UNIVERSITÉ Grenoble Alpes

INSTITUT DE FRANCE
Académie des sciences

On the structure of compact Kähler manifolds with nef anticanonical bundles

Jean-Pierre Demailly

Institut Fourier, Université Grenoble Alpes, France
& Académie des Sciences de Paris

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- $L \to X$ is pseudoeffective (psef) if $\exists h = e^{-\varphi}$, $\varphi \in L^1_{\rm loc}$, s.t.

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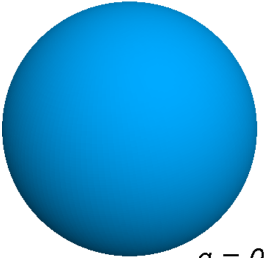
- L is nef if $\forall \varepsilon > 0$, $\exists h_\varepsilon = e^{-\varphi_\varepsilon}$ smooth such that

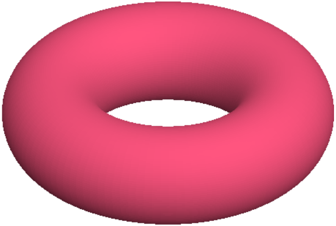
$$\Theta_{L,h_\varepsilon} = -dd^c \log h_\varepsilon \ge -\varepsilon\omega \text{ on } X$$

$\Leftrightarrow$ (for X projective) $L \cdot C \ge 0$, $\forall C$ algebraic curve.

Complex curves ($n = 1$) : genus and curvature



$g = 0$, $K_X < 0$
(positive curvature)



$g = 1$, $K_X = 0$
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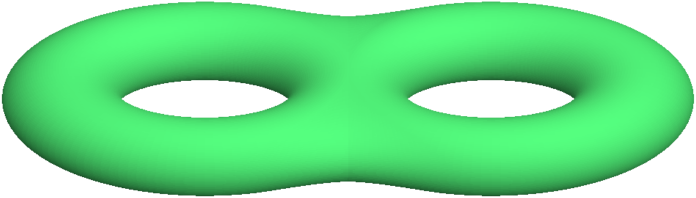

$g > 1$, $K_X > 0$
(negative curvature)

$K_X = \Lambda^n T_X^*$, $\deg(K_X) = 2g - 2$

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positive $\Leftrightarrow$ ample $\Rightarrow$ semi-ample $\Rightarrow$ semi-positive $\Rightarrow$ nef $\Rightarrow$ psef

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Example

Let X be the rational surface obtained by blowing up $\mathbb{P}^2$ in 9 distinct points $\{p_i\}$ on a smooth (cubic) elliptic curve $C \subset \mathbb{P}^2$, $\mu : X \to \mathbb{P}^2$ and $\hat{C}$ the strict transform of C.

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$$K_X = \mu^* K_{\mathbb{P}^2} \otimes \mathcal{O}(\sum E_i) \Rightarrow -K_X = \mu^* \mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}(-\sum E_i),$$

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One has

$$-K_X \cdot \Gamma = \hat{C} \cdot \Gamma \geq 0 \qquad \text{if } \Gamma \neq \hat{C},$$

$$-K_X \cdot \hat{C} = (-K_X)^2 = (\hat{C})^2 = C^2 - 9 = 0 \quad \Rightarrow \quad -K_X \text{ nef.}$$

In fact

$$G := (-K_X)_{|\hat{C}} \simeq \mathcal{O}_{\mathbb{P}^2|C}(3) \otimes \mathcal{O}_C(-\sum p_i) \in \mathrm{Pic}^0(C)$$

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Remark. $X = \mathbb{P}^2$ blown-up in ≥ 10 points is RC but $-K_X$ not nef.

Ex. of compact Kähler manifolds with $-K_X \geq 0$

(Recall: By Yau, $-K_X \geq 0 \Leftrightarrow \exists\omega$ Kähler with Ricci$(\omega) \geq 0$.)

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- all products $T \times \prod S_j \times \prod Y_k \times \prod Z_\ell$.

Main result. Essentially, this is a complete list !

Structure theorem for manifolds with $-K_X \geq 0$

Theorem [Campana, D., Peternell, 2012]

Let X be a compact Kähler manifold with $-K_X \geq 0$. Then:

(a) $\exists$ holomorphic and isometric splitting in irreducible factors

$$\widetilde{X} = \text{universal cover of } X \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times \prod Z_\ell$$

where $Y_j =$ Calabi-Yau (holonomy $\mathrm{SU}(n_j)$),
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(b) There exists a finite étale Galois cover $\widehat{X} \to X$ such that the Albanese map $\alpha : \widehat{X} \to \mathrm{Alb}(\widehat{X})$ is an (isometrically) locally trivial holomorphic fiber bundle whose fibers are products $\prod Y_j \times \prod S_k \times \prod Z_\ell$, as described in (a).

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(c) $\pi_1(\widehat{X}) \simeq \mathbb{Z}^{2q} \rtimes \Gamma$, Γ finite ("almost abelian" group).

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(d) For some (resp. for any) ample line bundle A on X, there exists a constant $C_A > 0$ such that

$$H^0(X, (T_X^*)^{\otimes m} \otimes A^{\otimes k}) = 0 \quad \forall m, k \in \mathbb{N}^* \text{ with } m \geq C_A k.$$

Proof of the RC criterion

Proof (essentially from Peternell 2006)

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By BDPP (2004), Y not uniruled $\Rightarrow K_Y$ psef. Then $\pi^* K_Y \hookrightarrow \Omega_X^p$ where $p = \dim Y$, which is a contradiction unless $p = 0$, and therefore X is RC.

Generalized holonomy principle

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Let $(E, h) \to X$ be a hermitian holomorphic vector bundle of rank r over X compact$/\mathbb{C}$. Assume that

$$\Theta_{E,h} \wedge \frac{\omega^{n-1}}{(n-1)!} = B\,\frac{\omega^n}{n!}, \quad B \in \mathrm{Herm}(E,E), \quad B \geq 0 \ \text{ on } X.$$

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Let H the restricted holonomy group of (E, h). Then

(a) If there exists a psef invertible sheaf $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$, then $\mathcal{L}$ is flat and invariant under parallel transport by the connection of $(E^*)^{\otimes m}$ induced by the Chern connection ∇ of (E, h) ; moreover, H acts trivially on $\mathcal{L}$.

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Proof. $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$ which has trace of curvature ≤ 0 while $\Theta_{\mathcal{L}} \geq 0$, use Bochner formula. □

Surjectivity of the Albanese morphism

Recall that if X is a compact Kähler manifold, the Albanese map

$$\alpha_X : X \to \mathrm{Alb}(X) := \mathbb{C}^q/\Lambda$$

is the holomorphic map given by

$$z \mapsto \alpha_X(z) = \Big(\int_{z_0}^{z} u_j\Big)_{1\le j\le q} \quad \text{mod period subgroup } \Lambda \subset \mathbb{C}^q,$$

where $(u_1, \ldots, u_q)$ is a basis of $H^0(X, \Omega^1_X)$.

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Proof. Based on characteristic p techniques.

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Proof. Based on variation arguments for twisted Kähler-Einstein metrics.

Approach via generically nef vector bundles (J.Cao)

Definition

Let X compact Kähler manifold, $\mathcal{E} \to X$ torsion free sheaf.

(a) $\mathcal{E}$ is generically nef with respect to a Kähler class ω if

$$\mu_\omega(\mathcal{S}) = \omega\text{-slope of } \mathcal{S} := \frac{\int_X c_1(\mathcal{S}) \wedge \omega^{n-1}}{\mathrm{rank}\,\mathcal{S}} \geq 0$$

for all torsion free quotients $\mathcal{E} \to \mathcal{S} \to 0$.
If $\mathcal{E}$ is ω-generically nef for all ω, we simply say that $\mathcal{E}$ is generically nef.

Approach via generically nef vector bundles (J.Cao)

Definition

Let X compact Kähler manifold, $\mathcal{E} \to X$ torsion free sheaf.

(a) $\mathcal{E}$ is generically nef with respect to a Kähler class ω if

$$\mu_\omega(\mathcal{S}) = \omega\text{-slope of } \mathcal{S} := \frac{\int_X c_1(\mathcal{S}) \wedge \omega^{n-1}}{\operatorname{rank} \mathcal{S}} \geq 0$$

for all torsion free quotients $\mathcal{E} \to \mathcal{S} \to 0$.
If $\mathcal{E}$ is ω-generically nef for all ω, we simply say that $\mathcal{E}$ is generically nef.

(b) Let

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s = \mathcal{E}$$

be a filtration of $\mathcal{E}$ by torsion free coherent subsheaves such that the quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ are ω-stable subsheaves of $\mathcal{E}/\mathcal{E}_i$ of maximal rank. We call such a sequence a refined Harder-Narasimhan (HN) filtration w.r.t. ω.

Characterization of generically nef vector bundles

It is a standard fact that refined HN-filtrations always exist, moreover

$$\mu_\omega(\mathcal{E}_i/\mathcal{E}_{i-1}) \geq \nu_\omega(\mathcal{E}_{i+1}/\mathcal{E}_i)$$

for all i.

Proposition

Let (X, ω) be a compact Kähler manifold and $\mathcal{E}$ a torsion free sehaf on X. Then $\mathcal{E}$ is ω-generically nef if and only if

$$\mu_\omega(\mathcal{E}_{i+1}/\mathcal{E}_i) \geq 0$$

for some refined HN-filtration.

Proof. Easy arguments on filtrations. □

A result of J. Cao about manifolds with $-K_X$ nef

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(Junyan Cao, 2013) Let X be a compact Kähler manifold with $-K_X$ nef. Then the tangent bundle T_X is ω-generically nef for all Kähler classes ω.

Proof. use the fact that $\forall\varepsilon > 0$, $\exists$ Kähler metric with $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\,\omega_\varepsilon$ (Yau, DPS 1995).

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From this, one can deduce

Theorem

Let X be a compact Kähler manifold with nef anticanonical bundle. Then the bundles $T_X^{\otimes m}$ are ω-generically nef for all Kähler classes ω and all positive integers m. In particular, the bundles $S^k T_X$ and $\bigwedge^p T_X$ are ω-generically nef.

A lemma on sections of contravariant tensors

Lemma

Let (X, ω) be a compact Kähler manifold with $-K_X$ nef and

$$\eta \in H^0(X, (\Omega_X^1)^{\otimes m} \otimes \mathcal{L})$$

where $\mathcal{L}$ is a numerically trivial line bundle on X.

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Proof. By Cao's theorem there exists a refined HN-filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s = T_X^{\otimes m}$$

with ω-stable quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ such that $\mu_\omega(\mathcal{E}_{i+1}/\mathcal{E}_i) \geq 0$ for all i. Then we use the fact that any section in a (semi-)negative slope reflexive sheaf $\mathcal{E}_{i+1}/\mathcal{E}_i \otimes \mathcal{L}$ is parallel w.r.t. its Bando-Siu metric (resp. vanishes). □

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Corollary

Let X be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map $\alpha_X : X \to \mathrm{Alb}(X)$ is a submersion on the complement of the HN filtration singular locus in X [$\Rightarrow \alpha_X$ surjects onto $\mathrm{Alb}(X)$].

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Cao's thm $\Rightarrow$ rank of $(du_1, \ldots, du_q)$ is $= q$ generically. □

Isotriviality of the Albanese map

Theorem [J. Cao, arXiv:1612.05921]

Let X be a projective manifold with nef anti-canonical bundle. Then the Albanese map $\alpha_X : X \to Y = \mathrm{Alb}(X)$ is locally isotrivial, i.e., for any small open set $U \subset Y$, $\alpha_X^{-1}(U)$ is biholomorphic to the product $U \times F$, where F is the generic fiber of α_X.

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Proof. Let A be a (large) ample line bundle on X and $E = (\alpha_X)_* A$ its direct image. Then $E = (\alpha_X)_*(mK_{X/Y} + L)$ with $L = A - mK_{X/Y} = A - mK_X$ nef. By results of Berndtsson-Păun on direct images, one can show that $\det E$ is pseudoeffective.

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Using arguments of [DPS95], one can infer that $E' = E \otimes (\det E)^{-1/r}$, $r = \mathrm{rank}(E)$, is numerically flat, hence a locally constant coefficient system (Simpson, Deng Ya). However, if $A \gg 0$, E provides equations of the fibers.

The simply connected case

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Corollary [Junyan Cao, Andreas Höring]

Let X be a projective manifold such that $-K_X$ is nef. Then after replacing X with a finite étale cover, the Albanese map α_X is isotrivial and its fibers are of the form $\prod S_j \times \prod Y_k \times \prod Z_\ell$ with S_j holomorphic symplectic, Y_k Calabi-Yau and Z_ℓ rationally connected.

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Expected more general definition

A compact Kähler manifold X is a singular Calabi-Yau if X has a non singular model X' satisfying $\pi_1(X') = 0$ and $K_{X'} = E$ for an effective divisor E of numerical dimension 0, and $H^0(X', \Omega^p_{X'}) = 0$ for $0 < p < \dim X$.

Further problems (II)

Definition

A compact Kähler manifold $X = X^{2p}$ is a singular hyperkähler manifold if X has a non singular model X' satisfying $\pi_1(X') = 0$ and possessing a section $\sigma \in H^0(X', \Omega^2_{X'})$ such that the zero divisor $E = \mathrm{div}(\sigma^p)$ has numerical dimension 0 (so that $K_{X'} = E$ again).

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Conjecture (known by BDPP for X projective!)

Let X be compact Kähler, and let $X \to Y$ be the MRC fibration (after taking suitable blow-ups to make it a genuine morphism). Then K_Y is psef.

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Let X be compact Kähler, and let $X \to Y$ be the MRC fibration (after taking suitable blow-ups to make it a genuine morphism). Then K_Y is psef.

Proof ? Take the part of slope > 0 in the HN filtration of T_X, w.r.t. to classes in the dual of the psef cone, show that this corresponds to the MRC fibration, and apply duality.

Further problems (III)

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They might possibly include all rationally connected varieties.

Thank you for your attention!