Towards the Green-Griffiths-Lang conjecture

Jean-Pierre Demailly

Institut Fourier, Université de Grenoble I, France
& Académie des Sciences de Paris

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Entire curves

- **Definition.** By an entire curve we mean a non constant holomorphic map $f : \mathbb{C} \to X$ into a complex n-dimensional manifold.
  X is said to be (Brody) hyperbolic if $\nexists$ such $f : \mathbb{C} \to X$.
- If X is a bounded open subset $\Omega \subset \mathbb{C}^n$, then there are no entire curves $f : \mathbb{C} \to \Omega$ (Liouville's theorem),
  $\Rightarrow$ every bounded open set $\Omega \subset \mathbb{C}^n$ is hyperbolic
- $X = \overline{\mathbb{C}} \smallsetminus \{0, 1, \infty\} = \mathbb{C} \smallsetminus \{0, 1\}$ has no entire curves, so it is hyperbolic (Picard's theorem)
- A complex torus $X = \mathbb{C}^n/\Lambda$ (Λ lattice) has a lot of entire curves. As $\mathbb{C}$ simply connected, every $f : \mathbb{C} \to X = \mathbb{C}^n/\Lambda$ lifts as $\tilde{f} : \mathbb{C} \to \mathbb{C}^n$, $\tilde{f}(t) = (\tilde{f}_1(t), \ldots, \tilde{f}_n(t))$, and $\tilde{f}_j : \mathbb{C} \to \mathbb{C}$ can be arbitrary entire functions.

Projective algebraic varieties

- Consider now the complex projective n-space

$$\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} \smallsetminus \{0\})/\mathbb{C}^*, \qquad [z] = [z_0 : z_1 : \ldots : z_n].$$

- An entire curve $f : \mathbb{C} \to \mathbb{P}^n$ is given by a map

$$t \longmapsto [f_0(t) : f_1(t) : \ldots : f_n(t)]$$

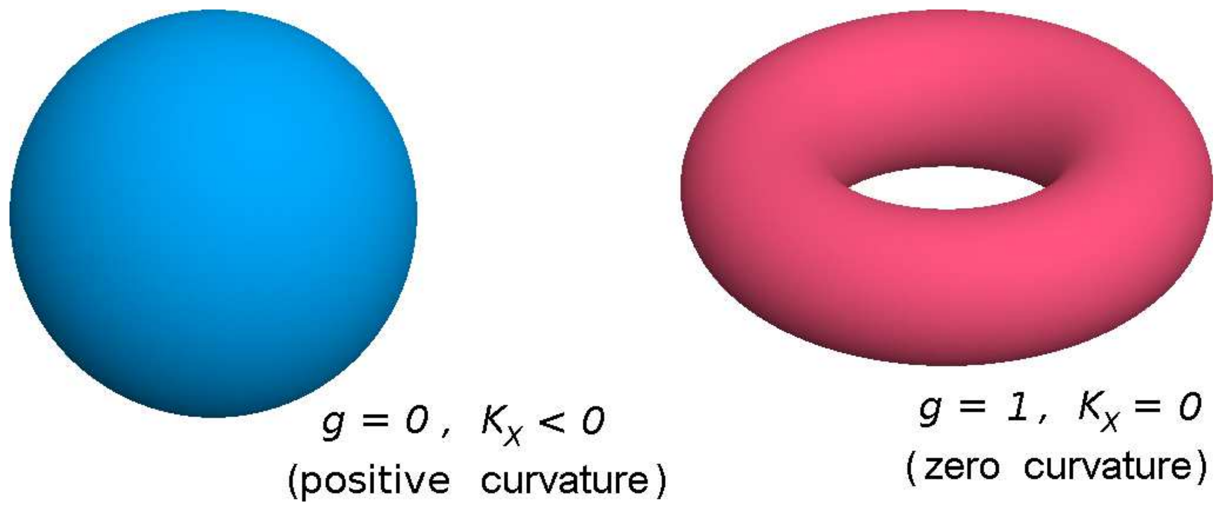
  where $f_j : \mathbb{C} \to \mathbb{C}$ are holomorphic functions without common zeroes (so there are a lot of them).

- More generally, look at a (complex) projective manifold, i.e.

$$X^n \subset \mathbb{P}^N, \qquad X = \{[z]\,;\ P_1(z) = ... = P_k(z) = 0\}$$

  where $P_j(z) = P_j(z_0, z_1, \ldots, z_N)$ are homogeneous polynomials (of some degree d_j), such that X is non singular.

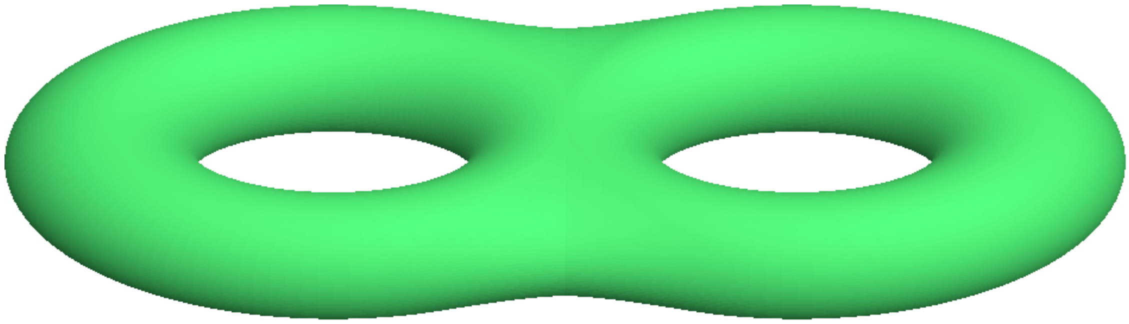
Complex curves (genus 0 and 1)



Canonical bundle $K_X = \Lambda^n T^*_X$ (here $K_X = T^*_X$)

- $g = 0$: $X = \mathbb{P}^1$ curvature $T_X > 0$: not hyperbolic
- $g = 1$: $X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ curvature $T_X = 0$: not hyperbolic

Complex curves (genus $g \geq 2$)



$\deg K_X = 2g - 2$

If $g \geq 2$, $X \simeq \mathbb{D}/\Gamma$ $(T_X < 0)$ $\Rightarrow$ X is hyperbolic.

In fact every curve $f : \mathbb{C} \to X \simeq \mathbb{D}/\Gamma$ lifts to $\widetilde{f} : \mathbb{C} \to \mathbb{D}$, and so $\widetilde{f}$ and f must be constant by Liouville.

Positivity concepts/General type varieties

Definition *Let $L \to X$ be a line bundle on a nonsingular complex projective variety X.*

- *L is said to be ample if for $m \gg 1$ the space of sections $S_m = H^0(X, K_X^{\otimes m})$ gives an embedding*

$$\Phi_m : X \hookrightarrow P(S_m^*) = \mathbb{P}^{N_m - 1}, \quad N_m = \dim S_m.$$

- *L is said to be big if the dimensions of sections*

$$\dim H^0(X, L^{\otimes m}) \sim cm^n$$

have maximal growth $\kappa(L) = n = \dim X$.

- *X is said to be of general type if K_X is big.*

Example: A non singular hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree d has $K_X = \mathcal{O}(d - n - 2)$, X is of general type iff $d > n + 2$.

Conjecture CGT. *If a compact variety X is hyperbolic, then it should be of general type, and if X is non singular, then $K_X = \Lambda^n T_X^*$ should be ample, i.e. $K_X > 0$ (Kodaira).*

Green-Griffiths-Lang conjecture

- **Conjecture (Green-Griffiths-Lang = GGL)**
  Let X be a projective variety of general type. Then $\exists Y \subsetneq X$ algebraic such that all entire curves $f : \mathbb{C} \to X$ satisfy $f(\mathbb{C}) \subset Y$.
- **Expected consequence of GGL**:
  If every subvariety Y of X is of general type, then X is hyperbolic.

  By CGT conjecture, this should be a necessary and sufficient characterization of hyperbolicity for projective varieties.
- **Arithmetic counterpart** (Lang 1987). *If X is a variety of general type defined over a number field and Y is the Green-Griffiths locus (Zariski closure of $\bigcup f(\mathbb{C})$), then $X(\mathbb{K}) \smallsetminus Y$ is finite for every number field $\mathbb{K}$.*

Results obtained so far

- Using "jet technology" and deep results of McQuillan for curve foliations on surfaces, D. – El Goul proved
  Theorem (solution of Kobayashi conjecture, 1998).
  A very generic surface $X \subset \mathbb{P}^3$ of degree ≥ 21 is hyperbolic.
  Independently McQuillan got degree ≥ 35.
  Recently improved to degree ≥ 18 (Păun, 2008).

  Still far : for $X \subset \mathbb{P}^{n+1}$, the optimal bound should be degree $\geq 2n+1$ for $n \geq 2$ **(Zaidenberg)**.
- **Generic GGL conjecture** (S. Diverio, J. Merker, E. Rousseau, 2009). *A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n := 2^{n^5}$ satisfies GGL (no entire curves).*
  (D-, 2012) *bound improved to* $d_n = \left\lfloor \frac{n^4}{3}\big(n \log(n \log(24n))\big)^n \right\rfloor$.

  Moreover (S. Diverio, S. Trapani, 2009) A generic hypersurface $X \subset \mathbb{P}^4$ of degree ≥ 593 is hyperbolic.

Category of directed manifolds

- **Goal.** More generally, we are interested in curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$ where V is a subbundle of T_X (or singular linear subspace, i.e. a closed irreducible analytic subspace such that $\forall x \in X$, $V_x := V \cap T_{X,x}$ linear).
- **Definition.** *Category of directed manifolds :*
  - Objects : pairs (X, V), X manifold/$\mathbb{C}$ and $V \subset T_X$
  - Arrows $\psi : (X, V) \to (Y, W)$ holomorphic s.t. $\psi_* V \subset W$
  - "Absolute case" (X, T_X), i.e. $V = T_X$
  - "Relative case" $(X, T_{X/S})$ where $X \to S$
  - "Integrable case" when $[V, V] \subset V$ (foliations)
- **Fonctor "1-jet" :** $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :

$$\tilde{X} = P(V) = \text{bundle of projective spaces of lines in } V$$
$$\pi : \tilde{X} = P(V) \to X, \quad (x, [v]) \mapsto x, \quad v \in V_x$$
$$\tilde{V}_{(x,[v])} = \{\xi \in T_{\tilde{X},(x,[v])} \,;\, \pi_*\xi \in \mathbb{C}v \subset T_{X,x}\}$$

Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$
$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \quad \text{(projectivized 1st-jet)}$$

- **Definition.** *Semple jet bundles :*
  - $(X_k, V_k) = k$-th iteration of fonctor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
  - $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized k-jet of f.
- **Basic exact sequences**

$$0 \to T_{\tilde{X}/X} \to \tilde{V} \xrightarrow{\pi_\star} \mathcal{O}_{\tilde{X}}(-1) \to 0 \qquad \Rightarrow \text{rk}\, \tilde{V} = r = \text{rk}\, V$$
$$0 \to \mathcal{O}_{\tilde{X}} \to \pi^\star V \otimes \mathcal{O}_{\tilde{X}}(1) \to T_{\tilde{X}/X} \to 0 \quad \text{(Euler)}$$
$$0 \to T_{X_k/X_{k-1}} \to V_k \xrightarrow{(\pi_k)_\star} \mathcal{O}_{X_k}(-1) \to 0 \qquad \Rightarrow \text{rk}\, V_k = r$$
$$0 \to \mathcal{O}_{X_k} \to \pi_k^\star V_{k-1} \otimes \mathcal{O}_{X_k}(1) \to T_{X_k/X_{k-1}} \to 0 \quad \text{(Euler)}$$

- For $n = \dim X$ and $r = \mathrm{rk}\, V$, get a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\mathrm{rk}\, V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

- **Theorem.** *X_k is a smooth compactification of*

$$X_k^{\mathrm{GG,reg}}/\mathbb{G}_k = J_k^{\mathrm{GG,reg}}/\mathbb{G}_k$$

where $\mathbb{G}_k$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C},0)$, acting on the right by reparametrization: $(f,\varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k-jets of regular curves.

- **Direct image formula.** $(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = E_{k,m} V^* =$ *invariant algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.*

Let $(\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, $\quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$ be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X. It has a local Taylor expansion

$$f(t) = x + t\xi_1 + \ldots + t^k \xi_k + O(t^{k+1}), \quad \xi_s = \frac{1}{s!}\nabla^s f(0)$$

for some connection ∇ on V.

One considers the Green-Griffiths bundle $E_{k,m}^{\mathrm{GG}} V^*$ of polynomials of weighted degree m written locally in coordinate charts as

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(x)\xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k}, \quad \xi_s \in V,$$

also viewed as algebraic differential operators

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k}. \end{aligned}$$

Definition of algebraic differential operators (2)

Here $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet, and $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are supposed to holomorphic functions on X.

The $\mathbb{G}_k$-action : $(f,\varphi) \mapsto f \circ \varphi$, yields in particular, $\varphi_\lambda(t) = \lambda t \Rightarrow (f \circ \varphi_\lambda)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$, whence a $\mathbb{C}^*$-action

$$\lambda \cdot (\xi_1, \xi_1, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k).$$

$E^{\mathrm{GG}}_{k,m}$ is precisely the set of polynomials of weighted degree m, corresponding to coefficients $a_{\alpha_1\ldots\alpha_k}$ with $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$.

$E_{k,m}V^* \subset E^{\mathrm{GG}}_{k,m}V^*$ is the bundle of $\mathbb{G}_k$-"invariant" operators, i.e. such that

$$P((f \circ \varphi)_{[k]}) = \varphi'^m P(f_{[k]}) \circ \varphi, \quad \forall \varphi \in \mathbb{G}_k.$$

Canonical sheaf of a singular pair (X,V)

When V is nonsingular, we simply set $K_V = \det(V^*)$.
When V is singular, the canonical sheaf K_V is the rank 1 analytic sheaf defined as the integral closure of the image of the natural morphism

$$\Lambda^r T^*_X \to \Lambda^r V^* \to \mathcal{L}_V := \text{invert. sheaf } (\Lambda^r V^*)^{**}$$

that is, if the image is $\mathcal{L}_V \otimes \mathcal{J}_V$, $\mathcal{J}_V \subset \mathcal{O}_X$, one sets

$$K_V = \mathcal{L}_V \otimes \overline{\mathcal{J}}_V.$$

Definition. We say that (X,V) is of general type if there exist proper modifications $\mu = \hat\mu \circ \tilde\mu : \widehat{X} \to \widetilde{X} \to X$ such that $\hat\mu^* K_{\widetilde{V}}$ is a big invertible sheaf on $\widehat{X}$, where $\widetilde{X}$ is equipped with the pull-back directed structure $\widetilde{V} = \overline{\tilde\mu^{-1}(V)}$.

Generalized GGL conjecture

- **Generalized GGL conjecture.** *If (X, V) is directed manifold of general type, i.e. K_V big, then $\exists Y \subsetneq X$ such that $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ non const., one has $f(\mathbb{C}) \subset Y$.*
- **Remark.** Elementary by Ahlfors-Schwarz if $r = \mathrm{rk}\, V = 1$. $t \mapsto \log \|f'(t)\|_{V,h}$ is strictly subharmonic if $r = 1$ and (V^*, h^*) has > 0 curvature in the sense of currents.
- **Strategy : fundamental vanishing theorem.** [Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996] *Let $P \in H^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-A))$ be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then $\forall f : \mathbb{C} \to (X, V)$, one has $P(f_{[k]}) \equiv 0$.*
- **Theorem** (D-, 2010). Let (X, V) be of general type, i.e. s.t. K_V is a big rank 1 sheaf. Then $\exists\, k \geq 1$ and $\exists$ algebraic hypersurface $\Sigma \subsetneq X_k$ such that every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \mapsto (X, V)$ satisfies $f_{[k]}(\mathbb{C}) \subset \Sigma$.

Finsler metric on the k-jet bundles

Let $J_k V$ be the bundle of k-jets of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ Assuming that V is equipped with a hermitian metric h, one defines a "weighted Finsler metric" on $J^k V$ by taking $p = k!$ and

$$\Psi_{h_k}(f) := \Big(\sum_{1 \leq s \leq k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \Big)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$, with curvature form $(x, \xi_1, \ldots, \xi_k) \mapsto$

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi} \sum_{1 \leq s \leq k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha} \overline{\xi}_{s\beta}}{|\xi_s|^2} dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor Θ_{V^*,h^*} and $\omega_{\mathrm{FS},k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\mathrm{GG}} \to X$. The expression gets simpler by using polar coordinates $x_s = |\xi_s|_h^{2p/s}$, $u_s = \xi_s / |\xi_s|_h = \nabla^s f(0) / |\nabla^s f(0)|$.

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi}\sum_{1\le s\le k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors u_s in the unit sphere bundle $SV\subset V$.

The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum|\xi_s|^{2p/s}=1$, so we can take here $x_s\ge 0$, $\sum x_s=1$. This is essentially a sum of the form $\sum\frac{1}{s}\gamma(u_s)$ where u_s are random points of the sphere, and so as $k\to+\infty$ this can be estimated by a "Monte-Carlo" integral

$$\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\int_{u\in SV}\gamma(u)\,du.$$

As γ is quadratic here, $\int_{u\in SV}\gamma(u)\,du=\frac{1}{r}\,\mathrm{Tr}(\gamma)$.

It follows that the leading term in the estimate only involves the trace of Θ_{V^*,h^*}, i.e. the curvature of $(\det V^*,\det h^*)$, which can be taken to be >0 if $\det V^*$ is big.

Corollary (D-, 2010). Let (X,V) be a directed manifold, $F\to X$ a $\mathbb{Q}$-line bundle, (V,h) and (F,h_F) hermitian. Define

$$L_k=\mathcal{O}_{X_k^{\mathrm{GG}}}(1)\otimes\pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big),$$

$$\eta=\Theta_{\det V^*,\det h^*}+\Theta_{F,h_F}.$$

Then for all $q\ge 0$ and all $m\gg k\gg 1$ such that m is sufficiently divisible, we have

$$h^q(X_k^{\mathrm{GG}},\mathcal{O}(L_k^{\otimes m}))\le\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q)}(-1)^q\eta^n+\frac{C}{\log k}\Big)$$

$$h^0(X_k^{\mathrm{GG}},\mathcal{O}(L_k^{\otimes m}))\ge\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,\le 1)}\eta^n-\frac{C}{\log k}\Big).$$

Induced directed structure on a subvariety

Let Z be an irreducible algebraic subset of some k-jet bundle X_k over X, such that Z projects onto X_{k-1}, i.e.

$$\pi_{k,k-1}(Z) = X_{k-1}.$$

We define the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure of $T_{Z'} \cap V_k$ taken on a suitable Zariski open set $Z' \subset Z_{\rm reg}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$.

Alternatively, one could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage $(\mathcal{X}_k, \mathcal{A}_k)$ of the "absolute Semple tower" associated with $(\mathcal{X}_0, \mathcal{A}_0) = (X, T_X)$.

This produces an induced directed pair

$$(Z, W) \subset (X_k, V_k),$$

and it is easy to show that $\operatorname{rk} W < \operatorname{rk} V_k = \operatorname{rk} V$.

Partial solution of GGL conjecture

Definition. *Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "strongly of general type" if it is of general type and for every irreducible component $Z \subsetneq X_k$ that projects onto X_{k-1}, $X_k \not\subset D_k := P(T_{X_{k-1}/X_{k-2}})$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, i.e. $K_W \otimes \mathcal{O}_{X_k}(m)_{|Z}$ is big for some $m \in \mathbb{Q}_+$.*

Theorem (D-, 2014) *If (X, V) is strongly of general type, the Green-Griffiths-Lang conjecture holds true for (X, V), namely there $\exists Y \subsetneq X$ such that every non constant holomorphic curve $f : (\mathbb{C}, T_{\mathbb{C}} \to (X, V)$ satisfies $f(\mathbb{C}) \subset Y$.*

Proof: Induction using existence theorem for jet differentials. Unfortunately by contradiction, and thus non constructive.

Definition. Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geq 1$, $Z \not\subset D_k$, and $Z = X = X_0$ for $k = 0$, we define the slope of the corresponding directed variety (Z, W) to be $\mu_A(Z, W) =$

$$\frac{\inf\left\{\lambda \in \mathbb{Q}\,;\ \exists m \in \mathbb{Q}_+,\ K_W \otimes \big(\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}(\lambda A)\big)_{|Z} \text{ big on } Z\right\}}{\operatorname{rank} W}.$$

Notice that (X, V) is of general type iff $\mu_A(X, V) < 0$.

We say that (X, V) is *A-jet-stable* (resp. *A-jet-semi-stable*) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leq \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above.

Observation. If (X, V) is of general type and A-jet-semi-stable, then (X, V) is strongly of general type.