On the cohomology of pseudoeffective line bundles

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Basic concepts (1)

Let X = compact Kähler manifold, $L \to X$ holomorphic line bundle, h a hermitian metric on L.
Locally $L_{|U} \simeq U \times \mathbb{C}$ and for $\xi \in L_x \simeq \mathbb{C}$, $\|\xi\|_h^2 = |\xi|^2 e^{-\varphi(x)}$.
Writing $h = e^{-\varphi}$ locally, one defines the curvature form of L to be the real $(1,1)$-form

$$\Theta_{L,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi = -dd^c \log h,$$
$$c_1(L) = \{\Theta_{L,h}\} \in H^2(X,\mathbb{Z}).$$

Any subspace $V_m \subset H^0(X, L^{\otimes m})$ define a meromorphic map

$$\begin{array}{rcll} \Phi_{mL} : X \smallsetminus Z_m & \longrightarrow & \mathbb{P}(V_m) & \text{(hyperplanes of } V_m) \\ x & \longmapsto & H_x = \{\sigma \in V_m\,;\ \sigma(x) = 0\} & \end{array}$$

where Z_m = base locus $B(mL) = \bigcap \sigma^{-1}(0)$.

Basic concepts (2)

Given sections $\sigma_1, \ldots, \sigma_n \in H^0(X, L^{\otimes m})$, one gets a singular hermitian metric on L defined by

$$|\xi|_h^2 = \frac{|\xi|^2}{\big(\sum |\sigma_j(x)|^2\big)^{1/m}},$$

its weight is the plurisubharmonic (psh) function

$$\varphi(x) = \frac{1}{m}\log\Big(\sum |\sigma_j(x)|^2\Big)$$

and the curvature is $\Theta_{L,h} = \frac{1}{m} dd^c \log\varphi \geq 0$
in the sense of currents, with logarithmic poles along the base locus

$$B = \bigcap \sigma_j^{-1}(0) = \varphi^{-1}(-\infty).$$

One has

$$(\Theta_{L,h})_{|X\smallsetminus B} = \frac{1}{m}\Phi_{mL}^*\omega_{\rm FS} \text{ where } \Phi_{mL} : X\smallsetminus B \to \mathbb{P}(V_m) \simeq \mathbb{P}^{N_m}.$$

Basic concepts (3)

Definition

- L is pseudoeffective (psef) if $\exists h = e^{-\varphi}$, $\varphi \in L^1_{\rm loc}$, (possibly singular) such that $\Theta_{L,h} = -dd^c \log h \geq 0$ on X, in the sense of currents.
- L is semipositive if $\exists h = e^{-\varphi}$ smooth such that $\Theta_{L,h} = -dd^c \log h \geq 0$ on X.
- L is positive if $\exists h = e^{-\varphi}$ smooth such that $\Theta_{L,h} = -dd^c \log h > 0$ on X.

The well-known Kodaira embedding theorem states that L is positive if and only if L is ample, namely:
$Z_m = B(mL) = \emptyset$ and

$$\Phi_{|mL|} : X \to \mathbb{P}(H^0(X, L^{\otimes m}))$$

is an embedding for $m \geq m_0$ large enough.

Positive cones

Definitions

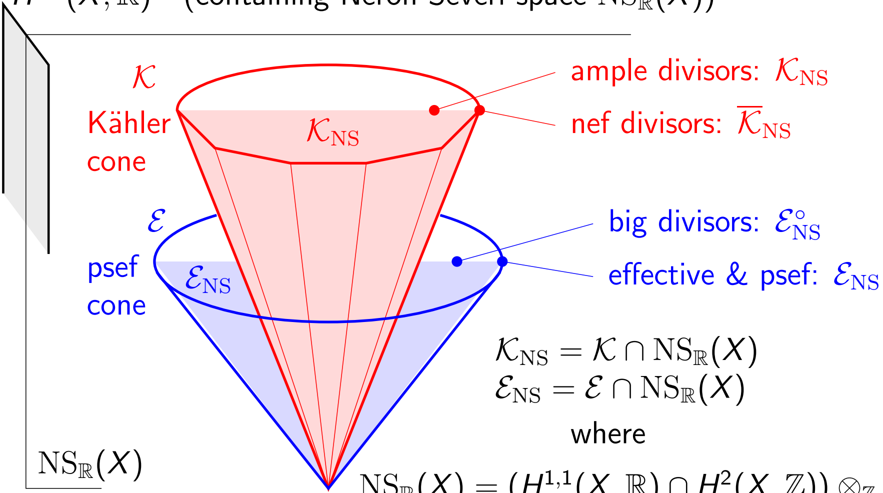
Let X be a compact Kähler manifold.

- The Kähler cone is the (open) set $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{\omega\}$ of positive Kähler forms.
- The pseudoeffective cone is the set $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{T\}$ of closed positive $(1,1)$ currents. This is a closed convex cone.
  (by weak compactness of bounded sets of currents).
- $\overline{\mathcal{K}}$ is the cone of "nef classes". One has $\overline{\mathcal{K}} \subset \mathcal{E}$.
- It may happen that $\overline{\mathcal{K}} \subsetneq \mathcal{E}$:
  if X is the surface obtained by blowing-up $\mathbb{P}^2$ in one point, then the exceptional divisor $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

Ample / nef / effective / big divisors

Positive cones can be visualized as follows :

$H^{1,1}(X,\mathbb{R})$ (containing Neron-Severi space $\mathrm{NS}_{\mathbb{R}}(X)$)



$$\mathcal{K}_{\mathrm{NS}} = \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X)$$
$$\mathcal{E}_{\mathrm{NS}} = \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X)$$

where

$$\mathrm{NS}_{\mathbb{R}}(X) = (H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{R}$$

Approximation of currents, Zariski decomposition

Definition

On X compact Kähler, a Kähler current T is a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.

Easy observation

$\alpha \in \mathcal{E}^\circ$ (interior of $\mathcal{E}$) $\iff$ $\alpha = \{T\}$, $T =$ a Kähler current.
We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.

Theorem on approximate Zariski decomposition (D92)

Any Kähler current can be written $T = \lim T_m$ where $T_m \in \alpha = \{T\}$ has logarithmic poles, i.e.
$\exists$ a modification $\mu_m : \widetilde{X}_m \to X$ such that $\mu_m^\star T_m = [E_m] + \beta_m$
where E_m is an effective $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coefficients in $\frac{1}{m}\mathbb{Z}$ and β_m is a Kähler form on $\widetilde{X}_m$.

Idea of proof of analytic Zariski decomposition (1)

- Write locally

$$T = i\partial\overline{\partial}\varphi$$

for some strictly plurisubharmonic psh potential φ on X.

- Approximate T (again locally) as

$$T_m = i\partial\overline{\partial}\varphi_m, \qquad \varphi_m(z) = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \Big\{f \in \mathcal{O}(\Omega)\,;\ \int_\Omega |f|^2 e^{-2m\varphi} dV < +\infty\Big\}.$$

- The Ohsawa-Takegoshi L^2 extension theorem (extending from a single isolated point) implies that there are enough such holomorphic functions, and thus $\varphi_m \geq \varphi - C/m$.
- Further, $\varphi = \lim_{m\to+\infty} \varphi_m$ by the mean value inequality.

Idea of proof of analytic Zariski decomposition (2)

• The Hilbert basis $(g_{\ell,m})$ is a family of local generators of the multiplier ideal sheaf $\mathcal{I}(m\varphi)$. Thanks to Hironaka, the modification $\mu_m : \widetilde{X}_m \to X$ is obtained by blowing-up $\mathcal{I}(m\varphi)$, with

$$\mu_m^\star \mathcal{I}(mT) = \mathcal{O}(-mE_m).$$

for some effective $\mathbb{Q}$-divisor E_m with normal crossings on $\widetilde{X}_m$.

• Now, we set

$$T_m = i\partial\overline{\partial}\varphi_m, \qquad \beta_m = \mu_m^* T_m - [E_m].$$

• Locally on $\widetilde{X}_m$ one has

$$\beta_m = i\partial\overline{\partial}\psi_m \quad \text{where} \quad \psi_m = \frac{1}{2m} \log \sum_\ell |g_{\ell,m} \circ \mu_m / h|^2$$

and h is a generator of $\mathcal{O}(-mE_m)$, thus $\beta_m \geq 0$ smooth.

• The construction can be made global by using a gluing technique, e.g. a partition of unity, and

• β_m can be made Kähler by a perturbation argument.

Concept of volume (very important !)

Definition (Boucksom 2002).
The volume (movable self-intersection) of a big class $\alpha \in \mathcal{E}^\circ$ *is*

$$\mathrm{Vol}(\alpha) = \sup_{T \in \alpha} \int_{\widetilde{X}} \beta^n > 0$$

where the supremum is taken over all Kähler currents $T \in \alpha$ *with logarithmic poles, and* $\mu^\star T = [E] + \beta$ *with respect to some modification* $\mu : \widetilde{X} \to X$.

If $\alpha \in \mathcal{K}$, then $\mathrm{Vol}(\alpha) = \alpha^n = \int_X \alpha^n$.

Theorem. (Boucksom 2002). *If L is a big line bundle and*
$\mu_m^*(mL) = [E_m] + [D_m]$
(where E_m = *fixed part,* D_m = *moving part), then*

$$\mathrm{Vol}(c_1(L)) = \lim_{m \to +\infty} \frac{n!}{m^n} h^0(X, mL) = \lim_{m \to +\infty} D_m^n.$$

Approximate Zariski decomposition

In other words, the volume measures the amount of sections and the growth of the degree of the images of the rational maps

$$\Phi_{|mL|} : X \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$$

By Fujita 1994 and Demailly-Ein-Lazarsfeld 2000, one has

Theorem. *Let L be a big line bundle on the projective manifold X. Let $\epsilon > 0$. Then there exists a modification $\mu : X_\epsilon \to X$ and a decomposition $\mu^*(L) = E + \beta$ with E an effective $\mathbb{Q}$-divisor and β a big and nef $\mathbb{Q}$-divisor such that*

$$\mathrm{Vol}(L) - \varepsilon \le \mathrm{Vol}(\beta) \le \mathrm{Vol}(L).$$

Movable intersection theory

Theorem (Boucksom 2002) *Let X be a compact Kähler manifold and*

$$H^{k,k}_{\geq 0}(X) = \{\{T\} \in H^{k,k}(X, \mathbb{R})\,;\ T \text{ closed} \geq 0\}.$$

- $\forall k = 1, 2, \ldots, n$, $\exists$ *canonical "movable intersection product"*

  $$\mathcal{E} \times \cdots \times \mathcal{E} \to H^{k,k}_{\geq 0}(X), \quad (\alpha_1, \ldots, \alpha_k) \mapsto \langle \alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1} \cdot \alpha_k \rangle$$

  such that $\mathrm{Vol}(\alpha) = \langle \alpha^n \rangle$ *whenever* α *is a big class.*
- *The product is increasing, homogeneous of degree* 1 *and superadditive in each argument, i.e.*

  $$\langle \alpha_1 \cdots (\alpha'_j + \alpha''_j) \cdots \alpha_k \rangle \geq \langle \alpha_1 \cdots \alpha'_j \cdots \alpha_k \rangle + \langle \alpha_1 \cdots \alpha''_j \cdots \alpha_k \rangle.$$

  It coincides with the ordinary intersection product when the $\alpha_j \in \overline{\mathcal{K}}$ *are nef classes.*

Movable intersection theory (continued)

- *For $k = 1$, one gets a "divisorial Zariski decomposition"*

$$\alpha = \{N(\alpha)\} + \langle\alpha\rangle$$

where :

Movable intersection theory (continued)

- *For* $k = 1$*, one gets a "divisorial Zariski decomposition"*

$$\alpha = \{N(\alpha)\} + \langle\alpha\rangle$$

where :

  - $N(\alpha)$ *is a uniquely defined effective divisor which is called the "negative divisorial part" of* α*. The map* $\alpha \mapsto N(\alpha)$ *is homogeneous and subadditive ;*
  - $\langle\alpha\rangle$ *is "nef outside codimension 2".*

Construction of the movable intersection product

First assume that all classes α_j are big, i.e. $\alpha_j \in \mathcal{E}^\circ$. Fix a smooth closed $(n-k, n-k)$ *semi-positive* form u on X. We select Kähler currents $T_j \in \alpha_j$ with logarithmic poles, and simultaneous more and more accurate log-resolutions $\mu_m : \widetilde{X}_m \to X$ such that

$$\mu_m^\star T_j = [E_{j,m}] + \beta_{j,m}.$$

We define

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{m \to +\infty} \uparrow \{(\mu_m)_\star(\beta_{1,m} \wedge \beta_{2,m} \wedge \ldots \wedge \beta_{k,m})\}$$

as a weakly convergent subsequence. The main point is to show that there is actually convergence and that the limit is unique in cohomology ; this is based on "monotonicity properties" of the Zariski decomposition.

Generalized abundance conjecture

Definition. *For a class $\alpha \in H^{1,1}(X, \mathbb{R})$, the numerical dimension $\nu(\alpha)$ is*

- $\nu(\alpha) = -\infty$ *if α is not pseudoeffective,*
- $\nu(\alpha) = \max\{p \in \mathbb{N} \,;\, \langle \alpha^p \rangle \neq 0\} \quad \in \{0, 1, \ldots, n\}$ *if α is pseudoeffective.*

Conjecture ("generalized abundance conjecture"). *For an arbitrary compact Kähler manifold X, the Kodaira dimension should be equal to the numerical dimension :*

$$\kappa(X) = \nu(c_1(K_X)).$$

Remark. The generalized abundance conjecture holds true when $\nu(c_1(K_X)) = -\infty$, 0, n (cases $-\infty$ and n being easy).

Orthogonality estimate

Theorem. *Let X be a projective manifold. Let $\alpha = \{T\} \in \mathcal{E}^{\circ}_{\rm NS}$ be a big class represented by a Kähler current T, and consider an approximate Zariski decomposition*

$$\mu_m^\star T_m = [E_m] + [D_m]$$

Then

$$(D_m^{n-1} \cdot E_m)^2 \le 20\,(C\omega)^n\big(\,\mathrm{Vol}(\alpha) - D_m^n\big)$$

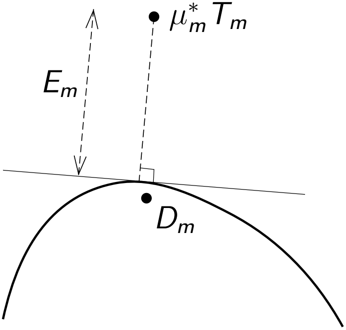
where $\omega = c_1(H)$ is a Kähler form and $C \ge 0$ is a constant such that $\pm\alpha$ is dominated by $C\omega$ (i.e., $C\omega \pm \alpha$ is nef).

By going to the limit, one gets

Corollary. $\alpha \cdot \langle \alpha^{n-1} \rangle - \langle \alpha^n \rangle = 0.$

Schematic picture of orthogonality estimate

The proof is similar to the case of projecting a point onto a convex set, where the segment to closest point is orthogonal to tangent plane.



Proof of duality between $\mathcal{E}_{\rm NS}$ and $\mathcal{M}_{\rm NS}$

Theorem (Boucksom-Demailly-Păun-Peternell 2004).
For X projective, a class α is in $\mathcal{E}_{\rm NS}$ (pseudoeffective) if and only if it is dual to the cone $\mathcal{M}_{\rm NS}$ of moving curves.

Proof of the theorem. We want to show that $\mathcal{E}_{\rm NS} = \mathcal{M}_{\rm NS}^{\vee}$. By obvious positivity of the integral pairing, one has in any case

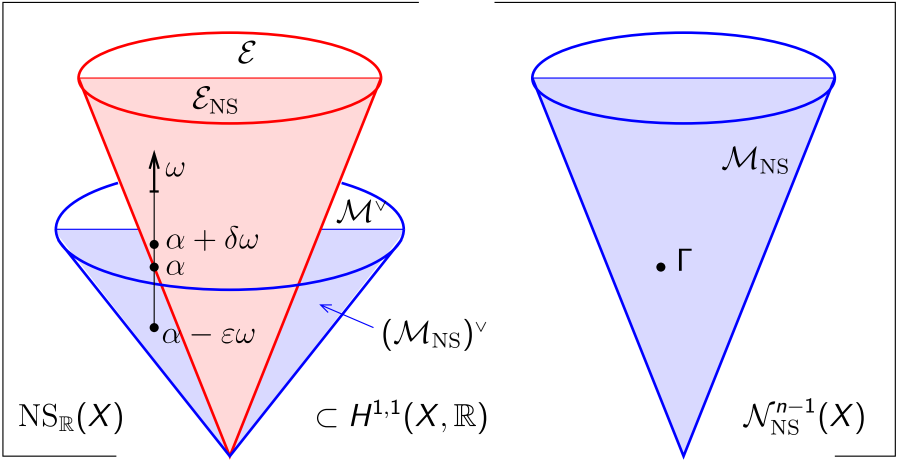
$$\mathcal{E}_{\rm NS} \subset (\mathcal{M}_{\rm NS})^{\vee}.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{\rm NS}$ on the boundary of $\mathcal{E}_{\rm NS}$ which is in the interior of $\mathcal{N}_{\rm NS}^{\vee}$. Hence

$$(*) \qquad \alpha \cdot \Gamma \geq \varepsilon\omega \cdot \Gamma$$

for every moving curve Γ, while $\langle \alpha^n \rangle = \mathrm{Vol}(\alpha) = 0$.

Schematic picture of the proof



Then use approximate Zariski decomposition of $\{\alpha + \delta\omega\}$ and orthogonality relation to contradict $(*)$ with $\Gamma = \langle \alpha^{n-1} \rangle$.

Characterization of uniruled varieties

Recall that a projective variety is called uniruled if it can be covered by a family of rational curves $C_t \simeq \mathbb{P}^1_{\mathbb{C}}$.

Theorem (Boucksom-Demailly-Paun-Peternell 2004)
A projective manifold X is not uniruled if and only if K_X is pseudoeffective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$.

Proof (of the non trivial implication). If $K_X \notin \mathcal{E}_{\mathrm{NS}}$, the duality pairing shows that there is a moving curve C_t such that $K_X \cdot C_t < 0$. The standard "bend-and-break" lemma of Mori then implies that there is family Γ_t of rational curves with $K_X \cdot \Gamma_t < 0$, so X is uniruled.

Plurigenera and the Minimal Model Program

Fundamental question. Prove that every birational class of non uniruled algebraic varieties contains a "minimal" member X with mild singularities, where "minimal" is taken in the sense of avoiding unnecessary blow-ups; minimality actually means that K_X is nef and not just pseudoeffective (pseudoeffectivity is known by the above results).

This requires performing certain birational transforms known as flips, and one would like to know whether
a) flips are indeed possible ("existence of flips"),
b) the process terminates ("termination of flips").
Thanks to Kawamata 1992 and Shokurov (1987, 1996), this has been proved in dimension 3 at the end of the 80's and more recently in dimension 4 (C. Hacon and J. McKernan also introduced in 2005 a new induction procedure).

Finiteness of the canonical ring

Basic questions.

- *Finiteness of the canonical ring:*
  Is the canonical ring $R = \bigoplus H^0(X, mK_X)$ of a variety of general type always finitely generated ?

  If true, Proj(R) of this graded ring R yields of course a "canonical model" in the birational class of X.
- *Boundedness of pluricanonical embeddings:*
  Is there a bound r_n depending only on dimension $\dim X = n$, *such that the pluricanonical map Φ_{mK_X} of a variety of general type yields a birational embedding in projective space for $m \geq r_n$?*
- *Invariance of plurigenera:*
  Are plurigenera $p_m = h^0(X, mK_X)$ always invariant under deformation ?

Recent results on extension of sections

The following is a very slight extension of results by M. Păun (2005) and B. Claudon (2006), which are themselves based on the ideas of Y.T. Siu 2000 and S. Takayama 2005.

Theorem. *Let* $\pi : \mathcal{X} \to \Delta$ *be a family of projective manifolds over the unit disk, and let* $(L_j, h_j)_{0 \le j \le m-1}$ *be (singular) hermitian line bundles with semipositive curvature currents* $i\Theta_{L_j,h_j} \ge 0$ *on* $\mathcal{X}$. *Assume that*

- *the restriction of* h_j *to the central fiber* X_0 *is well defined (i.e. not identically* $+\infty$*).*
- *additionally the multiplier ideal sheaf* $\mathcal{I}(h_{j|X_0})$ *is trivial for* $1 \le j \le m-1$.

Then any section σ *of* $\mathcal{O}(mK_{\mathcal{X}} + \sum L_j)_{|X_0} \otimes \mathcal{I}(h_{0|X_0})$ *over the central fiber* X_0 *extends to* $\mathcal{X}$.

Proof / invariance of plurigenera

The proof relies on a clever iteration procedure based on the Ohsawa-Takegoshi L^2 extension theorem, and a convergence process of an analytic nature (no algebraic proof at present !)

The special case of the theorem obtained by taking all bundles L_j trivial tells us in particular that any pluricanonical section σ of $mK_{\mathcal{X}}$ over X_0 extends to $\mathcal{X}$. By the upper semi-continuity of $t \mapsto h^0(X_t, mK_{X_t})$, this implies

Corollary (Siu 2000). *For any projective family $t \mapsto X_t$ of algebraic varieties, the plurigenera $p_m(X_t) = h^0(X_t, mK_{X_t})$ do not depend on t.*