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On Bergman bundles and their curvature properties

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Curvature tensor of a holomorphic vector bundle

Let X be a complex manifold, $n = \dim_{\mathbb{C}} X$, and (E, h) a holomorphic vector bundle of rank r equipped with a hermitian metric h. With respect to a local holomorphic frame $(e_\lambda)_{1\le\lambda\le r}$

$$\langle u, v\rangle = \sum h_{\lambda\mu}(z) u_\lambda \overline{v}_\mu, \quad u, v \in E_z.$$

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The Chern curvature tensor of (E, h) is defined to be the global $(1,1)$-form $\Theta_{E,h} \in C^\infty(X, \Lambda^{1,1} T^*_X \otimes \mathrm{End}(E))$

$$\Theta_{E,h} = \frac{i}{2\pi} \sum_{1\le j,k\le n,\, 1\le\lambda,\mu\le r} c_{jk\lambda\mu}(z)\, dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu$$

locally computed as the matrix $-\frac{i}{2\pi}\overline{\partial}(\overline{H}^{-1}\partial\overline{H})$ where $H = (h_{\lambda\mu})$.

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locally computed as the matrix $-\frac{i}{2\pi}\overline{\partial}(\overline{H}^{-1}\partial\overline{H})$ where $H = (h_{\lambda\mu})$. One has an associated hermitian form

$$\widetilde{\Theta}_{E,h}(\tau) = \sum c_{jk\lambda\mu}(z)\, \tau_{j\lambda}\overline{\tau}_{k\mu}, \quad \tau \in T_X \otimes E,$$

and one says that $\Theta_{E,h} > 0$ (in the sense of Nakano) if $\widetilde{\Theta}_{E,h}(\tau) > 0$ for all nonzero tensors $\tau \in T_X \otimes E$.

Kodaira embedding theorem

The special case of a holomorphic hermitian line bundle (L, h) is very interesting. Then one usually write the hermitian metric as $h = e^{-\varphi}$ locally on a trivializing open set $U \subset X$, so that

$$\Theta_{L,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi = \frac{i}{2\pi}\sum_{j,k}\frac{\partial^2\varphi}{\partial z_j\overline{\partial}z_k}\,dz_j \wedge \overline{z}_k,$$

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Theorem (Kodaira 1953 - main reason for his Fields medal!)

For X a compact complex manifold, TFAE :

(i) $L > 0$, i.e. L possesses a smooth hermitian metric s.t. $\Theta_{L,h} > 0$;

(ii) L is ample, i.e. there exists a tensor power $L^{\otimes m}$ and sections $\sigma_0, \ldots, \sigma_N \in H^0(X, L^{\otimes m})$ such that

$X \to \mathbb{P}^N, \quad x \mapsto [\sigma_0(x) : \sigma_1(x) : \ldots : \sigma_N(x)] \in \mathbb{P}^N$ is an embedding.

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Then X is in fact an algebraic submanifold $\{P_1 = \ldots = P_q = 0\}$ of $\mathbb{P}^N$, and one says that X is a projective algebraic manifold.

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Every compact complex manifold X carries a "very ample" complex Hilbert bundle, produced by means of a natural Bergman space construction; the curvature of this bundle is strongly positive and is given by a universal formula.

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Our goal is to investigate further this construction and explain potential applications to analytic geometry (Kähler invariance of plurigenera, transcendental holomorphic Morse inequalities...)

Tubular Stein neighborhoods

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$.

Denote by $\overline{X}$ its complex conjugate $(X, -J)$, so that $\mathcal{O}_{\overline{X}} = \overline{\mathcal{O}_X}$.

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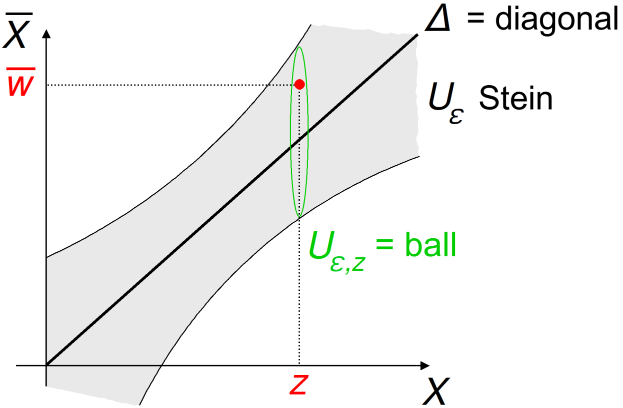

Tubular Stein neighborhoods (continued)

In the special case $X = \mathbb{C}^n$, $U_\varepsilon = \{(z,w)\,;\ |\overline{z} - w| < \varepsilon\}$ is of course Stein since

$$|\overline{z} - w|^2 = |z|^2 + |w|^2 - 2\,\mathrm{Re}\sum z_j w_j$$

and $(z,w) \mapsto \mathrm{Re}\sum z_j w_j$ is pluriharmonic.

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Technical lemma

Let $\exp : T_X \to X \times X$, $(z,\xi) \mapsto (z, \exp_z(\xi))$ be the exponential map associated with a real analytic hermitian metric γ on X, and exph its "holomorphic" part, so that

$$\exp_z(\xi) = \sum_{\alpha,\beta\in\mathbb{N}^n} a_{\alpha\,\beta}(z)\xi^\alpha\overline{\xi}^\beta, \quad \mathrm{exph}_z(\xi) = \sum_{\alpha\in\mathbb{N}^n} a_{\alpha\,0}(z)\xi^\alpha.$$

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Let $\mathrm{logh} : X \times X \supset W \to T_X$ be the inverse of exph and

$$U_\varepsilon = \{(z,w) \in X \times \overline{X}\,;\ |\,\mathrm{logh}_z(w)|_\gamma < \varepsilon\}, \quad \varepsilon > 0.$$

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$$U_\varepsilon = \{(z,w) \in X \times \overline{X}\,;\ |\mathrm{logh}_z(w)|_\gamma < \varepsilon\}, \quad \varepsilon > 0.$$

Then, for $\varepsilon \ll 1$, U_ε is Stein and $\mathrm{pr}_1 : U_\varepsilon \to X$ is a real analytic locally trivial bundle with fibers biholomorphic to complex balls.

Bergman sheaves

Let $U_\varepsilon = U_{\gamma,\varepsilon} \subset X \times \overline{X}$ be the ball bundle as above, and

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X, \qquad \overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to \overline{X}$$

the natural projections.

Definition

The "Bergman sheaf" $\mathcal{B}_\varepsilon = \mathcal{B}_{\gamma,\varepsilon}$ is the L^2 direct image

$$\mathcal{B}_\varepsilon = p_*^{L^2}(\overline{p}^*\mathcal{O}(K_{\overline{X}})),$$

i.e. the space of sections over an open subset $V \subset X$ defined by $\mathcal{B}_\varepsilon(V) =$ holomorphic sections f of $\overline{p}^*\mathcal{O}(K_{\overline{X}})$ on $p^{-1}(V)$,

$$f(z,w) = f_1(z,w)\, dw_1 \wedge \ldots \wedge dw_n, \quad z \in V,$$

that are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$:

$$\int_{p^{-1}(K)} i^{n^2} f \wedge \overline{f} \wedge \gamma^n < +\infty, \quad \forall K \Subset V.$$

Then $\mathcal{B}_\varepsilon$ is an $\mathcal{O}_X$-module, and by the Ohsawa-Takegoshi extension theorem applied to the subvariety $p^{-1}(z) \subset U_\varepsilon$, its fiber $B_{\varepsilon,z} = \mathcal{B}_{\varepsilon,z}/\mathfrak{m}_z\mathcal{B}_{\varepsilon,z}$ is isomorphic to the Hardy-Bergman space $\mathcal{H}^2(B(0,\varepsilon))$ of L^2 holomorphic n-forms on $p^{-1}(z) \simeq B(0,\varepsilon) \subset \mathbb{C}^n$.

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For this, consider the "Bergman Dolbeault" complex $\overline{\partial} : \mathcal{F}^q_\varepsilon \to \mathcal{F}^{q+1}_\varepsilon$ over X, with $\mathcal{F}^q_\varepsilon(V) =$ smooth (n,q)-forms

$$f(z,w) = \sum_{|J|=q} f_J(z,w)\, dw_1 \wedge \ldots \wedge dw_n \wedge d\overline{z}_J, \quad (z,w) \in U_\varepsilon \cap (V \times \overline{X}),$$

$f_J(z,w)$ holomorphic in w and all $\overline{\partial}_z f(z,w) \in L^2(p^{-1}(K))$, $K \Subset V$.

Very ampleness of Bergman bundles

By construction, $\overline{\partial}$ yields a complex of sheaves $(\mathcal{F}^\bullet, \overline{\partial}\,)$ and the kernel $\operatorname{Ker}\overline{\partial} : \mathcal{F}^0 \to \mathcal{F}^1$ coincides with $\mathcal{B}_\varepsilon$.

Theorem

Assume that $\varepsilon > 0$ is taken so small that $\psi(z,w) := |\log h_z(w)|^2$ is strictly plurisubharmonic up to the boundary on the compact set $\overline{U}_\varepsilon \subset X \times \overline{X}$. Then the complex of sheaves $(\mathcal{F}^\bullet, \overline{\partial})$ is a resolution of $\mathcal{B}_\varepsilon$ by soft sheaves over X (actually, by $\mathcal{C}^\infty_X$-modules), and for every holomorphic vector bundle $E \to X$ we have

$$H^q(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) = H^q\big(\Gamma(X, \mathcal{F}^\bullet_\varepsilon \otimes \mathcal{O}(E)), \overline{\partial}\big) = 0, \quad \forall q \geq 1.$$

Moreover the fibers $B_{\varepsilon,z} \otimes E_z$ are always generated by global sections of $H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$.

In other words, B_ε is a "very ample holomorphic vector bundle" (as a Hilbert bundle of infinite dimension).

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In other words, B_ε is a "very ample holomorphic vector bundle" (as a Hilbert bundle of infinite dimension).
But it is NOT holomorphically locally trivial.

Chern connection of Bergman bundles

Since we have a natural $\nabla^{0,1} = \overline{\partial}$ connection, and a natural hermitian metric on the Bergman bundle, it follows that B_ε can be equipped with a unique Chern connection.

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Then one sees that a (non holomorphic) orthonormal frame of B_ε is given by

$$e_\alpha(z,w) = \pi^{-n/2}\varepsilon^{-|\alpha|-n}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\,(w-\overline{z})^\alpha, \quad \alpha \in \mathbb{N}^n.$$

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The $(0,1)$-connection $\nabla^{0,1} = \overline{\partial}$ is given by

$$\nabla^{0,1}e_\alpha = \overline{\partial}_z e_\alpha(z,w) = \varepsilon^{-1}\sum_{1\le j\le n}\sqrt{\alpha_j(|\alpha|+n)}\;d\overline{z}_j\otimes e_{\alpha-c_j}$$

where $c_j = (0,\ldots,1,\ldots,0)\in\mathbb{N}^n$.

Curvature of Bergman bundles

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$$\widetilde{\Theta}_\varepsilon(v\otimes\xi) = \langle \Theta_{B_\varepsilon,h}\sigma(v,Jv)\xi,\xi\rangle_h$$

the associated quadratic form with $v \in T_X$, $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_\varepsilon$.

Formula

In the model case $X = \mathbb{C}^n$, the curvature tensor of the Bergman bundle (B_ε, h) is given by

$$\widetilde{\Theta}_\varepsilon(v\otimes\xi) = \varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\left(\Big|\sum_j \sqrt{\alpha_j}\;\xi_{\alpha-c_j}v_j\Big|^2 + \sum_j(|\alpha|+n)\;|\xi_\alpha|^2|v_j|^2\right).$$

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Observe that $\widetilde{\Theta}_\varepsilon(v \otimes \xi)$ is a positive but unbounded quadratic form on B_ε with respect to the standard norm $\|\xi\|^2 = \sum_\alpha |\xi_\alpha|^2$.

However there is convergence for all $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_{\rho\varepsilon}$, $\rho > 1$, since then $\sum_\alpha \rho^{2|\alpha|}|\xi_\alpha|^2 < +\infty$.

Curvature of Bergman bundles (general case)

Bergman curvature formula on a general hermitian manifold

Let X be a compact complex manifold equipped with a hermitian metric γ, and $B_\varepsilon = B_{\gamma,\varepsilon}$ the corresponding Bergman bundle. Then its curvature is given by an asymptotic expansion

$$\widetilde{\Theta}_\varepsilon(z, v \otimes \xi) = \sum_{p=0}^{+\infty} \varepsilon^{-2+p} Q_p(z, v \otimes \xi)$$

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where $Q_0(z, v \otimes \xi) = Q_0(v \otimes \xi)$ is given by the model case $\mathbb{C}^n$:

$$Q_0(v \otimes \xi) = \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \left(\Big| \sum_j \sqrt{\alpha_j}\, \xi_{\alpha - c_j} v_j \Big|^2 + \sum_j (|\alpha| + n)\, |\xi_\alpha|^2 |v_j|^2 \right).$$

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where $Q_0(z, v\otimes\xi) = Q_0(v\otimes\xi)$ is given by the model case $\mathbb{C}^n$:

$$Q_0(v\otimes\xi) = \varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\Bigg(\Big|\sum_j \sqrt{\alpha_j}\,\xi_{\alpha-c_j}v_j\Big|^2 + \sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2\Bigg).$$

The other terms $Q_p(z, v\otimes\xi)$ are real analytic and depend on the torsion and curvature tensor of γ, especially Q_1, Q_2.

Curvature of Bergman bundles (general case)

Bergman curvature formula on a general hermitian manifold

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A consequence of the above formula is that B_ε is strongly Nakano positive for $\varepsilon > 0$ small enough.

Invariance of plurigenera for polarized families of compact Kähler manifolds

Conjecture

Let $\pi : \mathcal{X} \to S$ be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base S. Assume that the family admits a polarization, i.e. a closed smooth $(1,1)$-form ω such that $\omega_{|X_t}$ is positive definite on each fiber $X_t := \pi^{-1}(t)$. Then the plurigenera

$$p_m(X_t) = h^0(X_t, mK_{X_t}) \text{ are independent of } t \text{ for all } m \geq 0.$$

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No algebraic proof is known in the latter case; one uses deeply the Ohsawa-Takegoshi L^2 extension theorem.

Invariance of plurigenera: strategy of proof (1)

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- $|s|_h^2 \leq 1$, i.e. $\varphi \geq \log|s|$ on X_0.

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To produce $h = e^{-\varphi}$, one defines inductively sections of $\sigma_{p,j}$ of $\mathcal{L}_p := \mathcal{A} + pK_{\mathcal{X}}$ such that:

- $(\sigma_{p,j})$ generates $\mathcal{L}_p$ for $0 \leq p < m$
- $\sigma_{p,j}$ extends $(\sigma_{p-m,j}s^m)_{|X_0}$ to $\mathcal{X}$ for $p \geq m$
- $\displaystyle\int_{\mathcal{X}} \frac{\sum_j |\sigma_{p,j}|^2}{\sum_j |\sigma_{p-1,j}|^2} \leq C$ for $p \geq 1$.

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By Hölder, the L^2 estimates imply $\int_{\mathcal{X}} \left(\sum_j |\sigma_{p,j}|^2\right)^{1/p} \le C$ for all p, and using the fact that $\lim \frac{1}{p}\Theta_{\mathcal{A}} = 0$, one can take

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Idea. In the polarized Kähler case, use the Bergman bundle $B_\varepsilon \to \mathcal{X}$ instead of an ample line bundle $\mathcal{A} \to \mathcal{X}$. This amounts to applying the Ohsawa-Takegoshi L^2 extension on Stein tubular neighborhoods $U_\varepsilon \subset \mathcal{X} \times \overline{\mathcal{X}}$, with projections $\mathrm{pr}_1 : U_\varepsilon \to \mathcal{X}$ and $\pi : \mathcal{X} \to \Delta$.

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Proposition

In the polarized Kähler case $(\mathcal{X}, \omega)$, shrinking from $U_{\rho\varepsilon}$, $\rho > 1$, to U_ε, one gets

$$i\partial\overline{\partial}\Big(\sum_j \|\sigma_{p,j}\|^2_{U_\varepsilon}\Big)^{\lambda/p} \geq -\varepsilon^{-2}(\log\rho)^{-1}\rho^{n\lambda/p}e^{C\lambda}\omega \quad \forall\lambda > 0.$$

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This is enough to imply the invariance of plurigenera if $\varepsilon > 0$ can be taken arbitrarily large.

Transcendental holomorphic Morse inequalities

Conjecture

Let X be a compact n-dimensional complex manifold and $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ a Bott-Chern class, represented by closed real $(1,1)$-forms modulo $\partial\overline{\partial}$ exact forms. Set

$$\mathrm{Vol}(\alpha) = \sup_{T=\alpha+i\partial\overline{\partial}\varphi\geq 0} \int_X T_{ac}^n, \quad T \geq 0 \text{ current.}$$

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Conjectural corollary (fundamental volume estimate)

Let X be compact Kähler, $\dim X = n$, and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be nef classes. Then

$$\mathrm{Vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

Transcendental Morse: known facts & beyond

The conjecture on Morse inequalities is known to be true when $\alpha = c_1(L)$ is the class of a line bundle ([D-1985]), and the corollary can be derived from this when α, β are integral classes (by [D-1993] and independently by [Trapani, 1993]).

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Idea. In the general case, one can find a sequence of non holomorphic hermitian line bundles (L_m, h_m) such that

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Thank you for your attention