A SHARP LOWER BOUND FOR THE LOG CANONICAL THRESHOLD

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May 11, 2012

MOTIVATION

PROBLEM

Let f be a holomorphic function on a manifold X. We want to understand the singularities of f at a given point.

- The two classical invariants of the singularity of f at a point $p \in H = \{f = 0\}$ are the *multiplicity* $\mathrm{ord}_p(f)$ and the *Milnor number* $\mu_p(f)$.
- In this seminar we are interested in the log canonical threshold, an invariant that can be thought of as a refinement of the reciprocal of the multiplicity.

The log canonical threshold of holomorphic functions

Definition

Let f be a holomorphic function in a neighborhood of a point $p \in \mathbb{C}^n$. The log canonical threshold or *complex singularity exponent* of f at p is the number $c_p(f)$ such that

- $|f|^{-s}$ is L^2 in a neighborhood of p for $s < c_p(f)$, and
- $|f|^{-s}$ is not L^2 in any neighborhood of p for $s > c_p(f)$.

Remark: We have $c_p(f) \in (0, 1]$.

Some historical facts

- The log canonical threshold appears implicitly in the works of Schwartz, Hörmander, Łojasiewicz and Gel′fand as the "division problem for distributions"(See e.g. Schwartz, Hörmander, Łojasiewicz and Gel′fand).
- Atiyah and Bernšteĭn made more extensive use of this invariant.
- Conjecture ([D-Kollár]): $\mathcal{C} = \{c_0(f) : f \in \mathcal{O}_{\mathbb{C}^n,0}\} \subset \mathbb{R}$ satisfies the ascending chain condition: any convergent increasing sequence in $\mathbb{C}$ should be stationary. It was proved by Phong and Sturm in complex dimension 2 ([Phong]). Recently, it was proved by T. Fernex, L. Ein and M. Mustaţa in higher dimension ([DEM10]).

Facts

- The negative of the log canonical threshold is the largest root of the Bernšteĭn-Sato polynomial associated with $|f|^{2s}$.
- Tian's α-invariant is an asymptotic version of the log canonical threshold that provides a criterion for the existence of Kähler-Einstein metrics.
- The log canonical threshold appears in many applications of vanishing theorems, due to its relation to multiplier ideals. An important example is the work of Angehrn and Siu on the global generation of adjoint line bundles.
- Lower bounds for the log canonical threshold has also received applications to birational geometry in recent years ([DEM10]).

The log canonical threshold for plurisubharmonic functions

In order to be able to use pluripotential theory we extend the definition of the log canonical threshold to plurisubharmonic functions.

Definition ([D-Kollár])

Let X be a complex manifold, $p \in X$, and φ be a plurisubharmonic function defined on X. The log canonical threshold or *complex singularity exponent* of φ at p is defined by

$$c_p(\varphi) = \sup\left\{c \geq 0 : e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } p\right\},$$

Notations and basic facts

- A domain $\Omega \subset \mathbb{C}^n$ is called *hyperconvex* if there exists $\psi \in \mathcal{PSH}(\Omega)$, $\psi \leq 0$, such that $\{z : \psi(z) < c\} \Subset \Omega$ for all $c < 0$.
- $\mathcal{E}(X)$ is the largest subclass of plurisubharmonic functions defined on a complex manifold X for which the complex Monge-Ampère operator is well-defined.
- $$\mathcal{E}_0(\Omega) = \left\{ \varphi \in \mathcal{PSH} \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0, \int_\Omega (dd^c\varphi)^n < +\infty \right\}.$$
- $$\mathcal{F}(\Omega) = \left\{ \varphi \in \mathcal{PSH}(\Omega) : \exists\, \mathcal{E}_0(\Omega) \ni \varphi_p \searrow \varphi, \text{ and } \sup_{p\geq 1} \int_\Omega (dd^c\varphi_p)^n < +\infty \right\}.$$

The main result

Main Theorem (Demailly & Phạm)

Let $\varphi \in \mathcal{E}(\Omega)$ and $p \in \Omega$. If $e_1(\varphi) = 0$, then $c_p(\varphi) = \infty$. *Otherwise, we have that*

$$c_p(\varphi) \geq \sum_{j=0}^{n-1} \frac{e_j(\varphi)}{e_{j+1}(\varphi)},$$

where

$$e_j(\varphi) = \int_{\{p\}} (dd^c\varphi)^j \wedge (dd^c \log\|z\|)^{n-j}.$$

Remark: The above theorem is optimal.

Remark: To simplify the notions we shall in the rest of this talk assume that $p = 0$.

LEMMA 1

LEMMA 1

Let $\varphi \in \mathcal{E}(\Omega)$ and $0 \in \Omega$. Then we have that

$$e_j(\varphi)^2 \leq e_{j-1}(\varphi)e_{j+1}(\varphi),$$

for all $j = 1, \dots, n-1$.

PROOF OF LEMMA 1

Without loss generality we can assume that Ω is the unit ball and $\varphi \in \mathcal{E}_0(\Omega)$. For h, $\psi \in \mathcal{E}_0(\Omega)$ we have by integration by parts, and the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\left[\int_\Omega -h(dd^c\varphi)^j \wedge (dd^c\psi)^{n-j}\right]^2 \\
&\quad = \left[\int_\Omega d\varphi \wedge d^c\psi \wedge (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j-1} \wedge dd^c h\right]^2 \\
&\quad \leq \int_\Omega d\psi \wedge d^c\psi \wedge (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j-1} \wedge dd^c h \\
&\qquad \int_\Omega d\varphi \wedge d^c\varphi \wedge (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j-1} \wedge dd^c h \\
&= \int_\Omega -h(dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j+1} \int_\Omega -h(dd^c\varphi)^{j+1} \wedge (dd^c\psi)^{n-j-1},
\end{aligned}$$

PROOF OF LEMMA 1, CONTINUED

Now, as $p \to +\infty$, take

$$h(z) = h_p(z) = \max\Big(-1, \frac{1}{p}\log\|z\|\Big) \nearrow \begin{cases} 0 & \text{if } z \in \Omega\backslash\{0\} \\ -1 & \text{if } z = 0. \end{cases}$$

By the monotone convergence theorem we get in the limit that

$$\left[\int_{\{0\}} (dd^c\varphi)^j \wedge (dd^c\psi)^{n-j}\right]^2 \leq \int_{\{0\}} (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j+1} \int_{\{0\}} (dd^c\varphi)^{j+1} \wedge (dd^c\psi)^{n-j-1}.$$

For $\psi(z) = \ln\|z\|$, this is the desired estimate. □

LEMMA 2

LEMMA 2

Let $\varphi, \psi \in \mathcal{E}(\Omega)$ be such that $\varphi \leq \psi$ (i.e φ is "more singular" than ψ). Then we have that

$$\sum_{j=0}^{n-1} \frac{e_j(\varphi)}{e_{j+1}(\varphi)} \leq \sum_{j=0}^{n-1} \frac{e_j(\psi)}{e_{j+1}(\psi)}.$$

Proof of Lemma 2

Set

$$D = \{t = (t_1, \dots, t_n) \in [0, +\infty)^n : t_1^2 \leq t_2, t_j^2 \leq t_{j-1}t_{j+1},\ \forall j = 2, \dots, n-1\}.$$

Then D is a convex set in $\mathbb{R}^n$, as can be checked by a straightforward application of the Cauchy-Schwarz inequality. Next, consider the function $f : \operatorname{int} D \to [0, +\infty)$ defined by

$$f(t_1, \dots, t_n) = \frac{1}{t_1} + \frac{t_1}{t_2} \dots + \frac{t_{n-1}}{t_n}.$$

We have have that

$$\frac{\partial f}{\partial t_j}(t) = -\frac{t_{j-1}}{t_j^2} + \frac{1}{t_{j+1}} \leq 0, \qquad \forall t \in D.$$

Proof of Lemma 2, continued

For $a, b \in \operatorname{int} D$ such that $a_j \geq b_j, j = 1, \dots, n$, the function

$$[0, 1] \ni \lambda \to f(b + \lambda(a - b))$$

is decreasing. Hence,

$$f(a) \leq f(b) \qquad \text{for all } a, b \in \operatorname{int} D,\ a_j \geq b_j,\ j = 1, \dots, n.$$

On the other hand, the hypothesis $\varphi \leq \psi$ implies that $e_j(\varphi) \geq e_j(\psi)$, $j = 1, \dots, n$, by the comparison principle. Therefore we have that

$$f(e_1(\varphi), \dots, e_n(\varphi)) \leq f(e_1(\psi), \dots, e_n(\psi)).$$

□

It will be convenient here to introduce Kiselman's refined Lelong number.

DEFINITION

Let $\varphi \in \mathcal{PSH}(\Omega)$. Then the function defined by

$$\nu_\varphi(x) = \lim_{t \to -\infty} \frac{\max\left\{\varphi(z) : |z_1| = e^{x_1 t}, \dots, |z_n| = e^{x_n t}\right\}}{t}$$

is called the refined Lelong number of φ at 0.

The refined Lelong number of φ at 0 is increasing in each variable x_j, and concave on $\mathbb{R}^n{}_+$.

The proof is divided into the following steps:

- Proof of the theorem in the toric case, i.e. $\varphi(z_1, \dots, z_n) = \varphi(|z_1|, \dots, |z_n|)$ depends only on $|z_j|$ and therefore we can without loss of generality assume that $\Omega = \Delta^n$ is the unit polydisk.

- Reduction to the case of plurisubharmonic functions with analytic singularity, i.e. $\varphi = \log(|f_1|^2 + \dots + |f_N|^2)$, where $f_1, \dots, f_N$ are germs of holomorphic functions at 0.

- Reduction to the case of monomial ideals, i.e. for $\varphi = \log(|f_1|^2 + \dots. + |f_N|^2)$, where $f_1, \dots, f_N$ are germs of monomial elements at 0.

Proof of the theorem in the toric case

Set

$$\Sigma = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_j = 1 \right\} .$$

We choose $x^0 = (x_1^0, \dots, x_n^0) \in \Sigma$ such that

$$\nu_\varphi(x^0) = \max\{\nu_\varphi(x) : x \in S\}.$$

By Theorem 5.8 in [Kis94] we have the following formula

$$c(\varphi) = \frac{1}{\nu_\varphi(x^0)} .$$

Proof of the theorem in the toric case, continued

Set

$$\zeta(x) = \nu_\varphi(x^0) \min\left(\frac{x_1}{x_1^0}, \dots, \frac{x_n}{x_n^0} \right), \qquad \forall x \in \Sigma .$$

Then ζ is the smallest nonnegative concave increasing function on Σ such that $\zeta(x^0) = \nu_\varphi(x^0)$, hence $\zeta \leq \nu_\varphi$. This implies that

$$\begin{aligned} \varphi(z_1, \dots, z_n) &\leq -\nu_\varphi(-\ln|z_1|, \dots, -\ln|z_n|) \\ &\leq -\zeta(-\ln|z_1|, \dots, -\ln|z_n|) \\ &\leq \nu_\varphi(x^0) \max\left(\frac{\ln|z_1|}{x_1^0}, \dots, \frac{\ln|z_n|}{x_n^0} \right) := \psi(z_1, \dots, z_n). \end{aligned}$$

By Lemma 2 we get that

$$f(e_1(\varphi), \dots, e_n(\varphi)) \leq f(e_1(\psi), \dots, e_n(\psi)) = c_p(\psi) = \frac{1}{\nu_\varphi(x^0)} = c_p(\varphi) .$$

REDUCTION TO THE CASE OF PLURISUBHARMONIC FUNCTIONS WITH ANALYTIC SINGULARITY

Let $\mathcal{H}_{m\varphi}(\Omega)$ be the Hilbert space of holomorphic functions f on Ω such that

$$\int_{\Omega} |f|^2 e^{-2m\varphi} dV < +\infty ,$$

and let $\psi_m = \frac{1}{2m} \log \sum |g_{m,k}|^2$ where $\{g_{m,k}\}_{k\geq 1}$ be an orthonormal basis for $\mathcal{H}_{m\varphi}(\Omega)$. Using $\bar{\partial}$-equation with L^2-estimates (see Theorem 4.2 in [D-Kollár]), there are constants $C_1, C_2 > 0$ independent of m such that

$$\varphi(z) - \frac{C_1}{m} \leq \psi_m(z) \leq \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}$$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$

REDUCTION TO THE CASE OF PLURISUBHARMONIC FUNCTIONS WITH ANALYTIC SINGULARITY, CONTINUED

and

$$\nu(\varphi) - \frac{n}{m} \leq \nu(\psi_m) \leq \nu(\varphi), \qquad \frac{1}{c_p(\varphi)} - \frac{1}{m} \leq \frac{1}{c_p(\psi_m)} \leq \frac{1}{c_p(\varphi)}.$$

By Lemma 2, we have that

$$f(e_1(\varphi), \ldots, e_n(\varphi)) \leq f(e_1(\psi_m), \ldots, e_n(\psi_m)), \qquad \forall m \geq 1.$$

The above inequalities show that in order to prove the lower bound of $c_p(\varphi)$ in the Main Theorem, we only need prove it for $c_p(\psi_m)$ and then let $m \to \infty$.

Reduction to the case of monomial ideals

For $j = 0, \dots, n$ set

$$\mathcal{J} = (f_1, \dots, f_N),\ c_p(\mathcal{J}) = c_p(\varphi),\ \text{and } e_j(\mathcal{J}) = e_j(\varphi)\,.$$

Now, by fixing a multiplicative order on the monomials

$$z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$$

(see [Eis95] Chap. 15 and [DEM04]). It is well known that one can construct a flat family $(\mathcal{J}_s)_{s\in\mathbb{C}}$ of ideals of $\mathcal{O}_{\mathcal{C}^n,0}$ depending on a complex parameter $s \in \mathbb{C}$, such that $\mathcal{J}_0$ is a monomial ideal, $\mathcal{J}_1 = \mathcal{J}$ and

$$\dim(\mathcal{O}_{\mathcal{C}^n,0}/\mathcal{J}_s^t) = \dim(\mathcal{O}_{\mathcal{C}^n,0}/\mathcal{J}^t) \text{ for all } s, t \in \mathbb{N}\,.$$

Reduction to the case of monomial ideals, continued

In fact $\mathcal{J}_0$ is just the initial ideal associated to $\mathcal{J}$ with respect to the monomial order. Moreover, we can arrange by a generic rotation of coordinates $\mathbb{C}^p \subset \mathbb{C}^n$ that the family of ideals $\mathcal{J}_s \cap \mathcal{O}_{\mathbb{C}^p,0}$ is also flat, and that the dimensions

$$\dim\left(\mathcal{O}_{b^p,0}/(\mathcal{J}_s \cap \mathcal{O}_{\mathbb{C}^p,0})^t\right) = \dim\left(\mathcal{O}_{\mathbb{C}^p,0}/(\mathcal{J} \cap \mathcal{O}_{\mathbb{C}^p,0})^t\right)$$

compute the intermediate multiplicities

$$e_p(\mathcal{J}_s) = \lim_{t\to+\infty} \frac{p!}{t^p} \dim\left(\mathcal{O}_{\mathbb{C}^p,0}/(\mathcal{J}_s \cap \mathcal{O}_{\mathbb{C}^p,0})^t\right) = e_p(\mathcal{J}),$$

in particular, $e_p(\mathcal{J}_0) = e_p(\mathcal{J})$ for all p.
The semicontinuity property of the log canonical threshold (see for example [D-Kollár]) now implies that $c_p(\mathcal{J}_0) \leq c_p(\mathcal{J}_s) = c_p(\mathcal{J})$ for all s, so the lower bound is valid for $c_p(\mathcal{J})$ if it is valid for $c_p(\mathcal{J}_0)$.

The End

Any questions?

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