A brief survey on Seshadri constants

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Memorial Lectures for
Professor Conjeevaram Srirangachari Seshadri
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The starting point of Seshadri constant theory

The paper that started the whole theory:

C.S. Seshadri, *Annals of Mathematics*, Vol. 95 (May 1972) 511–556

Quotient spaces modulo reductive algebraic groups

By C. S. Seshadri

Introduction

In his book "Geometric invariant theory" [M], Mumford developed a theory of quotient spaces of algebraic schemes acted on by reductive algebraic groups when the ground field is of characteristic zero and showed how this can be used for several questions of moduli. In order to have this theory in arbitrary characteristic, he made the following conjecture:

(A): Let G be a reductive algebraic group (say over an algebraically closed field) and V a finite dimensional rational G-module. Then given a G-invariant point v, $v \neq 0$, there is a G-invariant homogeneous polynomial F on V such that $F(v) \neq 0$.

Seshadri's ampleness criterion

LEMMA 7.2. *Let X be a complete variety and x a point of X at which X is smooth. Let $p\colon X' \to X$ be the* BLOWING UP *of X at the point x (i.e. blowing up of X with respect to the sheaf of ideals defining the reduced subscheme of X consisting of the one point x). Let L be a line bundle on X which is* PSEUDO-AMPLE *i.e. $\forall\, C \hookrightarrow X$ where C is a closed integral curve of X,* $\deg(L/C) \geqq 0$. *Let $Z = p^{-1}(x)$ and E the line bundle on X' defined by the effective divisor Z (so that $-E$ i.e. E^{-1} is relatively ample with respect to p). Suppose that $aL - bE$ is also pseudo-ample for some $a, b \in \mathbf{Z}$, $a, b > 0$. Then if $n = \dim X$, we have*

$$L^{(n)} = L \cdots L \ (n\text{-fold intersection product}) > 0\,.$$

⋮

Remark 7.1. The above lemma can be interpreted (as has been remarked by C.P. Ramanujam) to give a criterion of ampleness as follows: Let X be a complete algebraic scheme. To every closed integral curve C, $C \hookrightarrow X$, define $m(C)$ to be the maximum of the multiplicities at the different points of C. Let L be a line bundle on X. Then L is ample on $X \Leftrightarrow \exists\, \varepsilon > 0$ such that $\forall$ closed integral curve $C \hookrightarrow X$, $\deg(L|_C) \geqq \varepsilon m(C)$.

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$$\varepsilon(L, x) = \sup\{\gamma \geq 0 \,/\, \pi^* L - \gamma E \text{ is nef on } \tilde{X}\}.$$

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Remark. In [D, 1990], over $\mathbb{K} = \mathbb{C}$, the Seshadri constant is related to more analytic invariants. For instance, if L is ample, it can be shown that $\varepsilon(L,x)$ is the supremum of $\gamma \geq 0$ for which L possesses a singular Hermitian metric h with $\Theta_{L,h} \geq 0$, that is smooth on $X \smallsetminus \{x\}$ with a logarithmic pole of Lelong number γ at x.

Relation to the Fujita conjecture

Proposition (D, 1990 – implied by the Kodaira vanishing theorem)

For $L \in \mathrm{Pic}(X)$ define

$$\sigma(L, x) = \limsup_{k \to +\infty} \frac{s(kL, x)}{k}$$

where

$$s(L, x) = \max\{m \,/\, H^0(X, L) \text{ generates } m\text{-jets at } x\}.$$

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Seshadri constants on surfaces

Miranda constructed a sequence of examples of smooth surfaces X_p, ample line bundles L_p on X_p and points $x_p \in X_p$ such that $\lim \varepsilon(L_p, x_p) = 0$, but is is unknown whether one can possibly have

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Improvement by Geng Xu (1995)

If L satisfies $L^2 \geq \frac{1}{3}(4a^2 - 4a + 5)$ and $L \cdot C \geq a$ for some every curve C and some integer $a > 1$, then $\varepsilon(L, x) \geq a$ for all $x \in X$ outside a finite union of curves.

Seshadri constants on surfaces (sequel)

Theorem (A. Steffens, 1998)

If L is ample on a smooth surface X with Picard number $\rho(X) = 1$, then the generic value $\varepsilon(L, x_{\rm very\ general}) \geq \lfloor \sqrt{L^2} \rfloor$, and if L^2 is an integer, there is equality.

Theorem (T. Szemberg, 2008)

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A. Broustet in his PhD thesis (Grenoble, 2006) studied the case of the anticanonical line bundle $L = -K_x$ on Del Pezzo surfaces

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Relation to the Nagata conjecture

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Nagata conjecture (1959), reformulated

Let $x_1, \ldots, x_p$ be p very general points in $\mathbb{P}^2$, $p \geq 9$. Then the multipoint Seshadri constant of $\mathcal{O}(1)$ on $\mathbb{P}^2$ satisfies

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A simple counting argument implies that $\varepsilon(\mathcal{O}(1), x_1, \ldots, x_p) \leq \frac{1}{\sqrt{p}}$, and the main difficulty is to find good configurations of points to get lower bounds. In case $p = q^2$ is a perfect square, a square grid works, hence equality. For $4 < p < 9$, one is in the Del Pezzo case, and the inequality turns out to be strict.

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By a more elaborate use of the EKL argument, M. Nakamaye got in 2004 the improved very generic lower bound

$$\varepsilon(L, x) \geq \frac{3n+1}{3n^2} \quad \left(\text{resp. } \frac{1}{2} \text{ if } n = 3\right).$$

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Theorem (Ein, Küchle, Lazarsfeld, 1995)

Let L be ample on a non singular n-dimensional projective variety X over $\mathbb{C}$. Then $\varepsilon(L,x) \geq 1/n$ at a very general point $x \in X$.

The proof uses a "differentiation argument" of sections of $kL - pE_x$, considered on the universal family, i.e., sections of $k(\mathrm{pr}_1^* L) - pE$ on the blow up of $X \times X$ along the diagonal.
By a more elaborate use of the EKL argument, M. Nakamaye got in 2004 the improved very generic lower bound

$$\varepsilon(L,x) \geq \frac{3n+1}{3n^2} \quad \left(\text{resp. } \frac{1}{2} \text{ if } n=3\right).$$

Question (even for $n = \dim X = 2$!). Is there a lower bound for $\varepsilon(X) = \inf_{L \in \mathrm{Pic}(X)\,\mathrm{ample}} \varepsilon(L)$ depending only on the geometry of X ?

More is known for special classes of varieties . . .

Case of Abelian varieties (Nakamaye, 1996)

Let $X = \mathbb{C}^n/\Lambda$ be an Abelian variety and $L \in \mathrm{Pic}(X)$ be ample. Then $\varepsilon(L, x) = \varepsilon(L) \geq 1$, and the equality occurs if and only if $X \simeq E \times Y$ where $E =$ elliptic curve and $Y =$ Abelian variety of dimension $n - 1$, with $L \equiv \mathrm{pr}_1^*\mathcal{O}_E([p_0]) + \mathrm{pr}_2{}^*A$ and $C = E \times \{y_0\}$.

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$$X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to X_0 = \{\mathrm{point}\},$$

so that each $X_k = \mathbb{P}(\mathcal{O}_{X_{k-1}} \oplus \mathcal{L})$ is a $\mathbb{P}^1$-bundle over X_{k-1}.

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One of the main points is to identify the nef cone of $\mathrm{Pic}(X_n) \simeq \mathbb{Z}^n$.

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Thank you for your attention

Professor C.S. Seshadri in Bangalore (2010)