H_0 , and $\log g$ denotes the logarithm of a positive Hermitian endomorphism g .

The reason is that the term $-\varepsilon \log \tilde{q}_\varepsilon$ is a “friction term” that prevents the explosion of the a priori estimates, similarly what happens for Monge-Ampère equations $(\omega_0 + i\partial\bar{\partial}\varphi)^n = e^{\varepsilon\varphi + f} \omega_0^n$.

The above matrix equation is equivalent to prescribing $\det q_\varepsilon = \det H_0$ and the trace free equation of rank $(r^2 - 1)$

$$\omega_0^{n-1} \wedge \Theta_{E, q_\varepsilon}^\circ = -\varepsilon \omega_0^n \otimes \log \tilde{q}_\varepsilon.$$

Search for an appropriate evolution equation

General setup

In this context, given $\alpha > 0$ large enough, it is natural to search for a time dependent family of metrics $h_t(z)$ on the fibers E_z of E , $t \in [0, 1]$, satisfying a generalized Monge-Ampère equation

$$(D) \quad \det_{T_X \otimes E^*} \left({}^T \Theta_{E, h_t} + (1-t)\alpha \omega_0 \otimes \text{Id}_{E^*} \right)^{1/r} = f_t \omega_0^n, \quad f_t > 0,$$

and trace free, rank $r^2 - 1$, Hermite-Einstein conditions

$$(T) \quad \omega_t^{n-1} \wedge \Theta_{E, h_t}^\circ = g_t$$

with smoothly varying families of functions $f_t \in C^\infty(X, \mathbb{R})$, Hermitian metrics $\omega_t > 0$ on X and sections

$$g_t \in C^\infty(X, \Lambda_{\mathbb{R}}^{n,n} T_X^* \otimes \text{Herm}_{h_t}^\circ(E, E)), \quad t \in [0, 1].$$

Observe that this is a determined (not overdetermined!) system.

Choice of the initial state ($t = 0$)

We start with the Uhlenbeck-Yau solution $h_0 = q_\varepsilon$ of the “cushioned” trace free Hermite-Einstein equation, so that $\det h_0 = \det H_0$, and take $\alpha > 0$ so large that

$${}^T\Theta_{E,h_0} + \alpha\omega_0 \otimes \text{Id}_{E^*} > 0 \text{ in the sense of Nakano.}$$

If conditions (D) and (T) can be met for all $t \in [0, 1]$, thus without any explosion of the solutions h_t , we infer from (D) that

$${}^T\Theta_{E,h_t} + (1-t)\alpha\omega_0 \otimes \text{Id}_{E^*} > 0 \text{ in the sense of Nakano}$$

for all $t \in [0, 1]$.

Observation

At time $t = 1$, we would then get a Hermitian metric h_1 on E such that Θ_{E,h_1} is dual Nakano positive !!

Possible choices of the right hand side

One still has the freedom of adjusting f_t , ω_t and g_t in the general setup. There are in fact many possibilities:

Proposition

Let (E, H_0) be a smooth Hermitian holomorphic vector bundle such that E is ample and $\omega_0 = \Theta_{\det E, \det H_0} > 0$. Then the system of determinantal and trace free equations

$$(D) \quad \det_{T_X \otimes E^*} ({}^T\Theta_{E,h_t} + (1-t)\alpha\omega_0 \otimes \text{Id}_{E^*})^{1/r} = F(t, z, h_t, D_z h_t)$$

$$(T) \quad \omega_0^{n-1} \wedge \Theta_{E,h_t}^\circ = G(t, z, h_t, D_z h_t, D_z^2 h_t) \in \text{Herm}^\circ(E, E)$$

(where $F > 0$), is a well determined system of PDEs.

It is **elliptic** whenever the symbol η_h of the linearized operator $u \mapsto DG_{D^2 h}(t, z, h, Dh, D^2 h) \cdot D^2 u$ has an Hilbert-Schmidt norm

$$\sup_{\xi \in T_X^*, |\xi|_{\omega_0} = 1} \|\eta_{h_t}(\xi)\|_{h_t} \leq (r^2 + 1)^{-1/2} n^{-1}$$

for any metric h_t involved, e.g. if G does not depend on $D^2 h$.

Proof of the ellipticity

The (long, computational) proof consists of analyzing the linearized system of equations, starting from the curvature tensor formula

$$\Theta_{E,h} = i\bar{\partial}(h^{-1}\partial h) = i\bar{\partial}(\tilde{h}^{-1}\partial_{H_0}\tilde{h}),$$

where $\partial_{H_0}s = H_0^{-1}\partial(H_0s)$ is the $(1,0)$ -component of the Chern connection on $\text{Hom}(E, E)$ associated with H_0 on E .

Let us recall that the ellipticity of an operator

$$P : C^\infty(V) \rightarrow C^\infty(W), \quad f \mapsto P(f) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha f(x)$$

means the invertibility of the principal symbol

$$\sigma_P(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \in \text{Hom}(V, W)$$

whenever $0 \neq \xi \in T_{X,x}^*$.

For instance, on the torus $\mathbb{R}^n/\mathbb{Z}^n$, $f \mapsto P_\lambda(f) = -\Delta f + \lambda f$ has an invertible symbol $\sigma_{P_\lambda}(x, \xi) = -|\xi|^2$, but P_λ is invertible only for $\lambda > 0$.

A more specific choice of the right hand side

Theorem

The elliptic differential system defined by

$$\det_{T_X \otimes E^*} \left({}^T \Theta_{E,h} + (1-t)\alpha\omega_0 \otimes \text{Id}_{E^*} \right)^{1/r} = \left(\frac{\det H_0(z)}{\det h_t(z)} \right)^\lambda a_0(z),$$

$$\omega_0^{n-1} \wedge \Theta_{E^\circ, h} = -\varepsilon \left(\frac{\det H_0(z)}{\det h_t(z)} \right)^\mu (\log \tilde{h}^\circ) \omega_0^n$$

possesses an **invertible elliptic linearization** for $\varepsilon \geq \varepsilon_0(h_t)$ and $\lambda \geq \lambda_0(h_t)(1 + \mu^2)$, with $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ large enough.

Corollary

Under the above conditions, starting from the Uhlenbeck-Yau solution h_0 such that $\det h_0 = \det H_0$ at $t = 0$, the PDE system **still has a solution** for $t \in [0, t_0]$ and $t_0 > 0$ small. (What for $t_0 = 1$?)

Here, the proof consists of analyzing the **total symbol** of the linearized operator, and the rest is just linear algebra.

Monge-Ampère volume for vector bundles

If $E \rightarrow X$ is an ample vector bundle of rank r that is dual Nakano positive, one can introduce its **Monge-Ampère volume** to be

$$\text{MAVol}(E) = \sup_h \int_X \det_{T_X \otimes E^*} \left((2\pi)^{-1} T \Theta_{E,h} \right)^{1/r},$$

where the supremum is taken over all smooth metrics h on E such that $T \Theta_{E,h}$ is Nakano positive.

This supremum is always finite, and in fact

Proposition

For any dual Nakano positive vector bundle E , one has

$$\text{MAVol}(E) \leq r^{-n} c_1(E)^n.$$

Taking $\omega_0 = \Theta_{\det E}$, the proof is a consequence of the inequality $(\prod \lambda_j)^{1/nr} \leq \frac{1}{nr} \sum \lambda_j$ between geometric and arithmetic means, for the eigenvalues λ_j of $(2\pi)^{-1} T \Theta_{E,h}$, after raising to power n .

Concluding remarks

- Siarhei Finski (PostDoc at Institut Fourier right now) has observed that the equality holds iff E is **projectively flat**.
- In the split case $E = \bigoplus_{1 \leq j \leq r} E_j$ and $h = \bigoplus_{1 \leq j \leq r} h_j$, the inequality reads

$$\left(\prod_{1 \leq j \leq r} c_1(E_j)^n \right)^{1/r} \leq r^{-n} c_1(E)^n,$$

with equality iff $c_1(E_1) = \dots = c_1(E_r)$.

- In the split case, it seems natural to conjecture that

$$\text{MAVol}(E) = \left(\prod_{1 \leq j \leq r} c_1(E_j)^n \right)^{1/r},$$

i.e. that the supremum is reached for split metrics $h = \bigoplus h_j$.

- The Euler-Lagrange equation for the maximizer is 4th order.

Thank you for your attention

