Une preuve effective simple de la conjecture de Kobayashi sur l'hyperbolicité des hypersurfaces algébriques générales

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Kobayashi hyperbolicity and entire curves

Kobayashi-Eisenman infinitesimal pseudometrics

Let X be a complex space, $\dim_{\mathbb{C}} X = n$, $\mathbb{B}_p =$ unit ball in $\mathbb{C}^p$, $1 \le p \le n$ and $\tau_0 = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_p \in \Lambda^p \mathbb{C}^p$. The Kobayashi-Eisenman infinitesimal pseudometric $\mathbf{e}_X^p$ is the pseudometric defined on decomposable p-vectors $\xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p T_{X,x}$, by

$$\mathbf{e}_X^p(\xi) = \inf \big\{ \lambda > 0 \,;\, \exists f : \mathbb{B}_p \to X,\ f(0) = x,\ \lambda f_*(\tau_0) = \xi \big\}.$$

We say that X is (infinitesimally) p-measure hyperbolic if $\mathbf{e}_X^p$ is everywhere locally uniformly positive definite on the tautological line bundle of the Grassmannian bundle of p-subspaces $\mathrm{Gr}(T_X, p)$.

Characterization of Kobayashi hyperbolicity (Brody, 1978)

For a compact complex manifold X, $\dim_{\mathbb{C}} X = n$, TFAE:
(i) The pseudometric $\mathbf{k}_X = \mathbf{e}_X^1$ is everywhere non degenerate ;
(ii) the integrated pseudodistance $\mathbf{d_{Kob}}$ of $\mathbf{e}_X^1$ is a distance ;
(iii) X Brody hyperbolic, i.e. $\nexists$ entire curves $f : \mathbb{C} \to X$, $f \ne$ const.

Main conjectures

Conjecture of General Type (CGT)

- A compact variety $X / \mathbb{C}$ is volume hyperbolic (w.r.t. $\mathbf{e}^n_X$) $\Leftrightarrow$ X is of general type, i.e. K_X big [implication $\Leftarrow$ is well known].
- X Kobayashi (or Brody) hyperbolic should imply K_X ample.

Green-Griffiths-Lang Conjecture (GGL)

Let X be a projective variety/$\mathbb{C}$ of general type. Then $\exists Y \subsetneq X$ algebraic such that all entire curves $f : \mathbb{C} \to X$ satisfy $f(\mathbb{C}) \subset Y$.

Arithmetic counterpart (Lang 1987) – very optimistic ?

For X projective defined over a number field $\mathbb{K}_0$, the exceptional locus $Y = \mathrm{Exc}(X)$ in GGL's conjecture equals $\mathrm{Mordel}(X) =$ smallest Y such that $X(\mathbb{K}) \smallsetminus Y$ is finite, $\forall \mathbb{K}$ number field $\supset \mathbb{K}_0$.

Consequence of CGT + GGL

A compact complex manifold X should be Kobayashi hyperbolic iff it is projective and every subvariety Y of X is of general type.

Kobayashi conjecture on generic hyperbolicity

Kobayashi conjecture (1970)

- Let $X^n \subset \mathbb{P}^{n+1}$ be a (very) generic hypersurface of degree $d \geq d_n$ large enough. Then X is Kobayashi hyperbolic.
- By results of Riemann, Poincaré, Zaidenberg, Clemens, Ein, Voisin, Pacienza, the optimal bound is expected to be $d_1 = 4$, $d_n = 2n + 1$ for $2 \leq n \leq 4$ and $d_n = 2n$ for $n \geq 5$.

Using "jet technology" and deep results of McQuillan for curve foliations on surfaces, the following has been proved:

Theorem (Work of McQuillan + D., El Goul, 1998)

A very generic surface $X^2 \subset \mathbb{P}^3$ of degree $d \geq 21$ is hyperbolic. Independently McQuillan got $d \geq 35$.

This has been improved to $d \geq 18$ (Păun, 2008).
In 2012, Yum-Tong Siu announced a proof of the case of arbitrary dimension n, with a non explicit d_n (and a rather involved proof).

Results on the generic Green-Griffiths conjecture

By combining an algebraic existence theorem for jet differentials and Y.T. Siu's technique of slanted vector fields (itself derived from ideas of H. Clemens, L. Ein and C. Voisin), the following was proved:

Theorem (S. Diverio, J. Merker, E. Rousseau, 2009, & followers)

A generic hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n := 2^{n^5}$ satisfies the GGL conjecture. Bound then improved to $d_n = O(e^{n^{1+\varepsilon}})$:

$d_n = 9n^n$ (Bérczi, 2010, using residue formulas),

$d_n = (5n)^2\, n^n$ (Darondeau, 2015, alternative method),

$d_n = \left\lfloor \frac{n^4}{3}\big(n\log(n\log(24n))\big)^n \right\rfloor$ (D-, 2012), weaker bound,

but special case of general result for arbitrary projective varieties.

Theorem (S. Diverio, S. Trapani, 2009)

Additionally, a generic hypersurface $X^3 \subset \mathbb{P}^4$ of degree $d \geq 593$ is hyperbolic.

Recent proof of the Kobayashi conjecture

Y.T. Siu's (Abel conf. 2002, Invent. Math. 2015) detailed a strategy for the proof of the Kobayashi conjecture. In 2016, Brotbek gave a more geometric proof, using Wronskian jet differentials.

Theorem (Brotbek, April 2016)

Let Z be a projective $n+1$-dimensional projective manifold and $A \to Z$ a very ample line bundle. Let $\sigma \in H^0(Z, dA)$ be a generic section. Then for $d \gg 1$ the hypersurface $X_\sigma = \sigma^{-1}(0)$ is hyperbolic.

The initial proof of Brotbek didn't provide effective bounds. Through various improvements, Deng Ya got in his PhD thesis (May 2016) the explicit bound $d_n = (n+1)^{n+2}(n+2)^{2n+7} = O(n^{3n+9})$.

Theorem (D-, 2018, with a much simplified proof)

In the above setting, a general hypersurface $X_\sigma = \sigma^{-1}(0)$ is hyperbolic as soon as

$$d \geq d_n = \lfloor (en)^{2n+2}/3 \rfloor.$$

Solution of a conjecture of Debarre (2005)

In the same vein, the following results have also been proved.

Solution of Debarre's conjecture (Brotbek-Darondeau & Xie, 2015)

Let Z be a projective $(n+c)$-dimensional projective manifold and $A \to Z$ a very ample line bundle. Let $\sigma_j \in H^0(Z, d_j A)$ be generic sections, $1 \le j \le c$. Then, for $c \ge n$ and $d_j \gg 1$ large, the n-dimensional complete intersection $X_\sigma = \bigcap \sigma_j^{-1}(0) \subset Z$ has an ample cotangent bundle $T^*_{X_\sigma}$.
In particular, such a generic complete intersection is hyperbolic.

S.Y. Xie got the sufficient lower bound $d_j \ge d_{n,c} = N^{N^2}$, $N = n+c$.

In his PhD thesis, Ya Deng obtained the improved lower bound

$$d_{n,c} = 4\nu(2N-1)^{2\nu+1} + 6N - 3 = O((2N)^{N+3}), \qquad \nu = \lfloor \tfrac{N+1}{2} \rfloor.$$

The proof is obtained by selecting carefully certain special sections σ_j associated with "lacunary" polynomials of high degree.

Category of directed manifolds

Goal. More generally, we are interested in curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$ where V is a subbundle of T_X, or possibly a singular linear subspace, i.e. a closed irreducible analytic subspace such that $\forall x \in X$, $V_x := V \cap T_{X,x}$ is linear.

Definition (Category of directed manifolds)

– Objects : pairs (X, V), X manifold/$\mathbb{C}$ and $V \subset T_X$
– Morphisms $\psi : (X, V) \to (Y, W)$ holomorphic s.t. $\psi_* V \subset W$
– "Absolute case" (X, T_X), i.e. $V = T_X$
– "Relative case" $(X, T_{X/S})$ where $X \to S$
– "Integrable case" when $[\mathcal{O}(V), \mathcal{O}(V)] \subset \mathcal{O}(V)$ (foliations)

Canonical sheaf of a directed manifold (X, V)

When V is nonsingular, i.e. a subbundle, one simply sets

$$K_V = \det(V^*) \quad \text{(as a line bundle)}.$$

Canonical sheaf of a singular pair (X,V)

When V is singular, we first introduce the rank 1 sheaf ${}^b\mathcal{K}_V$ of sections of $\det V^*$ that are locally bounded with respect to a smooth ambient metric on T_X. One can show that ${}^b\mathcal{K}_V$ is equal to the integral closure of the image of the natural morphism

$$\mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*) \to \mathcal{L}_V := \text{invert. sheaf } \mathcal{O}(\Lambda^r V^*)^{**}$$

that is, if the image is $\mathcal{L}_V \otimes \mathcal{J}_V$, $\mathcal{J}_V \subset \mathcal{O}_X$,

$${}^b\mathcal{K}_V = \mathcal{L}_V \otimes \overline{\mathcal{J}}_V, \quad \overline{\mathcal{J}}_V = \text{integral closure of } \mathcal{J}_V.$$

Consequence

If $\mu : \widetilde{X} \to X$ is a modification and $\widetilde{X}$ is equipped with the pull-back directed structure $\widetilde{V} = \widetilde{\mu}^{-1}(V)$, then

$${}^b\mathcal{K}_V \subset \mu_*({}^b\mathcal{K}_{\widetilde{V}}) \subset \mathcal{L}_V$$

and $\mu_*({}^b\mathcal{K}_{\widetilde{V}})$ increases with μ.

Canonical sheaf of a singular pair (X,V) [sequel]

By Noetherianity, one can define a sequence of rank 1 sheaves

$$\mathcal{K}_V^{[m]} = \lim_{\mu} \uparrow \mu_*({}^b\mathcal{K}_{\widetilde{V}})^{\otimes m}, \quad \mu_*({}^b\mathcal{K}_V)^{\otimes m} \subset \mathcal{K}_V^{[m]} \subset \mathcal{L}_V^{\otimes m}$$

which we call the pluricanonical sheaf sequence of (X, V).

Remark

The blow-up μ for which the limit is attained may depend on m. We do not know if there is a μ that works for all m.

This generalizes the concept of reduced singularities of foliations, which is known to work in that form only for surfaces.

Definition

We say that (X, V) is of general type if the pluricanonical sheaf sequence $\mathcal{K}_V^{[\bullet]}$ is big, i.e. $H^0(X, \mathcal{K}_V^{[m]})$ provides a generic embedding of X for a suitable $m \gg 1$.

Definition of algebraic differential operators

Let $(\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, $\quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$ be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X. It has a local Taylor expansion

$$f(t) = x + t\xi_1 + \ldots + t^k\xi_k + O(t^{k+1}), \quad \xi_s = \frac{1}{s!}\nabla^s f(0)$$

for some connection ∇ on V.

One considers the Green-Griffiths bundle $E^{\mathrm{GG}}_{k,m}V^*$ of polynomials of weighted degree m written locally in coordinate charts as

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(x)\xi_1^{\alpha_1}\ldots\xi_k^{\alpha_k}, \quad \xi_s \in V.$$

One can view them as algebraic differential operators

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}), \\ P(f_{[k]})(t) &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2}\ldots f^{(k)}(t)^{\alpha_k}. \end{aligned}$$

Definition of algebraic differential operators [cont.]

Here $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet, and $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are supposed to holomorphic functions on X.

The reparametrization action : $f \mapsto f \circ \varphi_\lambda$, $\varphi_\lambda(t) = \lambda t$, $\lambda \in \mathbb{C}^*$ yields $(f \circ \varphi_\lambda)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$, whence a $\mathbb{C}^*$-action

$$\lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k).$$

$E^{\mathrm{GG}}_{k,m}$ is precisely the set of polynomials of weighted degree m, corresponding to coefficients $a_{\alpha_1\ldots\alpha_k}$ with $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$.

Direct image formula

If $J_k^{\mathrm{nc}}V$ is the set of non constant k-jets, one defines the Green-Griffiths bundle to be $X_k^{\mathrm{GG}} = J_k^{\mathrm{nc}}V/\mathbb{C}^*$ and $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ to be the associated tautological rank 1 sheaf. Then we have

$$\pi_k : X_k^{\mathrm{GG}} \to X, \qquad E^{\mathrm{GG}}_{k,m}V^* = (\pi_k)_*\mathcal{O}_{X_k^{\mathrm{GG}}}(m)$$

Generalized GGL conjecture, strategy of attack

Generalized GGL conjecture

If (X,V) is directed manifold of general type, i.e. $\mathcal{K}_V^{[\bullet]}$ is big, then $\exists Y \subsetneq X$ such that $\forall f:(\mathbb{C},T_\mathbb{C})\to(X,V)$, one has $f(\mathbb{C})\subset Y$.

Fundamental vanishing theorem

[Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]
$\forall P\in H^0(X,E^{\rm GG}_{k,m}V^*\otimes\mathcal{O}(-A))$: global diff. operator on X (A ample divisor), $\forall f:(\mathbb{C},T_\mathbb{C})\to(X,V)$, one has $P(f_{[k]})\equiv 0$.
$\iff f_{[k]}(\mathbb{C})\subset\sigma^{-1}(0),\ \forall\sigma\in H^0(X_k^{\rm GG},\mathcal{O}_{X_k^{\rm GG}}(m)\otimes\pi_k^*\mathcal{O}(-A))$.

Corollary: exploit base locus of algebraic differential equations

Exceptional locus: $\mathrm{Exc}(X,V)=\overline{\bigcup_f f(\mathbb{C})}^{\rm Zar}$, $f:(\mathbb{C},T_\mathbb{C})\to(X,V)$,
Green-Griffiths locus: $\mathrm{GG}(X,V)=\bigcap_k\pi_k(\mathrm{GG}_k(X,V))$, where
$\mathrm{GG}_k(X,V)=\bigcap_\sigma\sigma^{-1}(0)$, $\sigma\in H^0(X_k^{\rm GG},\mathcal{O}_{X_k^{\rm GG}}(m)\otimes\pi_k^*\mathcal{O}(-A))$.
Then $\mathrm{Exc}(X,V)\subset\mathrm{GG}(X,V)$.

Proof of the fundamental vanishing theorem

Simple case. First assume that f is a Brody curve, i.e. that $\sup_{t\in\mathbb{C}}\|f'(t)\|_\omega<+\infty$ for some hermitian metric ω on X. By raising P to a power, we can assume A very ample, and view P as a $\mathbb{C}$ valued differential operator whose coefficients vanish on a very ample divisor A.

The Cauchy inequalities imply that all derivatives $f^{(s)}$ are bounded in any coordinate chart. Hence $u_A(t):=P(f_{[k]})(t)$ is bounded, and must be constant by Liouville's theorem.

Since A is very ample, we can move $A\in|A|$ such that A hits $f(\mathbb{C})\subset X$. Bu then u_A vanishes somewhere and so $u_A\equiv 0$.

General case of a general entire curve $f:(\mathbb{C},T_\mathbb{C})\to(X,V)$.
Instead, one makes use of Nevanlinna theory arguments (logarithmic derivative lemma).

Remark. Generalized GGL conjecture is easy if rank $V=1$.

And now ... the Semple jet bundles

- **Functor "1-jet" :** $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :
  $\tilde{X} = P(V) =$ bundle of projective spaces of lines in V
  $\pi : \tilde{X} = P(V) \to X, \quad (x, [v]) \mapsto x, \quad v \in V_x$
  $\tilde{V}_{(x,[v])} = \{\xi \in T_{\tilde{X},(x,[v])}\,;\ \pi_*\xi \in \mathbb{C}v \subset T_{X,x}\}$
- **For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V**
  f lifts as $\begin{cases} f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X} \\ f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \ \ \text{(projectivized 1st-jet)} \end{cases}$
- **Definition.** *Semple jet bundles :*
  – $(X_k, V_k) =$ k-th iteration of functor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
  – $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized k-jet of f.
- **Basic exact sequences**
  $0 \to T_{X_k/X_{k-1}} \to V_k \stackrel{(\pi_k)_\star}{\to} \mathcal{O}_{X_k}(-1) \to 0 \qquad \Rightarrow \text{rank } V_k = r$
  $0 \to \mathcal{O}_{X_k} \to \pi_k^\star V_{k-1} \otimes \mathcal{O}_{X_k}(1) \to T_{X_k/X_{k-1}} \to 0 \quad \text{(Euler)}$

Direct image formula for Semple bundles

For $n = \dim X$ and $r = \text{rank } V$, one gets a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \stackrel{\pi_k}{\to} X_{k-1} \to \cdots \to X_1 \stackrel{\pi_1}{\to} X_0 = X$$

with $\dim X_k = n + k(r-1)$, rank $V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

Theorem

X_k is a smooth compactification of $X_k^{\mathrm{GG,reg}}/\mathbb{G}_k = J_k^{\mathrm{GG,reg}}/\mathbb{G}_k$, where $\mathbb{G}_k$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization: $(f, \varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k-jets of regular curves.

Direct image formula for invariant differential operators

$E_{k,m}V^* := (\pi_{k,0})_*\mathcal{O}_{X_k}(m) =$ sheaf of algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ such that $P((f \circ \varphi)_{[k]}) = \varphi'^m P(f_{[k]}) \circ \varphi$.

Strategy of proof of the Kobayashi conjecture (Brotbek, simplified by D.)

Let $\pi : \mathcal{X} \to S$ be family of smooth projective varieties, and let $\mathcal{X}_k \to S$ be the relative Semple tower of $(\mathcal{X}, T_{\mathcal{X}/S})$.
If $X_t = \pi^{-1}(t)$, $t \in S$, is the general fiber, then the fiber of $\mathcal{X}_k \to S$ is the k-stage of the Semple tower $X_{t,k} \to X_t$
(the idea is to consider the universal family of hypersurfaces $X \subset \mathbb{P}^{n+1}$ of sufficiently high degree $d \gg 1$.)

Basic observation

Assume that there exists $t_0 \in S$ such that we get on $X_{t_0,k}$ a nef "twisted tautological sheaf" $\mathcal{G}_{|X_{t_0,k}}$ where

$$\mathcal{G} := \mathcal{O}_{\mathcal{X}_k}(m) \otimes \mathcal{I}_{k,m} \otimes \pi_{k,0} A^{-1}$$

(in the sense that a log resolution of $\mathcal{G}$ is nef), and $\mathcal{I}_{k,m}$ is a suitable "functorial" multiplier ideal with support in the set $\mathcal{X}_k^{\rm sing}$ of singular jets. Then X_t is Kobayashi hyperbolic for general $t \in S$.

Simplified proof of the Kobayashi conjecture

Proof. By hypothesis, One can take a resolution $\mu_{k,m} : \widehat{\mathcal{X}}_k \to \mathcal{X}_k$ of the ideal $\mathcal{I}_{k,m}$ as an invertible sheaf $\mu_{k,m}^* \mathcal{I}_{k,m}$ on $\widehat{\mathcal{X}}_{k,m}$, so that $\mu_{k,m}^* \mathcal{G}_{|\widehat{X}_{t_0,k}}$ is a nef line bundle.

Then one can add a small $\mathbb{Q}$-divisor $\mathcal{P}_\varepsilon$ that is a combination of the lower stages $\mathcal{O}_{\mathcal{X}_\ell}(m')$, $\ell < k$, and of the exceptional divisor of $\mu_{k,m}$ so that $(\mu_{k,m}^* \mathcal{G} \otimes \mathcal{P}_\varepsilon)_{|\widehat{X}_{t_0,k}}$ is an ample line bundle.

Since ampleness is a Zariski open property, one concludes that $(\mu_{k,m}^* \mathcal{G} \otimes G_\varepsilon)_{|\widehat{X}_{t,k}}$ is ample for general $t \in S$. The fundamental vanishing theorem then implies that X_t is Kobayashi hyperbolic. □

The next idea is to produce a very particular hypersurface X_{t_0} on which there are a lot of non trivial Wronskian operators that generate the required sheaf

$$\mathcal{G} = \mathcal{O}_{\mathcal{X}_k}(m) \otimes \mathcal{I}_{k,m} \otimes \pi_{k,0} A^{-1}.$$

Then $\mathcal{G}_{|X_{k,t_0}}$ is nef and we are done.

Wronskian operators

Let $L \to X$ be a line bundle, and let

$$s_0, \dots, s_k \in H^0(X, L)$$

be arbitrary sections. One defines Wronskian operators acting on $f : \mathbb{C} \to X$, $t \mapsto f(t)$ by $D = \frac{d}{dt}$ and

$$W(s_0, \dots, s_k)(f) = \begin{vmatrix} s_0(f) & s_1(f) & \dots & s_k(f) \\ D(s_0(f)) & D(s_1(f)) & \dots & D(s_k(f)) \\ \vdots & & & \vdots \\ D^k(s_0(f)) & D^k(s_1(f)) & \dots & D^k(s_k(f)) \end{vmatrix}$$

This actually does not depend on the trivialization of L and defines

$$W(s_0, \dots, s_k) \in H^0(X, E_{k,k'} T_X^* \otimes L^{k+1}), \quad k' = \frac{k(k+1)}{2}.$$

Problem. One has to take $L > 0$, hence $L^{k+1} > 0$: seems useless!

Wronskian operators can sometimes be divided !

Take e.g. $X = \mathbb{P}^N$, $A = \mathcal{O}(1)$ very ample, $k \le N$, $d \ge k$ and

$$s_j(z) = z_j^d q_j(z), \ \deg q_j = k \quad \Longrightarrow \quad s_j \in H^0(X, A^{d+k}).$$

Then derivatives $D^\ell(s_j \circ f)$ are divisible by z_j^{d-k} for $\ell \le k$, and (taking $L = A^{d+k}$) we find

$$\prod_{0 \le j \le k} z_j^{-(d-k)} W(s_0, \dots s_k) \in H^0(X, E_{k,k'} T_X^* \otimes A^{(d+k)(k+1)-(d-k)(k+1)})$$
$$= H^0(X, E_{k,k'} T_X^* \otimes A^{2k(k+1)}).$$

Not enough, but the exponent is independent of d and a division by one more factor z_j^{d-k} would suffice to reach $A^{<0}$, for $d \gg k$.

If we take the Fermat hypersurface $X = \{z_0^d + \dots + z_N^d = 0\}$ and $k = N-1$, $q_1 = \dots = q_k = q$, then $z_0^d = -\sum_{i>0} z_i^d$ implies that $W(s_0, \dots, s_k) = (-1)^k W(s_N, s_1, \dots, s_k)$ is also divisible by z_N^{d-k}, so

$$P := \prod_{0 \le i \le k+1} z_i^{-(d-k)} W(s_0, \dots s_k) \in H^0(X, E_{k,k'} T_X^* \otimes A^{k(2k+3)-d}).$$

Getting more jet differentials from Wronskians

A better choice than the Fermat hypersurface is to take $X = \sigma^{-1}(0) \subset \mathbb{P}^{n+1}$ with $\sigma \in H^0(\mathbb{P}^{n+1}, \mathcal{O}(d))$ given by

$$\sigma = \sum_{0\le i\le N} a_i(z)m_i(z)^\delta,\ a_i \text{ "random"},\ \deg a_i = \rho \ge k,\ m_i(z) = \prod_{J\ni i} \tau_J(z),$$

where the J's run over all subsets $J \subset \{0, 1, \ldots, N\}$ with $\mathrm{card}\, J = n$, $\tau_J \in H^0(\mathbb{P}^{n+1}, \mathcal{O}(1))$ is a sufficiently general linear section and $\delta \gg 1$.

An adequate choice to ensure smoothness of X is $N = n(n + 1)$.
Then, for $k \ge N$ and all $J \subset \{0, 1, ..., N\}$, $\mathrm{card}\, J = n$, the Wronskians

$$W_{q,\widehat\tau,k,J} = W(q_1\widehat\tau_1^{d-k}, ..., q_r\widehat\tau_r^{d-k}, (a_im_i^\delta)_{i\in\complement J}),\quad r = k - N + n$$

with $\deg q_j = k$ are divisible by $(\widehat\tau_j^{d-2k})_{1\le j\le n}$ and $(m_i^{\delta-k})_{i\in\complement J}$ $\Rightarrow$

$$P_{q,\widehat\tau,k,J} := \prod_{i\in\complement J} m_i^{-(\delta-k)} \prod_j \widehat\tau_j^{d-2k} W_{k,r} \in H^0(X, E_{k,k'}T^*_X \otimes A^{c_n})$$

where $c_n = k(k+1) \deg m_j = O((en)^{n+5/2})$. As $a_im_i^\delta = -\sum_{j\ne i} a_jm_j^\delta$ on X, we infer the divisibility of $P_{q,\widehat\tau,k,J}$ by the extra factor $\tau_J^{\delta-k}$.

Conclusion: analyzing base loci of Wronskians

We need $\delta > k + c_n$ to reach a negative exponent $A^{<0}$

$$\Rightarrow d \ge d_n = O((en)^{2n+2}).$$

A Bertini type lemma

For $k \ge n^3 + n^2 + 1$, the k-jets of the coefficients a_j are general enough, the simplified Wronskians $\widetilde P_{q,\widehat\tau,k,J}$ generate the universal Wronskian ideal $\mathcal{I}_{k,k'}$ outside of the hyperplane sections $\tau_J^{-1}(0)$.

The proof is achieved by induction on $\dim X$, taking $X' = \tau_J^{-1}(0)$. □

To generalize further, one needs stronger existence theorems for jets.

General existence theorem for jet differentials (D-, 2010)

Let (X, V) be of general type, such that ${}^b\mathcal{K}_V^{\otimes p}$ is a big rank 1 sheaf.
Then $\exists$ many global sections P, $m \gg k \gg 1 \Rightarrow \exists$ alg. hypersurface $Z \subsetneq X_k^{\mathrm{GG}}$ s.t. all entire $f : (\mathbb{C}, T_{\mathbb{C}}) \mapsto (X, V)$ satisfy $f_{[k]}(\mathbb{C}) \subset Z$.

1^{st} step: take a Finsler metric on k-jet bundles

Let $J_k V$ be the bundle of k-jets of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$
Assuming that V is equipped with a hermitian metric h, one defines a "weighted Finsler metric" on $J^k V$ by taking $p = k!$ and

$$\Psi_{h_k}(f) := \Big(\sum_{1\le s\le k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \Big)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$, with curvature form $(x, \xi_1, \ldots, \xi_k) \mapsto$

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor Θ_{V^*,h^*} and $\omega_{\mathrm{FS},k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\mathrm{GG}} \to X$.
The expression gets simpler by using polar coordinates

$$x_s = |\xi_s|_h^{2p/s}, \qquad u_s = \xi_s/|\xi_s|_h = \nabla^s f(0)/|\nabla^s f(0)|.$$

2^{nd} step: probabilistic interpretation of the curvature

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors u_s in the unit sphere bundle $SV \subset V$.
The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2p/s} = 1$, so we can take here $x_s \ge 0$, $\sum x_s = 1$.
This is essentially a sum of the form $\sum \frac{1}{s}\gamma(u_s)$ where u_s are random points of the sphere, and so as $k \to +\infty$ this can be estimated by a "Monte-Carlo" integral

$$\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big) \int_{u\in SV} \gamma(u)\, du.$$

As γ is quadratic here, $\int_{u\in SV} \gamma(u)\, du = \frac{1}{r}\,\mathrm{Tr}(\gamma)$.

3rd step: getting the main cohomology estimates

$\Rightarrow$ the leading term only involves the trace of Θ_{V^*,h^*}, i.e. the curvature of $(\det V^*, \det h^*)$, that can be taken > 0 if $\det V^*$ is big.

Corollary of holomorphic Morse inequalities (D-, 2010)

Let (X, V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V, h) and (F, h_F) hermitian. Define

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big),$$
$$\eta = \Theta_{\det V^*, \det h^*} + \Theta_{F,h_F}.$$

Then for all $q \geq 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have upper and lower bounds [$q = 0$ most useful!]

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \leq \frac{m^{n+kr-1}}{(n+kr-1)!}\,\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q)} (-1)^q \eta^n + \frac{C}{\log k}\Big)$$
$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \geq \frac{m^{n+kr-1}}{(n+kr-1)!}\,\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q,\,q\pm 1)} (-1)^q \eta^n - \frac{C}{\log k}\Big).$$

Induced directed structure on a subvariety

Let Z be an irreducible algebraic subset of some Semple k-jet bundle X_k over X (k arbitrary).

We define an induced directed structure $(Z, W) \hookrightarrow (X_k, V_k)$ by taking the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure of $T_{Z'} \cap V_k$ taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$.

Alternatively, one could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage $(\mathcal{X}_k, \mathcal{A}_k)$ of the "absolute Semple tower" associated with $(\mathcal{X}_0, \mathcal{A}_0) = (X, T_X)$ (so as to deal only with nonsingular ambient Semple bundles).

This produces an induced directed subvariety

$$(Z, W) \subset (X_k, V_k).$$

It is easy to show that

$\pi_{k,k-1}(Z) = X_{k-1} \Rightarrow \mathrm{rank}\, W < \mathrm{rank}\, V_k = \mathrm{rank}\, V$.

Sufficient criterion for the GGL conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "strongly of general type" if it is of general type and for every irreducible alg. subvariety $Z \subsetneq X_k$ that projects onto X, $X_k \not\subset D_k := P(T_{X_{k-1}/X_{k-2}})$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, i.e. ${}^b\mathcal{K}_W \otimes \mathcal{O}_{X_k}(m)_{|Z}$ is big for some $m \in \mathbb{Q}_+$, after a suitable blow-up.

Theorem (D-, 2014)

If (X, V) is strongly of general type, the Green-Griffiths-Lang conjecture holds true for (X, V), namely there $\exists Y \subsetneq X$ such that every non constant holomorphic curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfies $f(\mathbb{C}) \subset Y$.

Proof: Induction on rank V, using existence of jet differentials.

Related stability property

Definition

Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geq 1$, $Z \not\subset D_k$, and $Z = X = X_0$ for $k = 0$, we define the slope of the corresponding directed variety (Z, W) to be $\mu_A(Z, W) =$

$$\frac{\inf\left\{\lambda \in \mathbb{Q};\ \exists m \in \mathbb{Q}_+,\ {}^b\mathcal{K}_W \otimes \left(\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(\lambda A)\right)_{|Z} \text{ big on } Z\right\}}{\operatorname{rank} W}.$$

Notice that (X, V) is of general type iff $\mu_A(X, V) < 0$.

We say that (X, V) is A-jet-stable (resp. A-jet-semi-stable) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leq \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above.

Observation. If (X, V) is of general type and A-jet-semi-stable, then (X, V) is strongly of general type.

Criterion for the generalized Kobayashi conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "algebraically jet-hyperbolic" if for every irreducible alg. subvariety $Z \subsetneq X_k$ s.t. $X_k \not\subset D_k$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ either has $W = 0$ or is of general type modulo $X_k \to X$.

Theorem (D-, 2014)

If (X, V) is algebraically jet-hyperbolic, then (X, V) is Kobayashi (or Brody) hyperbolic, i.e. there are no entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.

Now, the hope is that a (very) generic complete intersection $X = H_1 \cap \ldots \cap H_c \subset \mathbb{P}^{n+c}$ of codimension c and degrees $(d_1, ..., d_c)$ s.t. $\sum d_j \geq 2n + c$ yields (X, T_X) algebraically jet-hyperbolic.

Invariance of "directed plurigenera" ?

One way to check the above property, at least with non optimal bounds, would be to show some sort of Zariski openness of the properties "strongly of general type" or "algebraically jet-hyperbolic". One would need e.g. to know the answer to

Question

Let $(\mathcal{X}, \mathcal{V}) \to S$ be a proper family of directed varieties over a base S, such that $\pi : \mathcal{X} \to S$ is a nonsingular deformation and the directed structure on $X_t = \pi^{-1}(t)$ is $V_t \subset T_{X_t}$, possibly singular. Under which conditions is

$$t \mapsto h^0(X_t, \mathcal{K}_{V_t}^{[m]})$$

locally constant over S ?

This would be very useful since one can easily produce jet sections for hypersurfaces $X \subset \mathbb{P}^{n+1}$ admitting meromorphic connections with low pole order (Siu, Nadel).

The end