General extension theorem for cohomology classes on non reduced analytic subspaces

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This is a joint work with Junyan Cao & Shin-ichi Matsumura

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The extension problem

Let (X,ω) be a possibly noncompact n-dimensional Kähler manifold, $\mathcal{J} \subset \mathcal{O}_X$ a coherent ideal sheaf, $Y = V(\mathcal{J})$ its zero variety and

$$\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}.$$

Here Y may be non reduced, i.e. $\mathcal{O}_Y$ may have nilpotent elements.

Also, let (L, h_L) be a hermitian holomorphic line bundle on X, and

$$\Theta_{L,h_L} = i\,\partial\overline{\partial}\log h_L^{-1}$$

its curvature current (we allow singular metrics, $h_L = e^{-\varphi}$, $\varphi \in L^1_{\rm loc}$, Θ_{L,h_L} being computed in the sense of currents).

Question

Under which conditions on X, $Y = V(\mathcal{J})$, (L, h_L) is

$$H^q(X, K_X \otimes L) \to H^q(Y, (K_X \otimes L)_{|Y}) = H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{O}_X/\mathcal{J})$$

a surjective restriction morphism?

Motivation: abundance conjecture and MMP

One potential application would be to study the Minimal Model Program (MMP) for arbitrary projective – or even Kähler – varieties, whereas only the case of general type varieties is known.

For a line bundle L, one defines the Kodaira-Iitaka dimension $\kappa(L) = \limsup_{m\to+\infty} \log\dim H^0(X, L^{\otimes m})/\log m$ and the numerical dimension $\mathrm{nd}(L)$ = maximum exponent p of non zero “positive intersections” $\langle T^p\rangle$ of a positive current $T \in c_1(L)$ when L is psef (pseudoeffective), and $\mathrm{nd}(L) = -\infty$ otherwise. They always satisfy

$$-\infty \le \kappa(L) \le \mathrm{nd}(L) \le n = \dim X.$$

Definition (abundance)

A line bundle L is said to be abundant if $\kappa(L) = \mathrm{nd}(L)$.

The fundamental abundance conjecture can be stated: for each nonsingular klt pair (X, Δ) the $\mathbb{Q}$-line bundle $K_X + \Delta$ is abundant.

Generalized base point free theorem ?

One can try to investigate the abundance of $L = K_X + \Delta$ by induction on the dimension $n = \dim X$, by extending sections of $K_X + L_m$, $L_m = (m-1)K_X + m\Delta$ from subvarieties (noticing that $K_X + \Delta$ psef implies L_m psef, and even $L_m - \Delta$ psef). Cf. BCHM and recent work of Demailly-Hacon-Păun, Fujino, Gongyo, Takayama.

Standard base point free theorem

Let (X, Δ) be a projective klt pair, and L be a nef line bundle such that $L - (K_X + \Delta)$ is nef and big. Then L is semiample, i.e. $|mL|$ is BPF for some $m > 0$.

Question (weak positivity variant of the BPF property ?)

Assume that X is not uniruled, i.e. that K_X is pseudoeffective, and let L be a line bundle such that $L - \varepsilon K_X$ is pseudoeffective for some $0 < \varepsilon \ll 1$. Does there exist $G \in \mathrm{Pic}^0(X)$ such that $L + G$ is abundant ?

First naive (and too restrictive) technique

Consider the exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to 0$$

twisted by $\mathcal{O}_X(K_X \otimes L)$, and the corresponding long exact sequence of cohomology groups

$$\cdots \to H^q(X, K_X \otimes L) \to H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{O}_X/\mathcal{J}) \to H^{q+1}(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{J}) \cdots$$

It is therefore enough to have

$$H^{q+1}(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{J}) = 0.$$

In order to kill H^{q+1} it is enough to make a strict positivity (ampleness) assumption, e.g. by the Kodaira-Nakano / Nadel vanishing theorems.

But we only want to make a weak semipositivity assumption!

In that case, one cannot expect to kill the cohomology group H^{q+1}.

Assumptions (1)

We assume X to be holomorphically convex. By the Cartan-Remmert theorem, this is the case iff X admits a proper holomorphic map $p: X \to S$ only a Stein complex space S.

Observation : cohomology is then always Hausdorff

Let X be a holomorphically convex complex space and $\mathcal{F}$ a coherent analytic sheaf over X. Then all cohomology groups $H^q(X, \mathcal{F})$ are Hausdorff with respect to their natural topology (local uniform convergence of holomorphic Čech cochains)

Proof. $H^q(X, \mathcal{F}) \simeq H^0(S, R^q p_* \mathcal{F})$ is a Fréchet space.

Corollary

To solve an equation $\overline{\partial} u = v$ on a holomorphically convex manifold X, it is enough to solve it approximately:

$$\overline{\partial} u_\varepsilon = v + w_\varepsilon, \qquad w_\varepsilon \to 0 \ \text{ as } \varepsilon \to 0$$

Assumptions (2)

We assume that the subvariety $Y \subset X$ is defined by

$$Y = V(\mathcal{I}(e^{-\psi})), \qquad \mathcal{O}_Y := \mathcal{O}_X / \mathcal{I}(e^{-\psi})$$

where ψ is a quasi-psh function with *analytic singularities*, i.e. locally on a neighborhood V of an arbitrary point $x_0 \in X$ we have

$$\psi(z) = c \log \sum |g_j(z)|^2 + v(z), \quad g_j \in \mathcal{O}_X(V), \ c > 0, \ v \in C^\infty(V),$$

and $\mathcal{I}(e^{-\psi}) \subset \mathcal{O}_X$ is the multiplier ideal sheaf

$$\mathcal{I}(e^{-\psi})_{x_0} = \big\{ f \in \mathcal{O}_{X,x_0} \,;\, \exists U \ni x_0 \,,\ \textstyle\int_U |f|^2 e^{-\psi} d\lambda < +\infty \big\}$$

Claim (Nadel)

$\mathcal{I}(e^{-\psi})$ is a coherent ideal sheaf.

Moreover $\mathcal{I}(e^{-\psi})$ is always an integrally closed ideal.

Typical choice: $\psi(z) = c \log |s(z)|^2_{h_E}, \ c > 0, \ s \in H^0(X, E)$.

Log resolution / reduction to the divisorial case

The simplest case is when $Y = \sum m_j Y_j$ is an effective simple normal crossing divisor and $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{O}_X(-Y)$. We can then take

$$\psi(z) = \sum c_j \log |\sigma_{Y_j}|^2_{h_j}, \quad c_j > 0, \ \lfloor c_j \rfloor = m_j,$$

for some smooth hermitian metric h_j on $\mathcal{O}_X(Y_j)$. Then

$$\mathcal{I}(e^{-\psi}) = \mathcal{O}_X\big(-\sum m_j Y_j\big), \qquad i\partial\overline{\partial}\psi = \sum c_j(2\pi[Y_j] - \Theta_{\mathcal{O}(Y_j),h_j})$$

The case of a higher codimensional multiplier ideal scheme $\mathcal{I}(e^{-\psi})$ can easily be reduced to the divisorial case by using a suitable log resolution (a composition of blow ups, thanks to Hironaka's desingularization theorem).

Main results

Theorem (JY. Cao, D– , S-i. Matsumura, January 2017)

Take (X, ω) to be Kähler and holomorphically convex,
and let (L, h_L) be a hermitian line bundle such that

$$(**) \qquad \Theta_{L,h_L} + (1+\alpha\delta) i\partial\overline{\partial}\psi \geq 0 \qquad \text{in the sense of currents}$$

for some $\delta(x) > 0$ continuous and $\alpha = 0, 1$. Then:
the morphism induced by the natural inclusion $\mathcal{I}(h_L e^{-\psi}) \to \mathcal{I}(h_L)$

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-\psi})) \to H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L))$$

is injective for every $q \geq 0$, in other words, the sheaf morphism $\mathcal{I}(h) \to \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})$ yields a surjection

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)) \to H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})).$$

Corollary (take h_L smooth $\Rightarrow \mathcal{I}(h_L) = \mathcal{O}_X$, and $Y = V(\mathcal{I}(e^{-\psi}))$)

If h_L is smooth, $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}(e^{-\psi})$ and h_L, ψ satisfy $(**)$, then $H^q(X, K_X \otimes L) \to H^q(Y, (K_X \otimes L)_{|Y})$ is surjective.

Comments / algebraic consequences

The exact sequence $0 \to \mathcal{I}(h_L e^{-\psi}) \to \mathcal{I}(h_L) \to \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi}) \to 0$ implies that both injectivity and surjectivity hold when

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-\psi})) = 0,$$

and for this it is enough to have a strict curvature assumption

$(***)$ $\qquad \Theta_{L,h_L} + i\partial\overline{\partial}\psi \geq \delta\omega > 0 \qquad$ in the sense of currents.

Corollary (purely algebraic)

Assume that X is projective (or that one has a projective morphism $X \to S$ over an affine algebraic base S). Let $Y = \sum m_j Y_j$ be an effective divisor and $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{O}_X(-Y)$. If (as $\mathbb{Q}$-divisors)

$(**)$ $\qquad L - (1+\delta)\sum c_j Y_j = G_\delta + U_\delta, \quad \lfloor c_j \rfloor = m_j$

with $\delta = 0$ or $\delta_0 \in \mathbb{Q}_+^*$, G_δ semiample and $U_\delta \in \mathrm{Pic}^0(X)$, then

$$H^q(X, K_X \otimes L) \to H^q(Y, (K_X \otimes L)_{|Y})$$

is surjective.

Twisted Bochner-Kodaira-Nakano inequality (Ohsawa-Takegoshi)

Let (X, ω) be a Kähler manifold and let η, $\lambda > 0$ be smooth functions on X.
For every compacted supported section $u \in \mathcal{C}_c^\infty(X, \Lambda^{p,q} T_X^* \otimes L)$ with values in a hermitian line bundle (L, h_L), one has

$$\|(\eta+\lambda)^{\frac{1}{2}}\overline{\partial}^* u\|^2 + \|\eta^{\frac{1}{2}}\overline{\partial}u\|^2 + \|\lambda^{\frac{1}{2}}\partial u\|^2 + 2\|\lambda^{-\frac{1}{2}}\partial\eta \wedge u\|^2 \geq \int_X \langle B^{p,q}_{L,h_L,\omega,\eta,\lambda} u, u\rangle dV_{X,\omega}$$

where $dV_{X,\omega} = \frac{1}{n!}\omega^n$ is the Kähler volume element and $B^{p,q}_{L,h_L,\omega,\eta,\lambda}$ is the Hermitian operator on $\Lambda^{p,q} T_X^* \otimes L$ such that

$$B^{p,q}_{L,h_L,\omega,\eta,\lambda} = [\eta\, i\Theta_L - i\,\partial\overline{\partial}\eta - i\lambda^{-1}\partial\eta \wedge \overline{\partial}\eta\ ,\ \Lambda_\omega].$$

Approximate solutions to $\overline{\partial}$-equations

Main L^2 estimate

Let (X,ω) be a Kähler manifold possessing a complete Kähler metric let (E,h_E) be a Hermitian vector bundle over X. Assume that $B=B^{n,q}_{E,h,\omega,\eta,\lambda}$ satisfies $B+\varepsilon\,\mathrm{Id}>0$ for some $\varepsilon>0$ (so that B can be just semi-positive or even slightly negative).
Take a section $v\in L^2(X,\Lambda^{n,q}T^*_X\otimes E)$ such that $\overline{\partial}v=0$ and

$$M(\varepsilon):=\int_X\langle(B+\varepsilon\,\mathrm{Id})^{-1}v,v\rangle\,dV_{X,\omega}<+\infty.$$

Then there exists an approximate solution $u_\varepsilon\in L^2(X,\Lambda^{n,q-1}T^*_X\otimes E)$ and a correction term $w_\varepsilon\in L^2(X,\Lambda^{n,q}T^*_X\otimes E)$ such that

$$\overline{\partial}u_\varepsilon=v+w_\varepsilon \qquad \text{and}$$

$$\int_X(\eta+\lambda)^{-1}|u_\varepsilon|^2\,dV_{X,\omega}+\frac{1}{\varepsilon}\int_X|w_\varepsilon|^2\,dV_{X,\omega}\le M(\varepsilon).$$

Proof: setting up the relevant $\overline{\partial}$ equation (1)

Every cohomology class in

$$H^q(X,\mathcal{O}_X(K_X\otimes L)\otimes\mathcal{I}(h_L)/\mathcal{I}(h_Le^{-\psi}))$$

is represented by a holomorphic Čech q-cocycle with respect to a Stein covering $\mathcal{U}=(U_i)$, say $(c_{i_0\ldots i_q})$,

$$c_{i_0\ldots i_q}\in H^0\big(U_{i_0}\cap\ldots\cap U_{i_q},\mathcal{O}_X(K_X\otimes L)\otimes\mathcal{I}(h_L)/\mathcal{I}(h_Le^{-\psi})\big).$$

By the standard sheaf theoretic isomorphism with Dolbeault cohomology, this class is represented by a smooth (n,q)-form

$$f=\sum_{i_0,\ldots,i_q}c_{i_0\ldots i_q}\rho_{i_0}\overline{\partial}\rho_{i_1}\wedge\ldots\overline{\partial}\rho_{i_q}$$

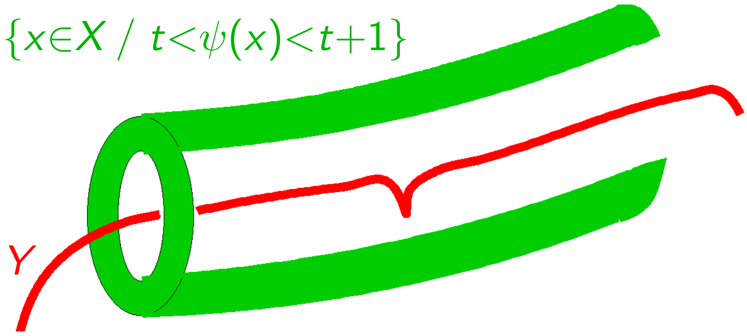
by means of a partition of unity (ρ_i) subordinate to (U_i). This form is to be interpreted as a form on the (non reduced) analytic subvariety Y associated with the colon ideal sheaf $\mathcal{J}=\mathcal{I}(he^{-\psi}):\mathcal{I}(h)$ and the structure sheaf $\mathcal{O}_Y=\mathcal{O}_X/\mathcal{J}$.

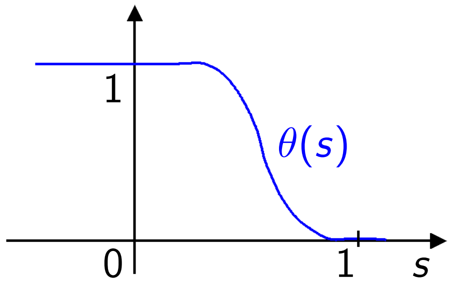
Proof: setting up the relevant $\overline{\partial}$ equation (2)

We get an extension of f as a smooth (no longer $\overline{\partial}$-closed) (n,q)-form on X by taking

$$\widetilde{f} = \sum_{i_0,\ldots,i_q} \widetilde{c}_{i_0\ldots i_q} \rho_{i_0} \overline{\partial}\rho_{i_1} \wedge \ldots \overline{\partial}\rho_{i_q}$$

where $\widetilde{c}_{i_0\ldots i_q}$ = extension of $c_{i_0\ldots i_q}$ from $U_{i_0}\cap\ldots\cap U_{i_q}\cap Y$ to $U_{i_0}\cap\ldots\cap U_{i_q}$





Now, truncate $\widetilde{f}$ as $\theta(\psi - t)\cdot\widetilde{f}$ on the green hollow tubular neighborhood, and solve an approximate $\overline{\partial}$-equation

$$(*) \qquad \overline{\partial} u_{t,\varepsilon} = \overline{\partial}(\theta(\psi - t)\cdot\widetilde{f}) + w_{t,\varepsilon}$$

Proof: setting up the relevant $\overline{\partial}$ equation (3)

Here we have

$$\overline{\partial}(\theta(\psi - t)\cdot\widetilde{f}) = \theta'(\psi - t)\overline{\partial}\psi \wedge \widetilde{f} + \theta(\psi - t)\cdot\overline{\partial}\widetilde{f}$$

where the first term vanishes near Y and the second one is L^2 with respect to $h_L e^{-\psi}$ (as the image of $\overline{\partial}\widetilde{f}$ in $\mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})$ is $\overline{\partial}f = 0$).

With ad hoc "twisting functions" $\eta = \eta_t := 1 - \delta\chi_t(\psi)$, $\lambda := \pi(1 + \delta^2\psi^2)$ and a suitable adjustment $\varepsilon = e^{(1+\beta)t}$, $\beta \ll 1$, we can find approximate L^2 solutions of the $\overline{\partial}$-equation such that

$$\overline{\partial} u_{t,\varepsilon} = \overline{\partial}(\theta(\psi - t)\cdot\widetilde{f}) + w_{t,\varepsilon}\ , \qquad \int_X |u_{t,\varepsilon}|^2_{\omega,h_L} e^{-\psi} dV_{X,\omega} < +\infty$$

and

$$\lim_{t\to-\infty} \int_X |w_{t,\varepsilon}|^2_{\omega,h_L} e^{-\psi} dV_{X,\omega} = 0.$$

The estimate on $u_{t,\varepsilon}$ with respect to the weight $h_L e^{-\psi}$ shows that $\theta(\psi - t)\cdot\widetilde{f} - u_{t,\varepsilon}$ is an approximate extension of f. □

Can one get estimates for the extension ?

The answer is yes if ψ is log canonical, namely $\mathcal{I}(e^{-(1-\varepsilon)\psi}) = \mathcal{O}_X$ for all $\varepsilon > 0$. Then $Y = V(\mathcal{I}(e^{-\psi}))$ is easily seen to be reduced.

Ohsawa's residue measure

If ψ is log canonical, one can also associate in a natural way a measure $dV_{Y^\circ,\omega}[\psi]$ on the set Y° of regular points of Y as follows. If $g \in \mathcal{C}_c(Y^\circ)$ is a compactly supported continuous function on Y° and $\widetilde{g}$ a compactly supported extension of g to X, one sets

$$\int_{Y^\circ} g\, dV_{Y^\circ,\omega}[\psi] = \lim_{t\to-\infty} \int_{\{x\in X\,,\ t<\psi(x)<t+1\}} \widetilde{g} e^{-\psi}\, dV_{X,\omega}$$

Theorem

If ψ is lc and the curvature hypothesis is satisfied, for any f in $H^0(Y, K_X \otimes L \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi}))$ s.t. $\int_{Y^\circ} |f|^2_{\omega,h_L} dV_{Y^\circ,\omega}[\psi] < +\infty$, there exists $\widetilde{f} \in H^0(X, K_X \otimes L \otimes \mathcal{I}(h_L))$ which extends f, such that

$$\int_X (1+\delta^2\psi^2)^{-1} e^{-\psi} |\widetilde{f}|^2_{\omega,h_L} dV_{X,\omega} \le \frac{34}{\delta} \int_{Y^\circ} |f|^2_{\omega,h_L} dV_{Y^\circ,\omega}[\psi].$$

Can one get estimates for the extension ? (sequel)

If ψ is not log canonical, consider the "last jumps" $m_{p-1} < m_p \le 1$ such that $\mathcal{I}(h_L e^{-m_{p-1}\psi}) \supsetneq \mathcal{I}(h_L e^{-m_p\psi}) = \mathcal{I}(h_L e^{-\psi})$ and assume

$$f \in H^0(Y, K_X \otimes L \otimes \mathcal{I}(h_L e^{-m_{p-1}\psi})/\mathcal{I}(h_L e^{-m_p\psi})),$$

i.e., f vanishes just a little bit less than prescribed by the sheaf $\mathcal{I}(h_L e^{-\psi}))$. Then there is still a corresponding residue measure:

Higher multiplicity residue measure

If f is as above, and $\widetilde{f}$ is a local extension, one can associate a higher multiplicity residue measure $|f|^2 dV_{Y^\circ,\omega}[\psi]$ (formal notation) as follows. If $g \in \mathcal{C}_c(Y^\circ)$ and $\widetilde{g}$ a compactly supported extension of g to X, one sets

$$\int_{Y^\circ} g\, |f|^2 dV_{Y^\circ,\omega}[\psi] = \lim_{t\to-\infty} \int_{\{x\in X\,,\ t<\psi(x)<t+1\}} \widetilde{g}\, |\widetilde{f}|^2 e^{-m_p\psi}\, dV_{X,\omega}$$

Then a global extension $\widetilde{f} \in H^0(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-m_{p-1}\psi}))$ exists, that satisfies the expected L^2 estimate.

Special case / limitations of the L^2 estimates

In the special case when ψ is given by $\psi(z) = r \log |s(z)|^2_{h_E}$ for a section $s \in H^0(X, E)$ generically transverse to the zero section of a rank r vector vector E on X, the subvariety $Y = s^{-1}(0)$ has codimension r, and one can check easily that

$$dV_{Y^\circ,\omega}[\psi] = \frac{dV_{Y^\circ,\omega}}{|\Lambda^r(ds)|^2_{\omega,h_E}}.$$

Thus one sees that the residue measure takes into account in a very precise manner the singularities of Y. It may happen that $dV_{Y^\circ,\omega}[\psi]$ has infinite mass near the singularities of Y, as is the case when Y is a simple normal crossing divisor.

Therefore, sections $s \in H^0(Y, (K_X \otimes L)_{|Y}$ may not be L^2 with respect to $dV_{Y^\circ,\omega}[\psi])$, and the L^2 estimate of the approximate extension can blow up as $\varepsilon \to 0$. The surprising fact is this is however sufficient to prove the qualitative extension theorem, but without any effective L^2 estimate in the limit.

Remarks: optimal L^2 estimates

In a series of fundamental papers, Z. Błocki and Guan-Zhou have shown that in the log canonical case, the L^2 estimate holds with an optimal constant.

This implies some interesting consequences, such as a proof of the Suita conjecture on the optimal comparison between the Bergman kernel and the logarithmic capacity of domains.

In a very recent paper (June 27), G. Hosono has proved that the L^2 estimate also holds with an optimal constant in case there is only one jump and the multiplier ideal sheaf is a power of the reduced ideal defining the subvariety Y.

In fact, this result can be seen to hold under the sole assumption that there is only one jump, by a variation of the known methods (Błocki, Guan-Zhou, Berndtsson-Lempert).

The end