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A brief survey on Seshadri constants

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The starting point of Seshadri constant theory

The paper that started the whole theory:

C.S. Seshadri, *Annals of Mathematics*, Vol. 95 (May 1972) 511–556

Quotient spaces modulo reductive algebraic groups

By C. S. Seshadri

Introduction

In his book "Geometric invariant theory" [M], Mumford developed a theory of quotient spaces of algebraic schemes acted on by reductive algebraic groups when the ground field is of characteristic zero and showed how this can be used for several questions of moduli. In order to have this theory in arbitrary characteristic, he made the following conjecture:

(A): Let G be a reductive algebraic group (say over an algebraically closed field) and V a finite dimensional rational G-module. Then given a G-invariant point v, $v \neq 0$, there is a G-invariant homogeneous polynomial F on V such that $F(v) \neq 0$.

Seshadri's ampleness criterion

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LEMMA 7.2. *Let X be a complete variety and x a point of X at which X is smooth. Let $p: X' \to X$ be the* BLOWING UP *of X at the point x (i.e. blowing up of X with respect to the sheaf of ideals defining the reduced subscheme of X consisting of the one point x). Let L be a line bundle on X which is* PSEUDO-AMPLE *i.e.* $\forall\, C \hookrightarrow X$ *where C is a closed integral curve of X,* $\deg(L/C) \geqq 0$. *Let $Z = p^{-1}(x)$ and E the line bundle on X' defined by the effective divisor Z (so that $-E$ i.e. E^{-1} is relatively ample with respect to p). Suppose that $aL - bE$ is also pseudo-ample for some $a, b \in \mathbf{Z}$, $a, b > 0$. Then if $n = \dim X$, we have*

$$L^{(n)} = L \cdots L \ (n\text{-fold intersection product}) > 0\,.$$

⋮

Remark 7.1. The above lemma can be interpreted (as has been remarked by C.P. Ramanujam) to give a criterion of ampleness as follows: Let X be a complete algebraic scheme. To every closed integral curve C, $C \hookrightarrow X$, define $m(C)$ to be the maximum of the multiplicities at the different points of C. Let L be a line bundle on X. Then L is ample on $X \Leftrightarrow \exists\, \varepsilon > 0$ such that $\forall$ closed integral curve $C \hookrightarrow X$, $\deg(L|_C) \geqq \varepsilon m(C)$.

Definition of the Seshadri constants

Definition (D, 1990)

Let X be a projective nonsingular variety and L a nef (or pseudo-ample) line bundle over X. Given a point $x \in X$, one defines the Seshadri constant $\varepsilon(L, x)$ of L at x to be

$$\varepsilon(L,x) = \inf_{\text{all alg. curves } C \ni x} \frac{L \cdot C}{\mathrm{mult}_x(C)}.$$

This is a very interesting numerical invariant that measures in a deep manner the "local positivity" of the line bundle L at point x.

Equivalent definition (already observed in Seshadri's paper !)

Let $\pi : \tilde{X} \to X$ be the blow-up of X at $x \in X$, and E the exceptional divisor in $\tilde{X}$. Then, for $L \in \mathrm{Pic}(X)$ assumed to be nef, one has

$$\varepsilon(L,x) = \sup\{\gamma \geq 0 \,/\, \pi^* L - \gamma E \text{ is nef on } \tilde{X}\}.$$

Reformulation of Seshadri's ampleness criterion

Reformulation of Seshadri's ampleness criterion

A nef line bundle $L \in \mathrm{Pic}(X)$ is ample if and only if one has $\varepsilon(L) := \inf_{x\in X} \varepsilon(L,x) > 0$.

A direct consequence of the fact $(\pi^*L - \gamma E)^n = L^n - \gamma^n \geq 0$ is that

$$\varepsilon(L,x) \leq (L^n)^{1/n}, \qquad \forall x \in X.$$

A curve C is said to be submaximal if $\dfrac{L\cdot C}{\mathrm{mult}_x(C)} < (L^n)^{1/n}$.

A large part of the investigations on Seshadri constants, especially in the case of surfaces, rests upon the study of submaximal curves.

Remark. In [D, 1990], over $\mathbb{K} = \mathbb{C}$, the Seshadri constant is related to more analytic invariants. For instance, if L is ample, it can be shown that $\varepsilon(L,x)$ is the supremum of $\gamma \geq 0$ for which L possesses a singular Hermitian metric h with $\Theta_{L,h} \geq 0$, that is smooth on $X \smallsetminus \{x\}$ with a logarithmic pole of Lelong number γ at x.

Relation to the Fujita conjecture

Proposition (D, 1990 – implied by the Kodaira vanishing theorem)

For $L \in \mathrm{Pic}(X)$ define

$$\sigma(L,x) = \limsup_{k\to+\infty} \frac{s(kL,x)}{k}$$

where

$$s(L,x) = \max\{m \,/\, H^0(X,L) \text{ generates } m\text{-jets at } x\}.$$

(1) For L ample, one has

$$\varepsilon(L,x) = \sigma(L,x).$$

(2) For L ample such that $p = \lceil \varepsilon(L,x) \rceil > n = \dim X$,

$$H^0(X, K_X + L) \text{ generates } (p-n-1)\text{-jets at } x.$$

(3) As a consequence, if $\varepsilon(L) \geq 1$, then $K_X + (n+1)L$ is generated by sections, and $K_X + (2n+1)L$ is very ample [the Fujita conj. states that L ample should imply $K_X + (n+2)L$ very ample.]

Seshadri constants on surfaces

Miranda constructed a sequence of examples of smooth surfaces X_p, ample line bundles L_p on X_p and points $x_p \in X_p$ such that $\lim \varepsilon(L_p, x_p) = 0$, but is is unknown whether one can possibly have

$$\varepsilon(X) := \inf_{L \in \mathrm{Pic}(X)\,\mathrm{ample},\ x \in X} \varepsilon(L, x) = 0 \qquad ???$$

However, many results are known for surfaces.

Theorem (Ein-Lazarsfeld,1993)

If L is ample on a smooth surface X, then $\varepsilon(L, x) \geq 1$ except for countably many points $x \in X$, and for finitely many if $L^2 > 1$.

Improvement by Geng Xu (1995)

If L satisfies $L^2 \geq \frac{1}{3}(4a^2 - 4a + 5)$ and $L \cdot C \geq a$ for some every curve C and some integer $a > 1$, then $\varepsilon(L, x) \geq a$ for all $x \in X$ outside a finite union of curves.

Seshadri constants on surfaces (sequel)

Theorem (A. Steffens, 1998)

If L is ample on a smooth surface X with Picard number $\rho(X) = 1$, then the generic value $\varepsilon(L, x_{\text{very general}}) \geq \lfloor \sqrt{L^2} \rfloor$,
and if L^2 is an integer, there is equality.

Theorem (T. Szemberg, 2008)

If L is ample on a smooth surface X with Picard number $\rho(X) = 1$, then

(1) $\forall x \in X,\ \ \varepsilon(L, x) \geq 1$ if X is not of general type.

(2) $\forall x \in X,\ \ \varepsilon(L, x) \geq \frac{1}{1 + (K_X^2)^{1/4}}$ if X is of general type.

Both bounds can be attained (and thus are sharp).

A. Broustet in his PhD thesis (Grenoble, 2006) studied the case of the anticanonical line bundle $L = -K_x$ on Del Pezzo surfaces

Seshadri constants over a subscheme

More general definition (Ein-Lazarsfeld)

Let X be a projective nonsingular variety and Z a (non necessarily reduced) subscheme, $\pi : \tilde{X} \to X$ the blow-up of X with centre Z, and E the exceptional divisor. Then, for $L \in \mathrm{Pic}(X)$ nef, one defines

$$\varepsilon(L, Z) = \sup\{\gamma \geq 0 \,/\, \pi^* L - \gamma E \text{ is nef on } \tilde{X}\}.$$

Remark 1. The case where $Z = \{x_1, \ldots, x_p\}$ is a finite set (or possibly a 0-dimensional scheme) is already very interesting.
One says that $\varepsilon(L, x_1, \ldots, x_p)$ is a multipoint Seshadri constant.

Remark 2. Ein, Lazarsfeld, Mustata, Nakamaye and Popa also extended the concept to pseudoeffective non nef line bundles, by introducing "moving" Seshadri numbers, based on a use of approximate Zariski decomposition of L as a $\mathbb{Q}$-divisor (2006).

Relation to the Nagata conjecture

The concept of Seshadri constant is already highly non trivial on rational surfaces. For instance, the famous Nagata conjecture, has attracted lot of work by Hirschowitz, Harbourne, Biran, Bauer, Szemberg, Dumnicki and others. It can be reformulated :

Nagata conjecture (1959), reformulated

Let $x_1, \ldots, x_p$ be p very general points in $\mathbb{P}^2$, $p \geq 9$. Then the multipoint Seshadri constant of $\mathcal{O}(1)$ on $\mathbb{P}^2$ satisfies

$$\varepsilon(\mathcal{O}(1), x_1, \ldots, x_p) = \frac{1}{\sqrt{p}}.$$

A simple counting argument implies that $\varepsilon(\mathcal{O}(1), x_1, \ldots, x_p) \leq \frac{1}{\sqrt{p}}$, and the main difficulty is to find good configurations of points to get lower bounds. In case $p = q^2$ is a perfect square, a square grid works, hence equality. For $4 < p < 9$, one is in the Del Pezzo case, and the inequality turns out to be strict.

Higher dimensional case

The geometry of curves makes the situation much more involved in the higher dimensional case. A result valid in arbitrary dimension is

Theorem (Ein, Küchle, Lazarsfeld, 1995)

Let L be ample on a non singular n-dimensional projective variety X over $\mathbb{C}$. Then $\varepsilon(L,x) \geq 1/n$ at a very general point $x \in X$.

The proof uses a "differentiation argument" of sections of $kL - pE_x$, considered on the universal family, i.e., sections of $k(\mathrm{pr}_1^* L) - pE$ on the blow up of $X \times X$ along the diagonal.
By a more elaborate use of the EKL argument, M. Nakamaye got in 2004 the improved very generic lower bound

$$\varepsilon(L,x) \geq \frac{3n+1}{3n^2} \quad \Big(\text{resp. } \frac{1}{2} \text{ if } n=3\Big).$$

Question (even for $n = \dim X = 2$!). Is there a lower bound for $\varepsilon(X) = \inf_{L\in \mathrm{Pic}(X)\,\mathrm{ample}} \varepsilon(L)$ depending only on the geometry of X ?

More is known for special classes of varieties . . .

Case of Abelian varieties (Nakamaye, 1996)

Let $X = \mathbb{C}^n/\Lambda$ be an Abelian variety and $L \in \mathrm{Pic}(X)$ be ample. Then $\varepsilon(L,x) = \varepsilon(L) \geq 1$, and the equality occurs if and only if $X \simeq E \times Y$ where E = elliptic curve and Y = Abelian variety of dimension $n-1$, with $L \equiv \mathrm{pr}_1^* \mathcal{O}_E([p_0]) + \mathrm{pr}_2{}^* A$ and $C = E \times \{y_0\}$.

The subject is still very much alive ! Since 2010, there have been 45 arXiv submissions dealing with Seshadri constants. The last one at this date is from June 2020, by I. Biswas, J. Dasgupta, K. Hanumanthu and B. Khan. It gives an estimate of Seshadri constants of nef line bundles on Bott towers, namely a particular class of projective non singular toric varieties of the form

$$X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to X_0 = \{\mathrm{point}\},$$

so that each $X_k = \mathbb{P}(\mathcal{O}_{X_{k-1}} \oplus \mathcal{L})$ is a $\mathbb{P}^1$-bundle over X_{k-1}.
One of the main points is to identify the nef cone of $\mathrm{Pic}(X_n) \simeq \mathbb{Z}^n$.

References

[1] C.S. Seshadri, *Quotient spaces modulo reductive algebraic groups*, Annals of Mathematics, Vol. 95 (May 1972) 511–556.

[2] J.-P. Demailly, *Singular Hermitian metrics on positive line bundles*, Complex algebraic varieties (Bayreuth, 1990), Lect. Notes Math. 1507, Springer-Verlag, 1992, pp. 87–104.

[3] T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 167–178.

[4] L. Ein & R. Lazarsfeld, *Seshadri constants on smooth surfaces*, In Journées de Géométrie Algébrique d'Orsay (Orsay, 1992); Astérisque No. 218 (1993), 177–186.

[5] Geng Xu, *Curves in* $\mathbb{P}^2$ *and symplectic packings*, Math. Ann. 299 (1994), 609–613.

[6] A. Steffens, *Remarks on Seshadri constants*, Math. Z. 227 (1998), 505-510.

[7] T. Szemberg, *An effective and sharp lower bound on Seshadri constants on surfaces with Picard number* 1, J. Algebra 319 (2008) 3112–3119.

[8] L. Ein, R. Lazarsfeld, M. Mustata, M. Nakamaye, M. Popa, *Asymptotic invariants of base loci*, Ann. Inst. Fourier (Grenoble) 56 (2006), 1701–1734.

[9] M. Nagata, Masayoshi, *On the* 14*-th problem of Hilbert*, American Journal of Mathematics, 81 (3): 766–772.

[10] L. Ein, O. Küchle, R. Lazarsfeld, *Local positivity of ample line bundles*, J. Differential Geom. 42 (1995), 193–219.

[11] M. Nakamaye, *Seshadri constants on abelian varieties*, American Journal of Math. 118 (1996), 621–635.

[12] I. Biswas, J. Dasgupta, K. Hanumanthu, B. Khan, *Seshadri constants on Bott towers*, math.AG, arXiv:2006.12723.

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Thank you for your attention

Professor C.S. Seshadri in Bangalore (2010)