

Hyperbolic algebraic varieties and holomorphic differential equations

Jean-Pierre Demailly

Institut Fourier, Université de Grenoble I, France
& Académie des Sciences de Paris

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Entire curves

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- **Definition.** By an **entire curve** we mean a non constant holomorphic map $f : \mathbb{C} \rightarrow X$ into a complex n -dimensional manifold.
 X is said to be **(Brody) hyperbolic** if \nexists such $f : \mathbb{C} \rightarrow X$.
- If X is a **bounded** open subset $\Omega \subset \mathbb{C}^n$, then there are no entire curves $f : \mathbb{C} \rightarrow \Omega$ (**Liouville's theorem**),
 \Rightarrow every bounded open set $\Omega \subset \mathbb{C}^n$ is hyperbolic
- $X = \overline{\mathbb{C}} \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$ has no entire curves, so it is hyperbolic (**Picard's theorem**)
- A complex torus $X = \mathbb{C}^n / \Lambda$ (Λ lattice) has a lot of entire curves. As \mathbb{C} simply connected, every $f : \mathbb{C} \rightarrow X = \mathbb{C}^n / \Lambda$ lifts as $\tilde{f} : \mathbb{C} \rightarrow \mathbb{C}^n$, $\tilde{f}(t) = (\tilde{f}_1(t), \dots, \tilde{f}_n(t))$, and $\tilde{f}_j : \mathbb{C} \rightarrow \mathbb{C}$ can be arbitrary entire functions.

- Consider now the complex projective n -space

$$\mathbb{P}^n = \mathbb{P}_{\mathbb{C}}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*, \quad [z] = [z_0 : z_1 : \dots : z_n].$$

- An entire curve $f : \mathbb{C} \rightarrow \mathbb{P}^n$ is given by a map

$$t \mapsto [f_0(t) : f_1(t) : \dots : f_n(t)]$$

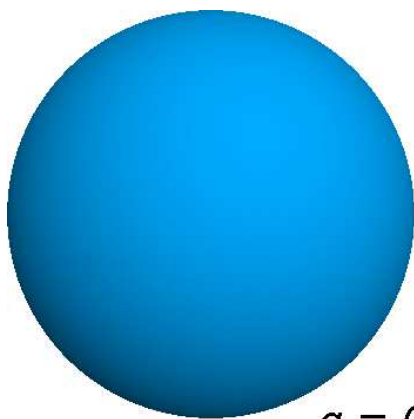
where $f_j : \mathbb{C} \rightarrow \mathbb{C}$ are holomorphic functions without common zeroes (so there are a lot of them).

- More generally, look at a (complex) **projective manifold**, i.e.

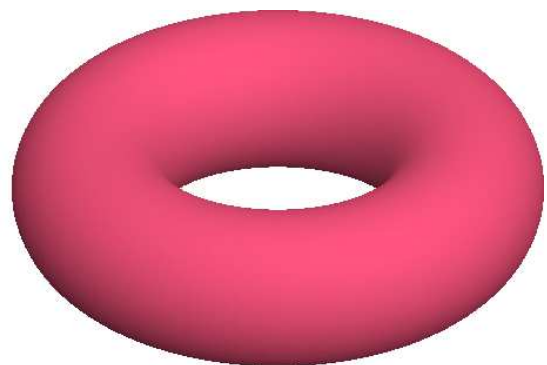
$$X^n \subset \mathbb{P}^N, \quad X = \{[z]; P_1(z) = \dots = P_k(z) = 0\}$$

where $P_j(z) = P_j(z_0, z_1, \dots, z_N)$ are homogeneous polynomials (of some degree d_j), such that X is **non singular**.

Complex curves (genus 0 and 1)



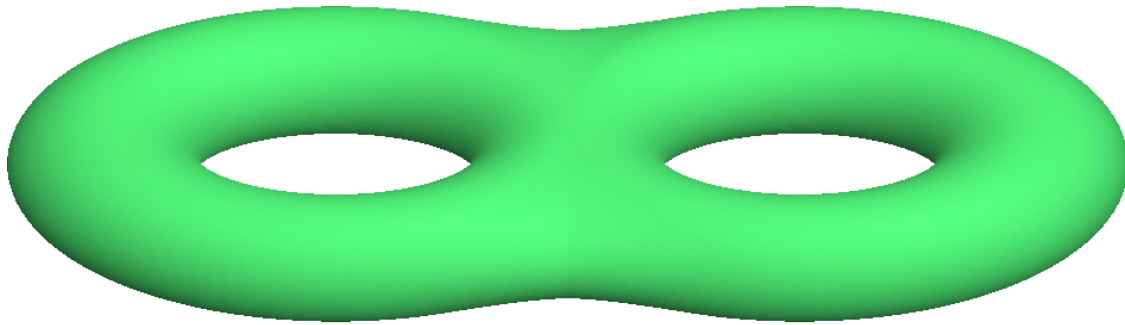
$g = 0, K_X < 0$
(positive curvature)



$g = 1, K_X = 0$
(zero curvature)

Canonical bundle $K_X = \Lambda^n T_X^*$ (here $K_X = T_X^*$)

- $g = 0 : X = \mathbb{P}^1$ courbure $T_X > 0$ **not hyperbolic**
- $g = 1 : X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ courbure $T_X = 0$ **not hyperbolic**



$g > 1, K_X > 0$
(negative curvature)

$$\deg K_X = 2g - 2$$

If $g \geq 2$, $X \simeq \mathbb{D}/\Gamma$ ($T_X < 0$) \Rightarrow X is hyperbolic.

In fact every curve $f : \mathbb{C} \rightarrow X \simeq \mathbb{D}/\Gamma$ lifts to $\tilde{f} : \mathbb{C} \rightarrow \mathbb{D}$,
and so must be constant by Liouville.

Kobayashi metric / hyperbolic manifolds

- For a complex manifold, $n = \dim_{\mathbb{C}} X$, one defines **the Kobayashi pseudo-metric** : $x \in X, \xi \in T_x$
 $\kappa_x(\xi) = \inf \{ \lambda > 0; \exists f : \mathbb{D} \rightarrow X, f(0) = x, \lambda f_*(0) = \xi \}$
 On $\mathbb{C}^n, \mathbb{P}^n$ or complex tori $X = \mathbb{C}^n/\Lambda$, one has $\kappa_X \equiv 0$.
- X is said to be **hyperbolic in the sense of Kobayashi** if the associated integrated pseudo-distance is a distance (i.e. it separates points – i.e. has Hausdorff topology).
- **Examples.** * $X = \Omega/\Gamma$, Ω bounded symmetric domain.
 * any product $X = X_1 \times \dots \times X_s$ where X_j hyperbolic.
- **Theorem (dimension n arbitrary)** (Kobayashi, 1970)
 T_X negatively curved ($T_X^* > 0$, i.e. ample) $\Rightarrow X$ hyperbolic.
 Recall that a holomorphic vector bundle E is **ample** iff its symmetric powers $S^m E$ have global sections which generate 1-jets of (germs of) sections at any point $x \in X$.

The proof of the above Kobayashi result depends crucially on:

Ahlfors-Schwarz lemma. Let $\gamma = i \sum \gamma_{jk} dt_j \wedge d\bar{t}_k$ be an almost everywhere positive hermitian form on the ball $B(0, R) \subset \mathbb{C}^p$, such that $-\text{Ricci}(\gamma) := i \partial \bar{\partial} \log \det \gamma \geq A\gamma$ in the sense of currents, for some constant $A > 0$ (this means in particular that $\det \gamma = \det(\gamma_{jk})$ is such that $\log \det \gamma$ is plurisubharmonic). Then the γ -volume form is controlled by the Poincaré volume form :

$$\det(\gamma) \leq \left(\frac{p+1}{AR^2} \right)^p \frac{1}{(1 - |t|^2/R^2)^{p+1}}.$$

Brody theorem

Brody reparametrization Lemma. Assume that X is *compact*, let ω be a hermitian metric on X and $f : \mathbb{D} \rightarrow X$ a holomorphic map. For every $\varepsilon > 0$, there exists a radius $R \geq (1 - \varepsilon) \|f'(0)\|_\omega$ and a homographic transformation ψ of the disk $D(0, R)$ onto $(1 - \varepsilon)\mathbb{D}$ such that $\|(f \circ \psi)'(0)\|_\omega = 1$ and $\|(f \circ \psi)'(t)\|_\omega \leq (1 - |t|^2/R^2)^{-1}$ for every $t \in D(0, R)$.
 \Rightarrow if f' unbounded, $\exists g = \lim f \circ \psi_\nu : \mathbb{C} \rightarrow X$ with $\|g'\|_\omega \leq 1$.

Brody theorem (1978). If X is *compact* then X is Kobayashi hyperbolic if and only if there are no entire holomorphic curves $f : \mathbb{C} \rightarrow X$ (*Brody hyperbolicity*).

Hyperbolic varieties are especially interesting for their expected diophantine properties :

Conjecture (S. Lang, 1986) An arithmetic projective variety X is hyperbolic iff $X(\mathbb{K})$ is finite for every number field \mathbb{K} .

- **Definition** A non singular projective variety X is said to be of *general type* if the growth of pluricanonical sections

$$\dim H^0(X, K_X^{\otimes m}) \sim cm^n, \quad K_X = \Lambda^n T_X^*$$

is maximal.

(sections locally of the form $f(z) (dz_1 \wedge \dots \wedge dz_n)^{\otimes m}$)

Example: A non singular hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree d satisfies $K_X = \mathcal{O}(d - n - 2)$,
 X is of general type iff $d > n + 2$.

- **Conjecture CGT.** If a compact variety X is hyperbolic, then it should be of general type, and if X is non singular, then $K_X = \Lambda^n T_X^*$ should be ample, i.e. $K_X > 0$ (Kodaira) (equivalently \exists Kähler metric ω such that $\text{Ricci}(\omega) < 0$).

Conjectural characterizations of hyperbolicity

- **Theorem.** Let X be projective algebraic. Consider the following properties :
 (GT) Every subvariety Y of X is of *general type*.
 (AH) $\exists \varepsilon > 0, \forall C \subset X$ algebraic curve

$$2g(\bar{C}) - 2 \geq \varepsilon \deg(C).$$

(X “*algebraically hyperbolic*”)

(HY) X is *hyperbolic*

(JC) X possesses a *jet-metric with negative curvature* on its k -jet bundle X_k [to be defined later], for $k \geq k_0 \gg 1$.

Then (JC) \Rightarrow (GT), (AH), (HY),

(HY) \Rightarrow (AH),

and if Conjecture CGT holds, (HY) \Rightarrow (GT).

- It is expected that all 4 properties are in fact equivalent for projective varieties.

- **Conjecture** (Green-Griffiths-Lang = GGL) *Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \rightarrow X$ one has $f(\mathbb{C}) \subset Y$.*
- Combining the above conjectures, we get :
Expected consequence (of CGT + GGL) Properties:
(HY) X is *hyperbolic*
(GT) Every subvariety Y of X is of *general type*
are equivalent if CGT + GGL hold.
- **Arithmetic counterpart** (Lang 1987). *If X is a variety of general type defined over a number field and Y is the Green-Griffiths locus (Zariski closure of $\bigcup f(\mathbb{C})$), then $X(\mathbb{K}) \setminus Y$ is finite for every number field \mathbb{K} .*

Results obtained so far

- Using “jet technology” and **deep results of McQuillan** for curve foliations on surfaces, D. – El Goul proved **Theorem** (solution of Kobayashi conjecture, 1998).
*A very generic surface $X \subset \mathbb{P}^3$ of **degree ≥ 21** is hyperbolic.*
Independently McQuillan got degree ≥ 35 .
Recently improved to **degree ≥ 18** (Păun, 2008).
For $X \subset \mathbb{P}^{n+1}$, the optimal bound should be **degree $\geq 2n + 1$** for $n \geq 2$ (**Zaidenberg**).
- **Generic GGL conjecture for $\dim_{\mathbb{C}} X = n$**
(S. Diverio, J. Merker, E. Rousseau, 2009).
*If $X \subset \mathbb{P}^{n+1}$ is a **generic n -fold of degree $d \geq d_n := 2^{n^5}$** ,
[also $d_3 = 593$, $d_4 = 3203$, $d_5 = 35355$, $d_6 = 172925$] then
 $\exists Y \subsetneq X$ s.t. \forall non const. $f : \mathbb{C} \rightarrow X$ satisfies $f(\mathbb{C}) \subset Y$*
Moreover (S. Diverio, S. Trapani, 2009) **$\text{codim}_{\mathbb{C}} Y \geq 2 \Rightarrow$**
generic hypersurface $X \subset \mathbb{P}^4$ of degree ≥ 593 is hyperbolic.

The main idea in order to attack GGL is to use differential equations. Let

$$\mathbb{C} \rightarrow X, \quad t \mapsto f(t) = (f_1(t), \dots, f_n(t))$$

be a curve written in some local holomorphic coordinates (z_1, \dots, z_n) on X .

Consider **algebraic differential operators** which can be written locally in multi-index notation

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \dots, f^{(k)}) \\ &= \sum a_{\alpha_1 \alpha_2 \dots \alpha_k}(f(t)) f'(t)^{\alpha_1} f''(t)^{\alpha_2} \dots f^{(k)}(t)^{\alpha_k} \end{aligned}$$

where $a_{\alpha_1 \alpha_2 \dots \alpha_k}(z)$ are holomorphic coefficients on X and $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \dots, f^{(k)})$ its **k -jet**. Obvious \mathbb{C}^* -action :

$$\lambda \cdot f(t) = f(\lambda t), \quad (\lambda \cdot f)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$$

\Rightarrow **weighted degree** $m = |\alpha_1| + 2|\alpha_2| + \dots + k|\alpha_k|$.

Vanishing theorem for differential operators

- **Definition.** $E_{k,m}^{\text{GG}}$ is the sheaf (bundle) of algebraic differential operators of order k and weighted degree m .
- **Fundamental vanishing theorem**
 [Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]
 Let $P \in H^0(X, E_{k,m}^{\text{GG}} \otimes \mathcal{O}(-A))$ be a global algebraic differential operator whose coefficients vanish on some ample divisor A . Then $\forall f : \mathbb{C} \rightarrow X, P(f_{[k]}) \equiv 0$.
- **Proof.** One can assume that A is very ample and intersects $f(\mathbb{C})$. Also assume f' bounded (this is not so restrictive by Brody !). Then all $f^{(k)}$ are bounded by Cauchy inequality. Hence

$$\mathbb{C} \ni t \mapsto P(f', f'', \dots, f^{(k)})(t)$$

is a bounded holomorphic function on \mathbb{C} which vanishes at some point. Apply Liouville's theorem ! □

- Let $X_k^{\text{GG}} = J_k(X)^*/\mathbb{C}^*$ be the **projectivized k -jet bundle** of X = quotient of non constant k -jets by \mathbb{C}^* -action. Fibers are weighted projective spaces.

Observation. If $\pi_k : X_k^{\text{GG}} \rightarrow X$ is canonical projection and $\mathcal{O}_{X_k^{\text{GG}}}(1)$ is the **tautological line bundle**, then

$$E_{k,m}^{\text{GG}} = (\pi_k)_* \mathcal{O}_{X_k^{\text{GG}}}(m)$$

- Saying that $f : \mathbb{C} \rightarrow X$ satisfies the differential equation $P(f_{[k]}) = 0$ means that

$$f_{[k]}(\mathbb{C}) \subset Z_P$$

where Z_P is the zero divisor of the section

$$\sigma_P \in H^0(X_k^{\text{GG}}, \mathcal{O}_{X_k^{\text{GG}}}(m) \otimes \pi_k^* \mathcal{O}(-A))$$

associated with P .

Consequence of fundamental vanishing theorem

- **Consequence of fundamental vanishing theorem.**

If $P_j \in H^0(X, E_{k,m}^{\text{GG}} \otimes \mathcal{O}(-A))$ is a basis of sections then the image $f(\mathbb{C})$ lies in $Y = \pi_k(\bigcap Z_{P_j})$, hence property asserted by the GGL conjecture holds true if there are “enough independent differential equations” so that

$$Y = \pi_k\left(\bigcap_j Z_{P_j}\right) \subsetneq X.$$

- However, **some differential equations are not very useful**. On a surface with coordinates (z_1, z_2) , a Wronskian equation $f_1' f_2'' - f_2' f_1'' = 0$ tells us that $f(\mathbb{C})$ sits on a line, but $f_2''(t) = 0$ says that the second component is linear affine in time, an essentially **meaningless information** which is lost by a change of parameter $t \mapsto \varphi(t)$.

- The k -th order Wronskian operator

$$W_k(f) = f' \wedge f'' \wedge \dots \wedge f^{(k)}$$

(locally defined in coordinates) has degree $m = \frac{k(k+1)}{2}$ and

$$W_k(f \circ \varphi) = \varphi'^m W_k(f) \circ \varphi.$$

- **Definition.** A differential operator P of order k and degree m is said to be invariant by reparametrization if

$$P(f \circ \varphi) = \varphi'^m P(f) \circ \varphi$$

for any parameter change $t \mapsto \varphi(t)$. Consider their set

$$E_{k,m} \subset E_{k,m}^{\text{GG}} \quad (\text{a subbundle})$$

(Any polynomial $Q(W_1, W_2, \dots, W_k)$ is invariant, but for $k \geq 3$ there are other invariant operators.)

Category of directed manifolds

- **Goal.** We are interested in curves $f : \mathbb{C} \rightarrow X$ such that $f'(\mathbb{C}) \subset V$ where V is a subbundle (or subsheaf) of T_X .
- **Definition.** Category of directed manifolds :
 - **Objects** : pairs (X, V) , X manifold/ \mathbb{C} and $V \subset \mathcal{O}(T_X)$
 - **Arrows** $\psi : (X, V) \rightarrow (Y, W)$ holomorphic s.t. $\psi_* V \subset W$
 - “**Absolute case**” (X, T_X)
 - “**Relative case**” $(X, T_{X/S})$ where $X \rightarrow S$
 - “**Integrable case**” when $[V, V] \subset V$ (foliations)
- **Fonctor “1-jet”** : $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :

$\tilde{X} = P(V) =$ bundle of projective spaces of lines in V

$\pi : \tilde{X} = P(V) \rightarrow X, \quad (x, [v]) \mapsto x, \quad v \in V_x$

$\tilde{V}_{(x, [v])} = \{\xi \in T_{\tilde{X}, (x, [v])} ; \pi_* \xi \in \mathbb{C}v \subset T_{X, x}\}$

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (\tilde{X}, \tilde{V}) \text{ (projectivized 1st-jet)}$$

- **Definition.** *Semple jet bundles :*

- $(X_k, V_k) = k$ -th iteration of fonctor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
- $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X_k, V_k)$ is the **projectivized k -jet of f** .

- **Basic exact sequences**

$$0 \rightarrow T_{\tilde{X}/X} \rightarrow \tilde{V} \xrightarrow{\pi^*} \mathcal{O}_{\tilde{X}}(-1) \rightarrow 0 \Rightarrow \text{rk } \tilde{V} = r = \text{rk } V$$

$$0 \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \pi^* V \otimes \mathcal{O}_{\tilde{X}}(1) \rightarrow T_{\tilde{X}/X} \rightarrow 0 \text{ (Euler)}$$

$$0 \rightarrow T_{X_k/X_{k-1}} \rightarrow V_k \xrightarrow{(\pi_k)^*} \mathcal{O}_{X_k}(-1) \rightarrow 0 \Rightarrow \text{rk } V_k = r$$

$$0 \rightarrow \mathcal{O}_{X_k} \rightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \rightarrow T_{X_k/X_{k-1}} \rightarrow 0 \text{ (Euler)}$$

Direct image formula

- For $n = \dim X$ and $r = \text{rk } V$, get a **tower of \mathbb{P}^{r-1} -bundles**

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r - 1)$, $\text{rk } V_k = r$,

and **tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$** .

- **Theorem.** X_k is a smooth compactification of

$$X_k^{\text{GG,reg}}/G_k = J_k^{\text{GG,reg}}/G_k$$

where G_k is the group of k -jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization:

$(f, \varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k -jets of regular curves.

- **Direct image formula.** $(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = E_{k,m} V^* =$
invariant algebraic differential operators $f \mapsto P(f_{[k]})$
acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$.

- Although very interesting, results are currently limited by **lack of knowledge on jet bundles and differential operators**
- **Theorem (Bérczi-Kirwan, 2009).** *The ring of germs of invariant differential operators on $(\mathbb{C}^n, T_{\mathbb{C}^n})$ at the origin*

$$\mathcal{A}_{k,n} = \bigoplus_m E_{k,m} T_{\mathbb{C}^n}^* \quad \text{is finitely generated.}$$

- Checked by direct calculations $\forall n, k \leq 2$ and $n = 2, k \leq 4$:

$$\mathcal{A}_{1,n} = \mathcal{O}[f'_1, \dots, f'_n]$$

$$\mathcal{A}_{2,n} = \mathcal{O}[f'_1, \dots, f'_n, W^{[ij]}], \quad W^{[ij]} = f'_i f''_j - f'_j f''_i$$

$$\mathcal{A}_{3,2} = \mathcal{O}[f'_1, f'_2, W_1, W_2][W]^2, \quad W_i = f'_i DW - 3f''_i W$$

$$\mathcal{A}_{4,2} = \mathcal{O}[f'_1, f'_2, W_{11}, W_{22}, S][W]^6, \quad W_{ii} = f'_i DW_i - 5f''_i W_i$$

where $W = f'_1 f''_2 - f'_2 f''_1$, $S = (W_1 DW_2 - W_2 DW_1)/W$.

Generalized GGL conjecture

- **Generalized GGL conjecture.** *If (X, V) is directed manifold of general type, i.e. $\det V^*$ big, then $\exists Y \subsetneq X$ such that $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ non const., $f(\mathbb{C}) \subset Y$.*
- **Remark.** Elementary by Ahlfors-Schwarz if $r = \text{rk } V = 1$. $t \mapsto \log \|f'(t)\|_{V,h}$ is strictly subharmonic if $r = 1$ and (V^*, h^*) has > 0 curvature in the sense of currents.
- **Strategy.** Try some sort of induction on $r = \text{rk } V$.
First try to get differential equations $f_{[k]}(\mathbb{C}) \subset Z \subsetneq X_k$.
Take **minimal such k** . If $k = 0$, we are done! Otherwise $k \geq 1$ and $\pi_{k,k-1}(Z) = X_{k-1}$, thus $V' = V_k \cap T_Z$ has $\text{rank} < \text{rk } V_k = r$ and should have again $\det V'^*$ big (unless some unprobable geometry situation occurs?).
- **Needed induction step.** *If (X, V) has $\det V^*$ big and $Z \subset X_k$ irreducible with $\pi_{k,k-1}(Z) = X_{k-1}$, then (Z, V') , $V' = V_k \cap T_Z$ has $\mathcal{O}_{Z_\ell}(1)$ big on (Z_ℓ, V'_ℓ) , $\ell \gg 0$.*

Holomorphic Morse inequalities (D-, 1985) Let $L \rightarrow X$ be a holomorphic line bundle on a compact complex manifold X , h a smooth hermitian metric on L and

$$\Theta_{L,h} = \frac{i}{2\pi} \nabla_{L,h}^2 = -\frac{i}{2\pi} \partial \bar{\partial} \log h$$

its curvature form. Then $\forall q = 0, 1, \dots, n = \dim_{\mathbb{C}} X$

$$\sum_{j=0}^q (-1)^{q-j} h^j(X, L^{\otimes k}) \leq \frac{k^n}{n!} \int_{X(L,h,\leq q)} (-1)^q \Theta_{L,h}^n + o(k^n).$$

where

$$X(L, h, q) = \{x \in X; \Theta_{L,h}(x) \text{ has signature } (n - q, q)\}$$

(q -index set), and

$$X(L, h, \leq q) = \bigcup_{0 \leq j \leq q} X(L, h, \leq j)$$

Holomorphic Morse inequalities (continued)

As a consequence, one gets **an upper bound**

$$h^0(X, L^{\otimes k}) \leq \frac{k^n}{n!} \int_{X(L,h,0)} \Theta_{L,h}^n + o(k^n)$$

and a **lower bound**

$$\begin{aligned} h^0(X, L^{\otimes k}) &\geq h^0(X, L^{\otimes k}) - h^1(X, L^{\otimes k}) \geq \\ &\geq \frac{k^n}{n!} \left(\int_{X(L,h,0)} \Theta_{L,h}^n - \int_{X(L,h,1)} |\Theta_{L,h}^n| \right) - o(k^n) \end{aligned}$$

and similar bounds for the higher cohomology groups H^q :

$$h^q(X, L^{\otimes k}) \leq \frac{k^n}{n!} \int_{X(L,h,q)} |\Theta_{L,h}^n| + o(k^n)$$

$$h^q(X, L^{\otimes k}) \geq \frac{k^n}{n!} \left(\int_{X(L,h,q)} - \int_{X(L,h,q-1)} - \int_{X(L,h,q+1)} |\Theta_{L,h}^n| \right) - o(k^n)$$

Let $J_k V$ be the bundle of k -jets of curves $f : (\mathbb{C}, T\mathbb{C}) \rightarrow (X, V)$. Assuming that V is equipped with a hermitian metric h , one defines a "weighted Finsler metric" on $J^k V$ by taking $p = k!$ and

$$\Psi_{h_k}(f) := \left(\sum_{1 \leq s \leq k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \right)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \dots \gg \varepsilon_k.$$

Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k^{\text{GG}}}(1)$, with curvature form $(x, \xi_1, \dots, \xi_k) \mapsto$

$$\Theta_{L_k, h_k} = \omega_{\text{FS}, k}(\xi) + \frac{i}{2\pi} \sum_{1 \leq s \leq k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i, j, \alpha, \beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha} \bar{\xi}_{s\beta}}{|\xi_s|^2} dz_i \wedge d\bar{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor Θ_{V^*, h^*} and $\omega_{\text{FS}, k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\text{GG}} \rightarrow X$. The expression gets simpler by using polar coordinates $x_s = |\xi_s|^{2p/s}$, $u_s = \xi_s / |\xi_s|_h = \nabla^s f(0) / |\nabla^s f(0)|$.

Probabilistic interpretation of the curvature

In such polar coordinates, one gets the formula

$$\Theta_{L_k, h_k} = \omega_{\text{FS}, p, k}(\xi) + \frac{i}{2\pi} \sum_{1 \leq s \leq k} \frac{1}{s} x_s \sum_{i, j, \alpha, \beta} c_{ij\alpha\beta}(z) u_{s\alpha} \bar{u}_{s\beta} dz_i \wedge d\bar{z}_j$$

where $\omega_{\text{FS}, k}(\xi)$ is positive definite in ξ . The other terms are a weighted average of the values of the curvature tensor $\Theta_{V, h}$ on vectors u_s in the unit sphere bundle $SV \subset V$. The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2p/s} = 1$, so we can take here $x_s \geq 0$, $\sum x_s = 1$. This is essentially a sum of the form $\sum \frac{1}{s} \gamma(u_s)$ where u_s are random points of the sphere, and so as $k \rightarrow +\infty$ this can be estimated by a "Monte-Carlo" integral

$$\left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right) \int_{u \in SV} \gamma(u) du.$$

As γ is quadratic here, $\int_{u \in SV} \gamma(u) du = \frac{1}{r} \text{Tr}(\gamma)$.

It follows that the leading term in the estimate only involves the trace of Θ_{V^*, h^*} , i.e. the curvature of $(\det V^*, \det h^*)$, which can be taken to be > 0 if $\det V^*$ is big.

Corollary (D-, 2010) Let (X, V) be a directed manifold, $F \rightarrow X$ a \mathbb{Q} -line bundle, (V, h) and (F, h_F) hermitian. Define

$$L_k = \mathcal{O}_{X_k^{\text{GG}}}(1) \otimes \pi_k^* \mathcal{O}\left(\frac{1}{kr} \left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right) F\right),$$

$$\eta = \Theta_{\det V^*, \det h^*} + \Theta_{F, h_F}.$$

Then for all $q \geq 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have

$$h^q(X_k^{\text{GG}}, \mathcal{O}(L_k^{\otimes m})) \leq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n! (k!)^r} \left(\int_{X(\eta, q)} (-1)^q \eta^n + \frac{C}{\log k} \right)$$

$$h^0(X_k^{\text{GG}}, \mathcal{O}(L_k^{\otimes m})) \geq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n! (k!)^r} \left(\int_{X(\eta, \leq 1)} \eta^n - \frac{C}{\log k} \right).$$

Partial solution of the GGL conjecture

Using the above cohomological estimate, we obtain:

Theorem (D-, 2010) Let (X, V) be of general type, i.e.

$K_V = (\det V)^*$ is a big line bundle. Then there exists $k \geq 1$ and an algebraic hypersurface $Z \subsetneq X_k$ such that every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \mapsto (X, V)$ satisfies $f_{[k]}(\mathbb{C}) \subset Z$ (in other words, f satisfies an algebraic differential equation of order k).

Another important consequence is:

Theorem (D-, 2012) A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n$ with

$$d_2 = 286, \quad d_3 = 7316, \quad d_n = \left\lfloor \frac{n^4}{3} (n \log(n \log(24n)))^n \right\rfloor$$

(for $n \geq 4$) satisfies the Green-Griffiths conjecture.

The proof of the last result uses an important idea due to Yum-Tong Siu, itself based on ideas of Claire Voisin and Herb Clemens, and then refined by M. Păun [Pau08], E. Rousseau [Rou06b] and J. Merker [Mer09].

The idea consists of studying vector fields on the **relative jet space of the universal family of hypersurfaces of \mathbb{P}^{n+1}** .

Let $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ be the universal hypersurface, i.e.

$$\mathcal{X} = \{(z, a); a = (a_\alpha) \text{ s.t. } P_a(z) = \sum a_\alpha z^\alpha = 0\},$$

let $\Omega \subset \mathbb{P}^{N_d}$ be the open subset of a 's for which $X_a = \{P_a(z) = 0\}$ is smooth, and let

$$p : \mathcal{X} \rightarrow \mathbb{P}^{n+1}, \quad \pi : \mathcal{X}|_\Omega \rightarrow \Omega \subset \mathbb{P}^{N_d}$$

be the natural projections.

Meromorphic vector fields on jet spaces

Let

$$p_k : \mathcal{X}_k \rightarrow \mathcal{X} \rightarrow \mathbb{P}^{n+1}, \quad \pi_k : \mathcal{X}_k \rightarrow \Omega \subset \mathbb{P}^{N_d}$$

be the relative Green-Griffiths k -jet space of $\mathcal{X} \rightarrow \Omega$. Then J. Merker [Mer09] has shown that global sections η_j of

$$\mathcal{O}(T_{\mathcal{X}_k}) \otimes p_k^* \mathcal{O}_{\mathbb{P}^{n+1}}(k^2 + 2k) \otimes \pi_k^* \mathcal{O}_{\mathbb{P}^{N_d}}(1)$$

generate the bundle at all points of $\mathcal{X}_k^{\text{reg}}$ for $k = n = \dim X_a$. From this, it follows that if P is a non zero global section over Ω of $E_{k,m}^{\text{GG}} T_{\mathcal{X}}^* \otimes p_k^* \mathcal{O}_{\mathbb{P}^{n+1}}(-s)$ for some s , then for a suitable collection of $\eta = (\eta_1, \dots, \eta_m)$, the m -th derivatives

$$D_{\eta_1} \dots D_{\eta_m} P$$

yield sections of $H^0(\mathcal{X}, E_{k,m}^{\text{GG}} T_{\mathcal{X}}^* \otimes p_k^* \mathcal{O}_{\mathbb{P}^{n+1}}(m(k^2 + 2k) - s))$ whose joint base locus is contained in $\mathcal{X}_k^{\text{sing}}$, whence the result.

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