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Recent progress towards the Green-Griffiths-Lang and Kobayashi conjectures

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Kobayashi hyperbolicity and entire curves

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A complex space X is said to be Kobayashi hyperbolic if the Kobayashi pseudodistance $d_{\mathrm{Kob}} : X \times X \to \mathbb{R}_+$ is a distance (i.e. everywhere non degenerate).

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For a compact complex manifold X, $dim_{\mathbb{C}} X = n$, TFAE:

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(ii) X is Brody hyperbolic, i.e. $\nexists$ entire curves $f : \mathbb{C} \to X$

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Our interest is the study of hyperbolicity for projective varieties. In dim 1, X is hyperbolic iff genus $g \geq 2$.

Main conjectures

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Arithmetic counterpart (Lang 1987): If X is projective and defined over a number field, the smallest locus $Y = \mathrm{GGL}(X)$ in GGL's conjecture is also the smallest Y such that $X(\mathbb{K}) \smallsetminus Y$ is finite $\forall \mathbb{K}$.

Results on the Kobayashi conjecture

Kobayashi conjecture (1970)

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Theorem (D., El Goul, 1998)

A very generic surface $X \subset \mathbb{P}^3$ of degree $d \geq 21$ is hyperbolic. Independently McQuillan got $d \geq 35$.

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In 2012, Yum-Tong Siu announced a proof of the case of arbitrary dimension n, with a very large d_n (and a rather involved proof).

Results on the generic Green-Griffiths conjecture

By a combination of an algebraic existence theorem for jet differentials and of Siu's technique of "slanted vector fields" (itself derived from ideas of H. Clemens, L. Ein and C. Voisin), the following was proved:

Theorem (S. Diverio, J. Merker, E. Rousseau, 2009)

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The bound was improved by (D-, 2012) to

$$d_n = \left\lfloor \frac{n^4}{3}\big(n\log(n\log(24n))\big)^n \right\rfloor = O(\exp(n^{1+\varepsilon})), \quad \forall\varepsilon > 0.$$

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Theorem (S. Diverio, S. Trapani, 2009)

Additionally, a generic hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 593$ is hyperbolic.

Category of directed manifolds

- **Goal.** More generally, we are interested in curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$ where V is a subbundle of T_X (or singular linear subspace, i.e. a closed irreducible analytic subspace such that $\forall x \in X,\ V_x := V \cap T_{X,x}$ linear).

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  - "Absolute case" (X, T_X), i.e. $V = T_X$
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- **Fonctor "1-jet" :** $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :

$$\tilde{X} = P(V) = \text{bundle of projective spaces of lines in } V$$

$$\pi : \tilde{X} = P(V) \to X, \quad (x, [v]) \mapsto x, \quad v \in V_x$$

$$\tilde{V}_{(x,[v])} = \{\xi \in T_{\tilde{X},(x,[v])}\ ;\ \pi_*\xi \in \mathbb{C}v \subset T_{X,x}\}$$

Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

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$$0 \to T_{\tilde{X}/X} \to \tilde{V} \xrightarrow{\pi_\star} \mathcal{O}_{\tilde{X}}(-1) \to 0 \qquad \Rightarrow \text{rk}\, \tilde{V} = r = \text{rk}\, V$$

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Direct image formula

For $n = \dim X$ and $r = \mathrm{rk}\, V$, one gets a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\mathrm{rk}\, V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

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Theorem

X_k is a smooth compactification of $X_k^{\mathrm{GG,reg}}/\mathbb{G}_k = J_k^{\mathrm{GG,reg}}/\mathbb{G}_k$, where $\mathbb{G}_k$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization: $(f, \varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k-jets of regular curves.

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Direct image formula

$(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = E_{k,m} V^* =$ invariant algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.

Definition of algebraic differential operators

Let $(\mathbb{C}, T_{\mathbb{C}}) \to (X, V), \quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$ be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X. It has a local Taylor expansion

$$f(t) = x + t\xi_1 + \ldots + t^k\xi_k + O(t^{k+1}), \quad \xi_s = \frac{1}{s!}\nabla^s f(0)$$

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One considers the Green-Griffiths bundle $E_{k,m}^{\mathrm{GG}}V^*$ of polynomials of weighted degree m written locally in coordinate charts as

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(x)\xi_1^{\alpha_1}\ldots\xi_k^{\alpha_k}, \quad \xi_s \in V,$$

also viewed as algebraic differential operators

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2}\ldots f^{(k)}(t)^{\alpha_k}. \end{aligned}$$

Definition of algebraic differential operators [cont.]

Here $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet, and $a_{\alpha_1 \alpha_2 \ldots \alpha_k}(z)$ are supposed to holomorphic functions on X.

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The $\mathbb{G}_k$-action : $(f, \varphi) \mapsto f \circ \varphi$, yields in particular, $\varphi_\lambda(t) = \lambda t \Rightarrow (f \circ \varphi_\lambda)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$, whence a $\mathbb{C}^*$-action

$$\lambda \cdot (\xi_1, \xi_1, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k).$$

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$E_{k,m}^{\mathrm{GG}}$ is precisely the set of polynomials of weighted degree m, corresponding to coefficients $a_{\alpha_1 \ldots \alpha_k}$ with $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$.

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$E_{k,m}V^* \subset E_{k,m}^{\mathrm{GG}}V^*$ is the bundle of $\mathbb{G}_k$-"invariant" operators, i.e. such that

$$P((f \circ \varphi)_{[k]}) = \varphi'^m P(f_{[k]}) \circ \varphi, \quad \forall \varphi \in \mathbb{G}_k.$$

Canonical sheaf of a singular pair (X,V)

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When V is singular, we first introduce the rank 1 sheaf ${}^b\mathcal{K}_V$ of sections of $\det V^*$ that are locally bounded with respect to a smooth ambient metric on T_X. One can show that ${}^b\mathcal{K}_V$ is equal to the integral closure of the image of the natural morphism

$$\Lambda^r T_X^* \to \Lambda^r V^* \to \mathcal{L}_V := \text{invert. sheaf } (\Lambda^r V^*)^{**}$$

that is, if the image is $\mathcal{L}_V \otimes \mathcal{J}_V$, $\mathcal{J}_V \subset \mathcal{O}_X$,

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Consequence

If $\mu : \widetilde{X} \to X$ is a modification and $\widetilde{X}$ is equipped with the pull-back directed structure $\widetilde{V} = \overline{\tilde{\mu}^{-1}(V)}$, then

$${}^b\mathcal{K}_V \subset \mu_*({}^b\mathcal{K}_{\widetilde{V}}) \subset \mathcal{L}_V$$

and $\mu_*({}^b\mathcal{K}_{\widetilde{V}})$ increases with μ.

Canonical sheaf of a singular pair (X,V) [cont.]

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$$\mathcal{K}_V^{[m]} = \lim_{\mu} \uparrow \mu_*({}^b\mathcal{K}_{\widetilde{V}})^{\otimes m}, \quad ({}^b\mathcal{K}_V)^{\otimes m}\mathcal{K}_V^{[m]} \subset \mathcal{L}_V^{\otimes m}$$

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This generalizes the concept of reduced singularities of foliations, which is known to work only for surfaces.

Definition

We say that (X, V) is of general type if the pluricanonical sheaf sequence is big, i.e. $H^0(X, \mathcal{K}_V^{[m]})$ provides a generic embedding of X for a suitable $m \gg 1$.

Generalized GGL conjecture

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Strategy : fundamental vanishing theorem

[Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]
$\forall P \in H^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-A))$: global diff. operator on X (A ample divisor), $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one has $P(f_{[k]}) \equiv 0$.

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Theorem (D-, 2010)

Let (X, V) be of general type, such that ${}^b\mathcal{K}_V$ is a big rank 1 sheaf. Then $\exists$ many global sections P, $m \gg k \gg 1 \Rightarrow \exists$ alg. hypersurface $Z \subsetneq X_k$ s.t. every entire $f : (\mathbb{C}, T_{\mathbb{C}}) \mapsto (X, V)$ satisfies $f_{[k]}(\mathbb{C}) \subset Z$.

Finsler metric on the k-jet bundles

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Assuming that V is equipped with a hermitian metric h, one defines a "weighted Finsler metric" on $J^k V$ by taking $p = k!$ and

$$\Psi_{h_k}(f) := \Big(\sum_{1 \le s \le k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \Big)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

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Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$, with curvature form $(x, \xi_1, \ldots, \xi_k) \mapsto$

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi} \sum_{1 \le s \le k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor Θ_{V^*,h^*} and $\omega_{\mathrm{FS},k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\mathrm{GG}} \to X$.

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The expression gets simpler by using polar coordinates

$$x_s = |\xi_s|_h^{2p/s}, \qquad u_s = \xi_s/|\xi_s|_h = \nabla^s f(0)/|\nabla^s f(0)|.$$

Probabilistic interpretation of the curvature

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors u_s in the unit sphere bundle $SV \subset V$.

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The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2p/s} = 1$, so we can take here $x_s \ge 0$, $\sum x_s = 1$. This is essentially a sum of the form $\sum \frac{1}{s}\gamma(u_s)$ where u_s are random points of the sphere, and so as $k \to +\infty$ this can be estimated by a "Monte-Carlo" integral

$$\left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right) \int_{u\in SV} \gamma(u)\, du.$$

As γ is quadratic here, $\int_{u\in SV} \gamma(u)\, du = \frac{1}{r} \operatorname{Tr}(\gamma)$.

Main cohomological estimate

$\Rightarrow$ the leading term only involves the trace of Θ_{V^*,h^*}, i.e. the curvature of $(\det V^*, \det h^*)$, that can be taken > 0 if $\det V^*$ is big.

Corollary (D-, 2010)

Let (X, V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V, h) and (F, h_F) hermitian. Define

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^* \mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big),$$
$$\eta = \Theta_{\det V^*, \det h^*} + \Theta_{F,h_F}.$$

Then for all $q \geq 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have upper and lower bounds [$q = 0$ most useful!]

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \leq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r} \Big(\int_{X(\eta,q)} (-1)^q \eta^n + \frac{C}{\log k}\Big)$$

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Alternatively, one could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage $(\mathcal{X}_k, \mathcal{A}_k)$ of the "absolute Semple tower" associated with $(\mathcal{X}_0, \mathcal{A}_0) = (X, T_X)$ (so as to deal only with nonsingular ambient Semple bundles).

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This produces an induced directed subvariety

$$(Z, W) \subset (X_k, V_k).$$

It is easy to show that $\pi_{k,k-1}(Z) = X_{k-1} \Rightarrow \text{rk}\, W < \text{rk}\, V_k = \text{rk}\, V$.

Partial solution of GGL conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "strongly of general type" if it is of general type and for every irreducible alg. subvariety $Z \subsetneq X_k$ that projects onto X, $X_k \not\subset D_k := P(T_{X_{k-1}/X_{k-2}})$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, i.e. ${}^{b}\mathcal{K}_W \otimes \mathcal{O}_{X_k}(m)_{|Z}$ is big for some $m \in \mathbb{Q}_+$, after a suitable blow-up.

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Theorem (D-, 2014)

If (X, V) is strongly of general type, the Green-Griffiths-Lang conjecture holds true for (X, V), namely there $\exists Y \subsetneq X$ such that every non constant holomorphic curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfies $f(\mathbb{C}) \subset Y$.

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Proof: Induction on rank V, using existence of jet differentials.

Related stability property

Definition

Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geq 1$, $Z \not\subset D_k$, and $Z = X = X_0$ for $k = 0$, we define the slope of the corresponding directed variety (Z, W) to be $\mu_A(Z, W) =$

$$\frac{\inf\left\{\lambda \in \mathbb{Q}\,;\ \exists m \in \mathbb{Q}_+,\ {}^b\mathcal{K}_W \otimes \left(\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(\lambda A)\right)_{|Z} \text{ big on } Z\right\}}{\operatorname{rank} W}.$$

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Notice that (X, V) is of general type iff $\mu_A(X, V) < 0$.

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Notice that (X, V) is of general type iff $\mu_A(X, V) < 0$.

We say that (X, V) is A-jet-stable (resp. A-jet-semi-stable) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leq \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above.

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Observation. If (X, V) is of general type and A-jet-semi-stable, then (X, V) is strongly of general type.

Approach of the Kobayashi conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "algebraically jet-hyperbolic" if for every irreducible alg. subvariety $Z \subsetneq X_k$ s.t. $X_k \not\subset D_k$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ either has $W = 0$ or is of general type modulo $X_k \to X$.

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Theorem (D-, 2014)

If (X, V) is algebraically jet-hyperbolic, then (X, V) is Kobayashi (or Brody) hyperbolic, i.e. there are no entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.

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Now, the hope is that a (very) generic complete intersection $X = H_1 \cap \ldots \cap H_c \subset \mathbb{P}^{n+c}$ of codimension c and degrees $(d_1, ..., d_c)$ s.t. $\sum d_j \geq 2n + c$ yields (X, T_X) algebraically jet-hyperbolic.

Invariance of plurigenera (?)

One way to check the above property, at least with non optimal bounds, would be to show some sort of Zariski openness of the properties “strongly of general type” or “algebraically jet-hyperbolic”.

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Question

Let $(\mathcal{X}, \mathcal{V}) \to S$ be a proper family of directed varieties over a base S, such that $\pi : \mathcal{X} \to S$ is a nonsingular deformation and the directed structure on $X_t = \pi^{-1}(t)$ is $V_t \subset T_{X_t}$, possibly singular. Under which conditions is

$$t \mapsto h^0(X_t, \mathcal{K}_{V_t}^{[m]})$$

locally constant over S ?

This would be very useful since one can easily produce jet sections for hypersurfaces $X \subset \mathbb{P}^{n+1}$ admitting meromorphic connections with low pole order (Siu, Nadel).