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Hermitian-Yang-Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles

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Ample vector bundles

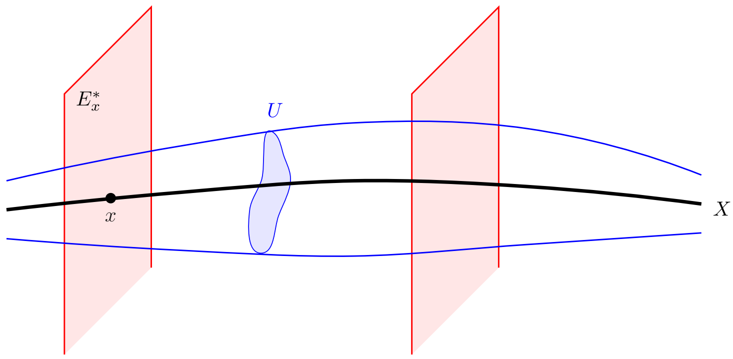
Let X be a projective n-dimensional manifold and $E \to X$ a holomorphic vector bundle of rank $r \geq 1$.

Ample vector bundles

$E \to X$ is said to be ample in the sense of Hartshorne if the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is ample, i.e. by Kodaira $\Longleftrightarrow \exists\ C^\infty$ hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ with positive curvature.

This is equivalent to the existence of a a strongly pseudoconvex tubular neighborhood U of the 0-section in E^*, i.e. of a negatively curved Finsler metric on E^*.

Geometric interpretation:
U can be taken S^1 invariant
$U \rightsquigarrow \bigcap_{|\lambda|=1} \lambda U$



Chern curvature tensor and positivity concepts

Chern curvature tensor

This is $\Theta_{E,h} = i\nabla^2_{E,h} \in C^\infty(\Lambda^{1,1}T^*_X \otimes \mathrm{Hom}(E,E))$, written

$$\Theta_{E,h} = i \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\, dz_j \wedge d\bar z_k \otimes e^*_\lambda \otimes e_\mu$$

in terms of an orthonormal frame $(e_\lambda)_{1\le\lambda\le r}$ of E.

Griffiths and Nakano positivity

One looks at the associated quadratic form on $S = T_X \otimes E$

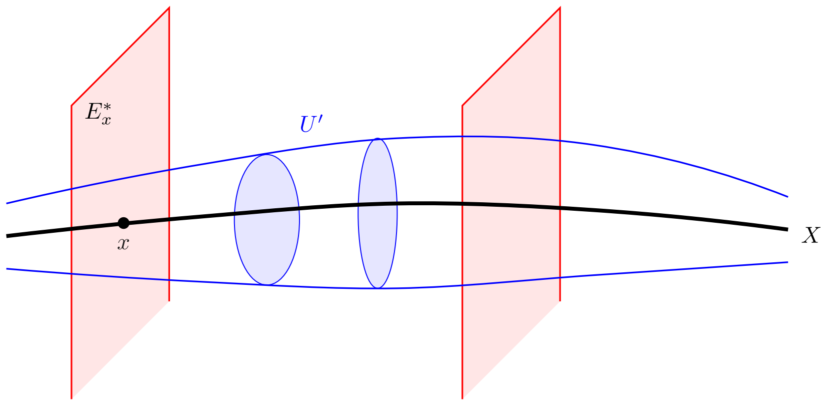
$$\widetilde\Theta_{E,h}(\xi\otimes v) := \langle \Theta_{E,h}(\xi,\bar\xi)\cdot v, v\rangle_h = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\xi_j\bar\xi_k v_\lambda \bar v_\mu.$$

Then E is said to be:

- Griffiths positive (Griffiths 1969) if at any point $z\in X$
$$\widetilde\Theta_{E,h}(\xi\otimes v) > 0, \quad \forall 0\ne\xi\in T_{X,z},\ \forall 0\ne v\in E_z$$
- Nakano positive (Nakano 1955) if at any point $z\in X$
$$\widetilde\Theta_{E,h}(\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j,\lambda}\bar\tau_{k,\mu} > 0, \quad \forall 0\ne\tau\in T_{X,z}\otimes E_z.$$

Geometric interpretation of Griffiths positivity

Griffiths positivity of E is equivalent to the existence of a strongly pseudoconvex neighborhood U' of the 0-section in E^* whose fibers are (varying) hermitian balls.



(Nakano >0 is more restrictive than strict pseudoconvexity of U'.)

Easy and well known facts

E Nakano positive $\Rightarrow$ E Griffiths positive $\Rightarrow$ E ample.

In fact, E Griffiths positive $\Rightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ positive.

Dual Nakano positivity – a conjecture

Curvature tensor of the dual bundle E^*

$$\Theta_{E^*,h} = -{}^T\Theta_{E,h} = -\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda}\, dz_j \wedge d\overline{z}_k \otimes (e_\lambda^*)^* \otimes e_\mu^*.$$

Dual Nakano positivity

One requires

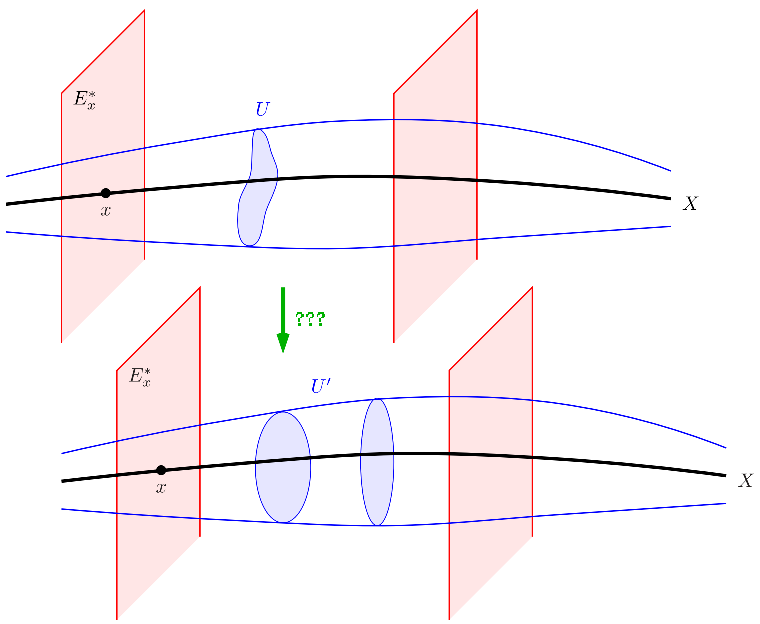
$$-\widetilde{\Theta}_{E^*,h}(\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda}\tau_{j\lambda}\overline{\tau}_{k\mu} > 0, \quad \forall 0 \ne \tau \in T_{X,z} \otimes E_z^*.$$

Dual Nakano positivity is clearly stronger than Griffiths positivity.
Also, it is better behaved than Nakano positivity, e.g.
E dual Nakano positive
$\Rightarrow$ any quotient $Q = E/S$ is also dual Nakano positive.

(Very speculative) conjecture

Is it true that E ample $\Rightarrow$ E dual Nakano positive ?

Geometric interpretation of the conjecture



Basic question. Is there a (geometric, analytic) procedure that turns the strictly pseudoconvex neighborhood U into another strictly pseudoconvex U' that would be a ball bundle ?

Answer is yes if $n = \dim X = 1$ (Umemura, Campana-Flenner) !!

Brief discussion around this positivity conjecture

If true, Griffiths conjecture would follow:

E ample $\Leftrightarrow$ E dual Nakano positive $\Leftrightarrow$ E Griffiths positive.

Remark

E ample $\not\Rightarrow$ E Nakano positive, in fact

E Griffiths positive $\not\Rightarrow$ E Nakano positive.

For instance, $T_{\mathbb{P}^n}$ is easy shown to be ample and Griffiths positive for the Fubini-Study metric, but it is not Nakano positive. Otherwise the Nakano vanishing theorem would then yield

$$H^{n-1,n-1}(\mathbb{P}^n,\mathbb{C}) = H^{n-1}(\mathbb{P}^n,\Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n}\otimes T_{\mathbb{P}^n}) = 0 \quad !!!$$

Let us mention here that there are already known subtle relations between ampleness, Griffiths and Nakano positivity are known to hold – for instance, B. Berndtsson has proved that the ampleness of E implies the Nakano positivity of $S^mE\otimes\det E$ for every $m\in\mathbb{N}$.

"Total" determinant of the curvature tensor

If the Chern curvature tensor $\Theta_{E,h}$ is dual Nakano positive, then one can introduce the $(n\times r)$-dimensional determinant of the corresponding Hermitian quadratic form on $T_X\otimes E^*$

$$\det{}_{T_X\otimes E^*}({}^T\Theta_{E,h})^{1/r} := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1\wedge d\overline{z}_1\wedge\ldots\wedge idz_n\wedge d\overline{z}_n.$$

This (n,n)-form does not depend on the choice of coordinates (z_j) on X, nor on the choice of the orthonormal frame (e_λ) on E.

Basic idea

Assigning a "matrix Monge-Ampère equation"

$$\det{}_{T_X\otimes E^*}({}^T\Theta_{E,h})^{1/r} = f > 0$$

where f is a positive (n,n)-form, may enforce the dual Nakano positivity of $\Theta_{E,h}$ if that assignment is combined with a continuity technique from an initial starting point where positivity is known.

Continuity method (case of rank 1)

For $r = 1$ and $h = h_0 e^{-\varphi}$, we have

$$^T\Theta_{E,h} = \Theta_{E,h} = -i\partial\overline{\partial}\log h = \omega_0 + i\partial\overline{\partial}\varphi,$$

and the equation reduces to a standard Monge-Ampère equation

$$(*) \qquad (\Theta_{E,h})^n = (\omega_0 + i\partial\overline{\partial}\varphi)^n = f.$$

If f is given and independent of h, Yau's theorem guarantees the existence of a unique solution $\theta = \Theta_{E,h} > 0$, provided E is an ample line bundle and $\int_X f = c_1(E)^n$.

When the right hand side $f = f_t$ of $(*)$ varies smoothly with respect to some parameter $t \in [0,1]$, one then gets a smoothly varying solution

$$\Theta_{E,h_t} = \omega_0 + i\partial\overline{\partial}\varphi_t > 0,$$

and the positivity of Θ_{E,h_0} forces the positivity of Θ_{E,h_t} for all t.

Undeterminacy of the equation

Assuming E to be ample of rank $r > 1$, the equation

$$(**) \qquad \det_{T_X \otimes E^*}(\,^T\Theta_{E,h})^{1/r} = f > 0$$

becomes underdetermined, as the real rank of the space of hermitian matrices $h = (h_{\lambda\mu})$ on E is equal to r^2, while $(**)$ provides only 1 scalar equation.

(Solutions might still exist, but lack uniqueness and a priori bounds.)

Conclusion

In order to recover a well determined system of equations, one needs an additional "matrix equation" of rank $(r^2 - 1)$.

Observation 1 (from the Donaldson-Uhlenbeck-Yau theorem)

Take a Hermitian metric η_0 on $\det E$ so that $\omega_0 := \Theta_{\det E,\eta_0} > 0$. If E is ω_0-polystable, $\exists h$ Hermitian metric h on E such that

$$\omega_0^{n-1} \wedge \Theta_{E,h} = \tfrac{1}{r}\,\omega_0^n \otimes \mathrm{Id}_E \quad \text{(Hermite-Einstein equation, slope } \tfrac{1}{r}\text{)}.$$

Resulting trace free condition

Observation 2

The trace part of the above Hermite-Einstein equation is "automatic", hence the equation is equivalent to the trace free condition

$$\omega_0^{n-1} \wedge \Theta^\circ_{E,h} = 0,$$

when decomposing any endomorphism $u \in \mathrm{Herm}(E,E)$ as

$$u = u^\circ + \tfrac{1}{r}\,\mathrm{Tr}(u)\,\mathrm{Id}_E \in \mathrm{Herm}^\circ(E,E) \oplus \mathbb{R}\,\mathrm{Id}_E, \quad \mathrm{tr}(u^\circ) = 0.$$

Observation 3

The trace free condition is a matrix equation of rank (r^2-1) !!!

Remark

In case $\dim X = n = 1$, the trace free condition means that E is projectively flat, and the Umemura proof of the Griffiths conjecture proceeds exactly in that way, using the fact that the graded pieces of the Harder-Narasimhan filtration are projectively flat.

Towards a "cushioned" Hermite-Einstein equation

In general, one cannot expect E to be ω_0-polystable, but Uhlenbeck-Yau have shown that there always exists a smooth solution q_ε to a certain "cushioned" Hermite-Einstein equation.

To make things more precise, let $\mathrm{Herm}(E)$ be the space of Hermitian (non necessarily positive) forms on E. Given a reference Hermitian metric $H_0 > 0$, let $\mathrm{Herm}_{H_0}(E,E)$ be the space of H_0-Hermitian endomorphisms $u \in \mathrm{Hom}(E,E)$; denote by

$$\mathrm{Herm}(E) \xrightarrow{\simeq} \mathrm{Herm}_{H_0}(E,E), \quad q \mapsto \widetilde{q} \;\text{ s.t. }\; q(v,w) = \langle \widetilde{q}(v), w\rangle_{H_0}$$

the natural isomorphism. Let also

$$\mathrm{Herm}^\circ_{H_0}(E,E) = \big\{q \in \mathrm{Herm}_{H_0}(E,E)\,;\; \mathrm{tr}(q) = 0\big\}$$

be the subspace of "trace free" Hermitian endomorphisms.

In the sequel, we fix H_0 on E such that

$$\Theta_{\det E, \det H_0} = \omega_0 > 0.$$

A basic result from Uhlenbeck and Yau

Uhlenbeck-Yau 1986, Theorem 3.1

For every $\varepsilon > 0$, there always exists a (unique) smooth Hermitian metric q_ε on E such that

$$\omega_0^{n-1} \wedge \Theta_{E,q_\varepsilon} = \omega_0^n \otimes \left(\frac{1}{r}\,\mathrm{Id}_E - \varepsilon \log \widetilde{q}_\varepsilon\right),$$

where $\widetilde{q}_\varepsilon$ is computed with respect to H_0, and $\log g$ denotes the logarithm of a positive Hermitian endomorphism g.

The reason is that the term $-\varepsilon \log \widetilde{q}_\varepsilon$ is a "friction term" that prevents the explosion of the a priori estimates, similarly what happens for Monge-Ampère equations $(\omega_0 + i\partial\overline{\partial}\varphi)^n = e^{\varepsilon\varphi+f}\omega_0^n$.

The above matrix equation is equivalent to prescribing $\det q_\varepsilon = \det H_0$ and the trace free equation of rank (r^2-1)

$$\omega_0^{n-1} \wedge \Theta^\circ_{E,q_\varepsilon} = -\varepsilon\, \omega_0^n \otimes \log \widetilde{q}_\varepsilon.$$

Search for an appropriate evolution equation

General setup

In this context, given $\alpha > 0$ large enough, it is natural to search for a time dependent family of metrics $h_t(z)$ on the fibers E_z of E, $t \in [0,1]$, satisfying a generalized Monge-Ampère equation

$$(D) \qquad \det\nolimits_{T_X \otimes E^*}\left({}^T\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_{E^*}\right)^{1/r} = f_t\,\omega_0^n, \quad f_t > 0,$$

and trace free, rank r^2-1, Hermite-Einstein conditions

$$(T) \quad \omega_t^{n-1} \wedge \Theta^\circ_{E,h_t} = g_t$$

with smoothly varying families of functions $f_t \in C^\infty(X,\mathbb{R})$, Hermitian metrics $\omega_t > 0$ on X and sections

$$g_t \in C^\infty(X, \Lambda^{n,n}_{\mathbb{R}} T^*_X \otimes \mathrm{Herm}^\circ_{h_t}(E,E)), \quad t \in [0,1].$$

Observe that this is a determined (not overdetermined!) system.

Choice of the initial state ($t = 0$)

We start with the Uhlenbeck-Yau solution $h_0 = q_\varepsilon$ of of the "cushioned" trace free Hermite-Einstein equation, so that $\det h_0 = \det H_0$, and take $\alpha > 0$ so large that

$$^T\Theta_{E,h_0} + \alpha\,\omega_0 \otimes \mathrm{Id}_{E^*} > 0 \text{ in the sense of Nakano.}$$

If conditions (D) and (T) can be met for all $t \in [0,1]$, thus without any explosion of the solutions h_t, we infer from (D) that

$$^T\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_{E^*} > 0 \quad \text{in the sense of Nakano}$$

for all $t \in [0,1]$.

Observation

At time $t = 1$, we would then get a Hermitian metric h_1 on E such that Θ_{E,h_1} is dual Nakano positive !!

Possible choices of the right hand side

One still has the freedom of adjusting f_t, ω_t and g_t in the general setup. There are in fact many possibilities:

Proposition

Let (E, H_0) be a smooth Hermitian holomorphic vector bundle such that E is ample and $\omega_0 = \Theta_{\det E,\det H_0} > 0$. Then the system of determinantal and trace free equations

$$(D)\quad \det{}_{T_X\otimes E^*}\big({}^T\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_{E^*}\big)^{1/r} = F(t,z,h_t,D_zh_t)$$

$$(T)\quad \omega_0^{n-1} \wedge \Theta^\circ_{E,h_t} = G(t,z,h_t,D_zh_t,D_z^2h_t) \in \mathrm{Herm}^\circ(E,E)$$

(where $F > 0$), is a well determined system of PDEs.

It is elliptic whenever the symbol η_h of the linearized operator $u \mapsto DG_{D^2h}(t,z,h,Dh,D^2h)\cdot D^2u$ has an Hilbert-Schmidt norm

$$\sup_{\xi\in T_X^*,|\xi|_{\omega_0}=1} \|\eta_{h_t}(\xi)\|_{h_t} \le (r^2+1)^{-1/2}\,n^{-1}$$

for any metric h_t involved, e.g. if G does not depend on D^2h.

Proof of the ellipticity

The (long, computational) proof consists of analyzing the linearized system of equations, starting from the curvature tensor formula

$$\Theta_{E,h} = i\overline{\partial}(h^{-1}\partial h) = i\overline{\partial}(\widetilde{h}^{-1}\partial_{H_0}\widetilde{h}),$$

where $\partial_{H_0}s = H_0^{-1}\partial(H_0 s)$ is the $(1,0)$-component of the Chern connection on $\mathrm{Hom}(E,E)$ associated with H_0 on E.

Let us recall that the ellipticity of an operator

$$P : C^\infty(V) \to C^\infty(W), \quad f \mapsto P(f) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha f(x)$$

means the invertibility of the principal symbol

$$\sigma_P(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\,\xi^\alpha \in \mathrm{Hom}(V,W)$$

whenever $0 \ne \xi \in T^*_{X,x}$.

For instance, on the torus $\mathbb{R}^n/\mathbb{Z}^n$, $f \mapsto P_\lambda(f) = -\Delta f + \lambda f$ has an invertible symbol $\sigma_{P_\lambda}(x,\xi) = -|\xi|^2$, but P_λ is invertible only for $\lambda > 0$.

A more specific choice of the right hand side

Theorem

The elliptic differential system defined by

$$\det_{T_X\otimes E^*}\left({}^T\Theta_{E,h} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_{E^*}\right)^{1/r} = \left(\frac{\det H_0(z)}{\det h_t(z)}\right)^\lambda a_0(z),$$

$$\omega_0^{n-1} \wedge \Theta_{E^\circ,h} = -\varepsilon\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^\mu (\log \widetilde{h}^\circ)\,\omega_0^n$$

possesses an invertible elliptic linearization for $\varepsilon \ge \varepsilon_0(h_t)$ and $\lambda \ge \lambda_0(h_t)(1+\mu^2)$, with $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ large enough.

Corollary

Under the above conditions, starting from the Uhlenbeck-Yau solution h_0 such that $\det h_0 = \det H_0$ at $t = 0$, the PDE system still has a solution for $t \in [0, t_0]$ and $t_0 > 0$ small. (What for $t_0 = 1$?)

Here, the proof consists of analyzing the total symbol of the linearized operator, and the rest is just linear algebra.

Monge-Ampère volume for vector bundles

If $E \to X$ is an ample vector bundle of rank r that is dual Nakano positive, one can introduce its Monge-Ampère volume to be

$$\mathrm{MAVol}(E) = \sup_h \int_X \det{}_{T_X \otimes E^*} \left((2\pi)^{-1}\, {}^T\Theta_{E,h}\right)^{1/r},$$

where the supremum is taken over all smooth metrics h on E such that ${}^T\Theta_{E,h}$ is Nakano positive.

This supremum is always finite, and in fact

Proposition

For any dual Nakano positive vector bundle E, one has

$$\mathrm{MAVol}(E) \le r^{-n} c_1(E)^n.$$

Taking $\omega_0 = \Theta_{\det E}$, the proof is a consequence of the inequality $(\prod \lambda_j)^{1/nr} \le \frac{1}{nr}\sum \lambda_j$ between geometric and arithmetic means, for the eigenvalues λ_j of $(2\pi)^{-1}\, {}^T\Theta_{E,h}$, after raising to power n.

Concluding remarks

- Siarhei Finski (PostDoc at Institut Fourier right now) has observed that the equality holds iff E is projectively flat.
- In the split case $E = \bigoplus_{1 \le j \le r} E_j$ and $h = \bigoplus_{1 \le j \le r} h_j$, the inequality reads

$$\Big(\prod_{1 \le j \le r} c_1(E_j)^n\Big)^{1/r} \le r^{-n} c_1(E)^n,$$

with equality iff $c_1(E_1) = \cdots = c_1(E_r)$.

- In the split case, it seems natural to conjecture that

$$\mathrm{MAVol}(E) = \Big(\prod_{1 \le j \le r} c_1(E_j)^n\Big)^{1/r},$$

i.e. that the supremum is reached for split metrics $h = \bigoplus h_j$.

- The Euler-Lagrange equation for the maximizer is 4th order.

Thank you for your attention