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Bergman bundles and transcendental holomorphic Morse inequalities, and invariance of plurigenera for Kähler manifolds

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Curvature tensor of a holomorphic vector bundle

Let X be a complex manifold, $n = \dim_{\mathbb{C}} X$, and (E,h) a holomorphic vector bundle of rank r equipped with a hermitian metric h. With respect to a local holomorphic frame $(e_\lambda)_{1\le\lambda\le r}$

$$\langle u,v\rangle = \sum h_{\lambda\mu}(z)u_\lambda\overline{v}_\mu, \quad u,v\in E_z.$$

The Chern curvature tensor of (E,h) is defined to be the global $(1,1)$-form $\Theta_{E,h}\in C^\infty(X,\Lambda^{1,1}T^*_X\otimes \mathrm{End}(E))$

$$\Theta_{E,h} = \frac{i}{2\pi}\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}(z)\,dz_j\wedge d\overline{z}_k\otimes e^*_\lambda\otimes e_\mu$$

locally computed as the matrix $-\frac{i}{2\pi}\overline{\partial}(\overline{H}^{-1}\partial\overline{H})$ where $H=(h_{\lambda\mu})$.
One has an associated hermitian form

$$\widetilde{\Theta}_{E,h}(\tau) = \sum c_{jk\lambda\mu}(z)\,\tau_{j\lambda}\overline{\tau}_{k\mu}, \quad \tau\in T_X\otimes E,$$

and one says that $\Theta_{E,h}>0$ (in the sense of Nakano) if $\widetilde{\Theta}_{E,h}(\tau)>0$ for all nonzero tensors $\tau\in T_X\otimes E$.

Kodaira embedding theorem

The special case of a holomorphic hermitian line bundle (L, h) is very interesting. Then one usually write the hermitian metric as $h = e^{-\varphi}$ locally on a trivializing open set $U \subset X$, so that

$$\Theta_{L,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi = \frac{i}{2\pi}\sum_{j,k}\frac{\partial^2\varphi}{\partial z_j\overline{\partial}z_k}\,dz_j \wedge \overline{z}_k,$$

and $\Theta_{L,h} > 0$ means that φ is strictly plurisubharmonic.

Theorem (Kodaira 1953 - main reason for his Fields medal!)

For X a compact complex manifold, TFAE :

(i) $L > 0$, i.e. L possesses a smooth hermitian metric s.t. $\Theta_{L,h} > 0$;
(ii) L is ample, i.e. there exists a tensor power $L^{\otimes m}$ and sections $\sigma_0, \ldots, \sigma_N \in H^0(X, L^{\otimes m})$ such that

$X \to \mathbb{P}^N$, $\;x \mapsto [\sigma_0(x) : \sigma_1(x) : \ldots : \sigma_N(x)] \in \mathbb{P}^N$ is an embedding.

Then X is in fact an algebraic submanifold $\{P_1 = \ldots = P_q = 0\}$ of $\mathbb{P}^N$, and one says that X is a projective algebraic manifold.

Projective vs Kähler vs non Kähler varieties

By Kodaira, non projective varieties do not have ample line bundles.

In the Kähler case, a Kähler class $\{\omega\} \in H^{1,1}(X, \mathbb{R})$, $\omega > 0$, may sometimes be used as a substitute for a polarization.

What for non Kähler compact complex manifolds?

Surprising fact (?)

Every compact complex manifold X carries a "very ample" complex Hilbert bundle, produced by means of a natural Bergman space construction; the curvature of this bundle is strongly positive and is given by a universal formula.
In particular, X can be embedded holomorphically in a "Hilbert Grassmannian" of infinite dimension and codimension.

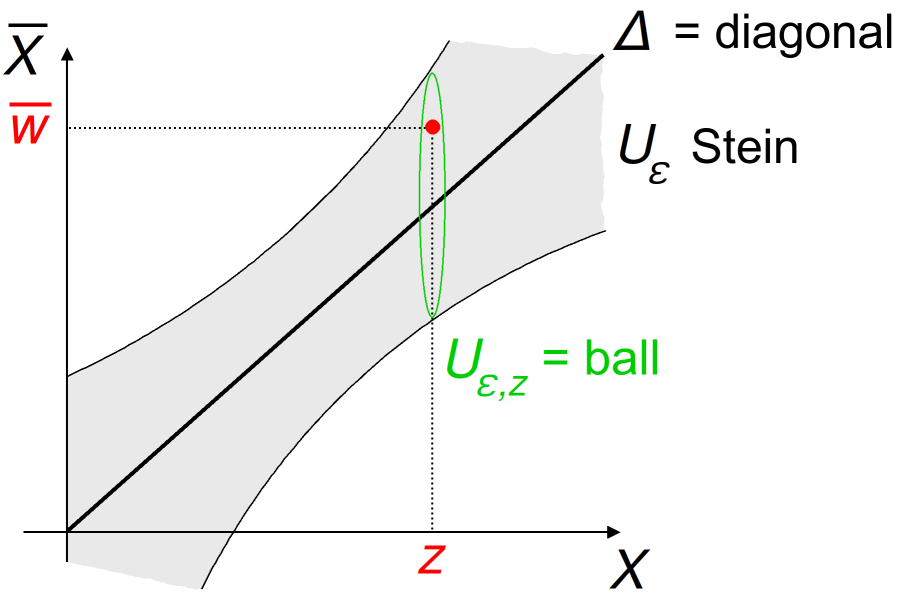
Our goal is to investigate further this construction and explain potential applications to analytic geometry (Kähler invariance of plurigenera, transcendental holomorphic Morse inequalities...)

Tubular Stein neighborhoods

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$.

Denote by $\overline{X}$ its complex conjugate $(X, -J)$, so that $\mathcal{O}_{\overline{X}} = \overline{\mathcal{O}_X}$.

The diagonal of $X \times \overline{X}$ is totally real, and by Grauert, we know that it possesses a fundamental system of Stein tubular neighborhoods.



Tubular Stein neighborhoods (continued)

In the special case $X = \mathbb{C}^n$, $U_\varepsilon = \{(z, w)\,;\ |\overline{z} - w| < \varepsilon\}$ is of course Stein since

$$|\overline{z} - w|^2 = |z|^2 + |w|^2 - 2\,\mathrm{Re} \sum z_j w_j$$

and $(z, w) \mapsto \mathrm{Re} \sum z_j w_j$ is pluriharmonic.

Technical lemma

Let $\exp : T_X \to X \times X$, $(z, \xi) \mapsto (z, \exp_z(\xi))$ be the exponential map associated with a real analytic hermitian metric γ on X, and exph its "holomorphic" part, so that

$$\exp_z(\xi) = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha\beta}(z)\xi^\alpha \overline{\xi}^\beta, \quad \mathrm{exph}_z(\xi) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha 0}(z)\xi^\alpha.$$

Let $\mathrm{logh} : X \times X \supset W \to T_X$ be the inverse of exph and

$$U_\varepsilon = \{(z, w) \in X \times \overline{X}\,;\ |\mathrm{logh}_z(w)|_\gamma < \varepsilon\}, \quad \varepsilon > 0.$$

Then, for $\varepsilon \ll 1$, U_ε is Stein and $\mathrm{pr}_1 : U_\varepsilon \to X$ is a real analytic locally trivial bundle with fibers biholomorphic to complex balls.

Bergman sheaves

Let $U_\varepsilon = U_{\gamma,\varepsilon} \subset X \times \overline{X}$ be the ball bundle as above, and

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X, \qquad \overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to \overline{X}$$

the natural projections.

Definition

The "Bergman sheaf" $\mathcal{B}_\varepsilon = \mathcal{B}_{\gamma,\varepsilon}$ is the L^2 direct image

$$\mathcal{B}_\varepsilon = p_*^{L^2}(\overline{p}^*\mathcal{O}(K_{\overline{X}})),$$

i.e. the space of sections over an open subset $V \subset X$ defined by
$\mathcal{B}_\varepsilon(V) =$ holomorphic sections f of $\overline{p}^*\mathcal{O}(K_{\overline{X}})$ on $p^{-1}(V)$,

$$f(z,w) = f_1(z,w)\, dw_1 \wedge \ldots \wedge dw_n, \quad z \in V,$$

that are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$:

$$\int_{p^{-1}(K)} i^{n^2} f \wedge \overline{f} \wedge \gamma^n < +\infty, \quad \forall K \Subset V.$$

Associated Bergman bundle and holom structure

Then $\mathcal{B}_\varepsilon$ is an $\mathcal{O}_X$-module, and by the Ohsawa-Takegoshi extension theorem applied to the subvariety $p^{-1}(z) \subset U_\varepsilon$, its fiber $B_{\varepsilon,z} = \mathcal{B}_{\varepsilon,z}/\mathfrak{m}_z\mathcal{B}_{\varepsilon,z}$ is isomorphic to the Hardy-Bergman space $\mathcal{H}^2(B(0,\varepsilon))$ of L^2 holomorphic n-forms on $p^{-1}(z) \simeq B(0,\varepsilon) \subset \mathbb{C}^n$.

Question

By putting $\|f(z)\|^2 = \int_{p^{-1}(z)} i^{n^2} f \wedge \overline{f}$, we get a (real analytic) locally trivial Hilbert bundle $B_\varepsilon \to X$.
Is there a "complex structure" on B_ε such that $\mathcal{B}_\varepsilon = \mathcal{O}(B_\varepsilon)$?

For this, consider the "Bergman Dolbeault" complex $\overline{\partial} : \mathcal{F}_\varepsilon^q \to \mathcal{F}_\varepsilon^{q+1}$ over X, with $\mathcal{F}_\varepsilon^q(V) =$ smooth (n,q)-forms

$$f(z,w) = \sum_{|J|=q} f_J(z,w)\, dw_1 \wedge \ldots \wedge dw_n \wedge d\overline{z}_J, \quad (z,w) \in U_\varepsilon \cap (V \times \overline{X}),$$

$f_J(z,w)$ holomorphic in w and all $\overline{\partial}_z f(z,w) \in L^2(p^{-1}(K))$, $K \Subset V$.

Very ampleness of Bergman bundles

By construction, $\overline{\partial}$ yields a complex of sheaves $(\mathcal{F}^\bullet, \overline{\partial})$ and the kernel $\mathrm{Ker}\,\overline{\partial} : \mathcal{F}^0 \to \mathcal{F}^1$ coincides with $\mathcal{B}_\varepsilon$.

Theorem

Assume that $\varepsilon > 0$ is taken so small that $\psi(z,w) := |\log h_z(w)|^2$ is strictly plurisubharmonic up to the boundary on the compact set $\overline{U}_\varepsilon \subset X \times \overline{X}$. Then the complex of sheaves $(\mathcal{F}^\bullet, \overline{\partial})$ is a resolution of $\mathcal{B}_\varepsilon$ by soft sheaves over X (actually, by $\mathcal{C}^\infty_X$-modules), and for every holomorphic vector bundle $E \to X$ we have

$$H^q(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) = H^q\big(\Gamma(X, \mathcal{F}^\bullet_\varepsilon \otimes \mathcal{O}(E)), \overline{\partial}\big) = 0, \quad \forall q \ge 1.$$

Moreover the fibers $B_{\varepsilon,z} \otimes E_z$ are always generated by global sections of $H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$.

In other words, B_ε is a "very ample holomorphic vector bundle" (as a Hilbert bundle of infinite dimension).
But it is NOT holomorphically locally trivial.

Chern connection of Bergman bundles

Since we have a natural $\nabla^{0,1} = \overline{\partial}$ connection, and a natural hermitian metric on the Bergman bundle, it follows that B_ε can be equipped with a unique Chern connection.

Model case: $X = \mathbb{C}^n$, $\gamma =$ standard hermitian metric.
Then one sees that a (non holomorphic) orthonormal frame of B_ε is given by

$$e_\alpha(z,w) = \pi^{-n/2}\varepsilon^{-|\alpha|-n}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\,(w-\overline{z})^\alpha, \quad \alpha \in \mathbb{N}^n.$$

The $(0,1)$-connection $\nabla^{0,1} = \overline{\partial}$ is given by

$$\nabla^{0,1} e_\alpha = \overline{\partial}_z e_\alpha(z,w) = \varepsilon^{-1}\sum_{1\le j\le n}\sqrt{\alpha_j(|\alpha|+n)}\; d\overline{z}_j \otimes e_{\alpha-c_j}$$

where $c_j = (0,\ldots,1,\ldots,0) \in \mathbb{N}^n$.

Curvature of Bergman bundles

Let $\Theta_{B_\varepsilon,h} = \nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0}$ be the curvature tensor of B_ε with its natural Hilbertian metric h, and

$$\widetilde{\Theta}_\varepsilon(v \otimes \xi) = \langle \Theta_{B_\varepsilon,h}\sigma(v, Jv)\xi, \xi\rangle_h$$

the associated quadratic form with $v \in T_X$, $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_\varepsilon$.

Formula

In the model case $X = \mathbb{C}^n$, the curvature tensor of the Bergman bundle (B_ε, h) is given by

$$\widetilde{\Theta}_\varepsilon(v \otimes \xi) = \varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\Bigg(\Big|\sum_j \sqrt{\alpha_j}\,\xi_{\alpha-c_j}v_j\Big|^2 + \sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2\Bigg).$$

Observe that $\widetilde{\Theta}_\varepsilon(v \otimes \xi)$ is a positive but unbounded quadratic form on B_ε with respect to the standard norm $\|\xi\|^2 = \sum_\alpha |\xi_\alpha|^2$.

However there is convergence for all $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_{\rho\varepsilon}$, $\rho > 1$, since then $\sum_\alpha \rho^{2|\alpha|}|\xi_\alpha|^2 < +\infty$.

Curvature of Bergman bundles (general case)

Bergman curvature formula on a general hermitian manifold

Let X be a compact complex manifold equipped with a hermitian metric γ, and $B_\varepsilon = B_{\gamma,\varepsilon}$ the corresponding Bergman bundle. Then its curvature is given by an asymptotic expansion

$$\widetilde{\Theta}_\varepsilon(z, v \otimes \xi) = \sum_{p=0}^{+\infty} \varepsilon^{-2+p} Q_p(z, v \otimes \xi)$$

where $Q_0(z, v \otimes \xi) = Q_0(v \otimes \xi)$ is given by the model case $\mathbb{C}^n$:

$$Q_0(v \otimes \xi) = \varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\Bigg(\Big|\sum_j \sqrt{\alpha_j}\,\xi_{\alpha-c_j}v_j\Big|^2 + \sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2\Bigg).$$

The other terms $Q_p(z, v \otimes \xi)$ are real analytic and depend on the torsion and curvature tensor of γ, especially Q_1, Q_2.

A consequence of the above formula is that B_ε is strongly Nakano positive for $\varepsilon > 0$ small enough.

Transcendental holomorphic Morse inequalities

Conjecture

Let X be a compact n-dimensional complex manifold and $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ a Bott-Chern class, represented by closed real $(1,1)$-forms modulo $\partial\overline{\partial}$ exact forms. Set

$$\mathrm{Vol}(\alpha) = \sup_{T=\alpha+i\partial\overline{\partial}\varphi\geq 0} \int_X T^n_{ac}, \quad T \geq 0 \text{ current.}$$

Then

$$\mathrm{Vol}(\alpha) \geq \sup_{u\in\{\alpha\},\ u\in C^\infty} \int_{X(u,0)} u^n$$

where

$$X(u,0) = 0\text{-index set of } u = \big\{x \in X\,;\ u(x) \text{ positive definite}\big\}.$$

Conjectural corollary (fundamental volume estimate)

Let X be compact Kähler, $\dim X = n$, and $\alpha, \beta \in H^{1,1}(X,\mathbb{R})$ be nef classes. Then

$$\mathrm{Vol}(\alpha-\beta) \geq \alpha^n - n\alpha^{n-1}\cdot\beta.$$

Transcendental Morse: known facts & beyond

The conjecture on Morse inequalities is known to be true when $\alpha = c_1(L)$ is the class of a line bundle ([D-1985]), and the corollary can be derived from this when α, β are integral classes (by [D-1993] and independently by [Trapani, 1993]).

Recently, the volume estimate for α, β transcendental has been established by D. Witt-Nyström when X is projective, and Xiao-Popovici even proved in general that $\mathrm{Vol}(\alpha-\beta) > 0$ if $\alpha^n - n\alpha^{n-1}\cdot\beta > 0$.

Idea. In the general case, one can find a sequence of non holomorphic hermitian line bundles (L_m, h_m) such that

$$m\alpha = \Theta_{L_m,h_m} + \gamma_m^{2,0} + \overline{\gamma}_m^{0,2}, \quad \gamma_m \to 0.$$

As U_ε is Stein, $\overline{\gamma}_m^{0,2} = \overline{\partial} v_m$, $v_m \to 0$, and $\mathrm{pr}_1^* L_m$ becomes a holomorphic line bundle with curvature form $\Theta_{\mathrm{pr}_1^* L_m} \simeq m\,\mathrm{pr}_1^*\alpha$.

Then apply L^2 direct image $(\mathrm{pr}_1)_*^{L^2}$ and use Bergman estimates instead of dimension counts in Morse inequalities.

Invariance of plurigenera for polarized families of compact Kähler manifolds

Conjecture

Let $\pi : \mathcal{X} \to S$ be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base S. Assume that the family admits a polarization, i.e. a closed smooth $(1,1)$-form ω such that $\omega_{|X_t}$ is positive definite on each fiber $X_t := \pi^{-1}(t)$. Then the plurigenera

$$p_m(X_t) = h^0(X_t, mK_{X_t}) \text{ are independent of } t \text{ for all } m \geq 0.$$

The conjecture is known to be true for a projective family $\mathcal{X} \to S$:

- Siu and Kawamata (1998) in the case of varieties of general type
- Siu (2000) and Păun (2004) in the arbitrary projective case

No algebraic proof is known in the latter case; one uses deeply the Ohsawa-Takegoshi L^2 extension theorem.

Invariance of plurigenera: strategy of proof (1)

It is enough to consider the case of a family $\mathcal{X} \to \Delta$ over the disc, such that there exists a relatively ample line bundle $\mathcal{A}$ over $\mathcal{X}$.

Given $s \in H^0(X_0, mK_{X_0})$, the point is to show that it extends into $\widetilde{s} \in H^0(\mathcal{X}, mK_{\mathcal{X}})$, and for this, one only needs to produce a hermitian metric $h = e^{-\varphi}$ on $K_{\mathcal{X}}$ such that:

- $\Theta_h = i\partial\overline{\partial}\varphi \geq 0$ in the sense of currents
- $|s|_h^2 \leq 1$, i.e. $\varphi \geq \log|s|$ on X_0.

The Ohsawa-Takegoshi theorem then implies the existence of $\widetilde{s}$.

To produce $h = e^{-\varphi}$, one defines inductively sections of $\sigma_{p,j}$ of $\mathcal{L}_p := \mathcal{A} + pK_{\mathcal{X}}$ such that:

- $(\sigma_{p,j})$ generates $\mathcal{L}_p$ for $0 \leq p < m$
- $\sigma_{p,j}$ extends $(\sigma_{p-m,j}s^m)_{|X_0}$ to $\mathcal{X}$ for $p \geq m$
- $\displaystyle\int_{\mathcal{X}} \frac{\sum_j |\sigma_{p,j}|^2}{\sum_j |\sigma_{p-1,j}|^2} \leq C$ for $p \geq 1$.

Invariance of plurigenera: strategy of proof (2)

By Hölder, the L^2 estimates imply $\int_{\mathcal{X}} \big(\sum_j |\sigma_{p,j}|^2\big)^{1/p} \le C$ for all p, and using the fact that $\lim \frac{1}{p}\Theta_{\mathcal{A}} = 0$, one can take

$$\varphi = \limsup_{p\to+\infty} \frac{1}{p}\log \sum_j |\sigma_{p,j}|^2.$$

Idea. In the polarized Kähler case, use the Bergman bundle $B_\varepsilon \to \mathcal{X}$ instead of an ample line bundle $\mathcal{A} \to \mathcal{X}$. This amounts to applying the Ohsawa-Takegoshi L^2 extension on Stein tubular neighborhoods $U_\varepsilon \subset \mathcal{X} \times \overline{\mathcal{X}}$, with projections $\mathrm{pr}_1 : U_\varepsilon \to \mathcal{X}$ and $\pi : \mathcal{X} \to \Delta$.

Proposition

In the polarized Kähler case $(\mathcal{X},\omega)$, shrinking from $U_{\rho\varepsilon}$, $\rho > 1$, to U_ε, one gets

$$i\partial\overline{\partial}\Big(\sum_j \|\sigma_{p,j}\|^2_{U_\varepsilon}\Big)^{\lambda/p} \ge -\varepsilon^{-2}(\log\rho)^{-1}\rho^{n\lambda/p}e^{C\lambda}\omega \quad \forall\lambda > 0.$$

This is enough to imply the invariance of plurigenera if $\varepsilon > 0$ can be taken arbitrarily large.

The end

Thank you for your attention