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# On the structure of compact Kähler manifolds with nef anticanonical bundles


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Complex Analysis and Geometry – XXIII, June 13, 2017



## Goals / main positivity concepts

- Analyze the structure of projective or compact Kähler manifolds $X$ with $-K_X$ nef.
- As is well known since the beginning of the XX$^{\rm th}$ century at least, the geometry depends on the sign of the curvature of the canonical line bundle

  $K_X = \Lambda^n T^*_X, \quad n = \dim_{\mathbb{C}} X.$
- $L \to X$ is pseudoeffective (psef) if $\exists h = e^{-\varphi}$, $\varphi \in L^1_{\rm loc}$, s.t.

  $\Theta_{L,h} = -dd^c \log h \geq 0$ on $X$ in the sense of currents

$\Leftrightarrow$ (for $X$ projective) $c_1(L) \in \overline{\rm Eff}$.
- $L \to X$ is semi-positive if $\exists h = e^{-\varphi}$ smooth ($C^\infty$) such that

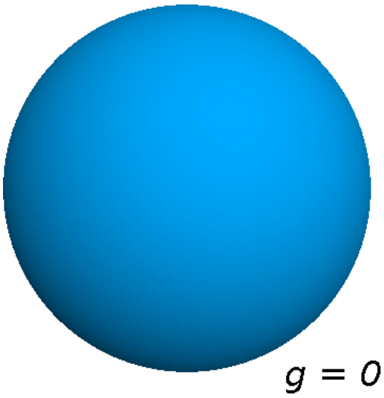
  $\Theta_{L,h} = -dd^c \log h \geq 0$ on $X$.

$\Leftarrow$ (for $X$ projective) $L^{\otimes m} = G \otimes H$, $G$ semi-ample, $H \in {\rm Pic}^0(X)$.
- $L$ is nef if $\forall \varepsilon > 0$, $\exists h_\varepsilon = e^{-\varphi_\varepsilon}$ smooth such that

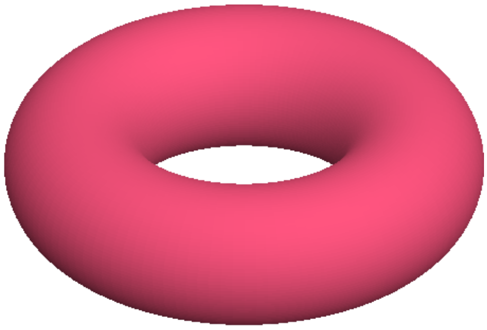
  $\Theta_{L,h_\varepsilon} = -dd^c \log h_\varepsilon \geq -\varepsilon\omega$ on $X$

$\Leftrightarrow$ (for $X$ projective) $L \cdot C \geq 0$, $\forall C$ algebraic curve.

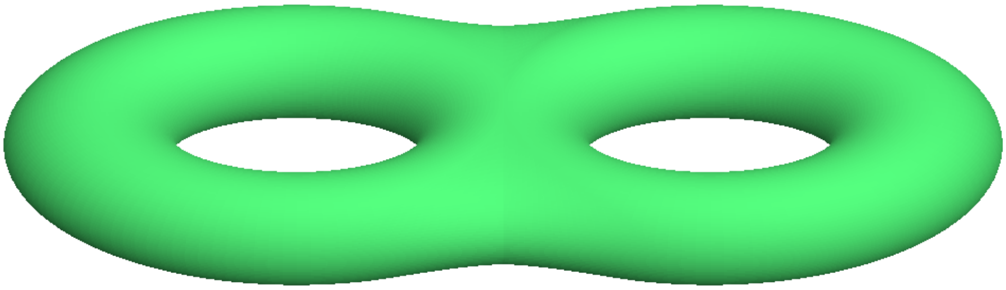
## Complex curves ($n = 1$) : genus and curvature



$g = 0\,,\ K_X < 0$
(positive curvature)



$g = 1\,,\ K_X = 0$
(zero curvature)



$g > 1\,,\ K_X > 0$
(negative curvature)

$K_X = \Lambda^n T^*_X, \quad \deg(K_X) = 2g - 2$



## Comparison of positivity concepts

Recall that for a line bundle

positive $\Leftrightarrow$ ample $\Rightarrow$ semi-ample $\Rightarrow$ semi-positive $\Rightarrow$ nef $\Rightarrow$ psef

but none of the reverse implications in red holds true.

**Example**

Let $X$ be the rational surface obtained by blowing up $\mathbb{P}^2$ in 9 distinct points $\{p_i\}$ on a smooth (cubic) elliptic curve $C \subset \mathbb{P}^2$, $\mu : X \to \mathbb{P}^2$ and $\hat{C}$ the strict transform of $C$. Then

$$K_X = \mu^* K_{\mathbb{P}^2} \otimes \mathcal{O}(\textstyle\sum E_i) \Rightarrow -K_X = \mu^* \mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}(-\textstyle\sum E_i),$$

thus

$$-K_X = \mu^* \mathcal{O}_{\mathbb{P}^2}(C) \otimes \mathcal{O}(-\textstyle\sum E_i) = \mathcal{O}_X(\hat{C}).$$

One has

$$-K_X \cdot \Gamma = \hat{C} \cdot \Gamma \geq 0 \qquad \text{if } \Gamma \neq \hat{C},$$

$$-K_X \cdot \hat{C} = (-K_X)^2 = (\hat{C})^2 = C^2 - 9 = 0 \quad \Rightarrow \quad -K_X \text{ nef.}$$

## Rationally connected manifolds

In fact

$$G := (-K_X)_{|\hat{C}} \simeq \mathcal{O}_{\mathbb{P}^2|C}(3) \otimes \mathcal{O}_C(-\sum p_i) \in \mathrm{Pic}^0(C)$$

If $G$ is a torsion point in $\mathrm{Pic}^0(C)$, then one can show that $-K_X$ is semi-ample, but otherwise it is not semi-ample.

Brunella has shown that $-K_X$ is $C^\infty$ semi-positive if $c_1(G)$ satisfies a diophantine condition found by T. Ueda, but that otherwise it may not be semi-positive (although nef).

$\mathbb{P}^2 \# 9$ points is an example of rationally connected manifold:

**Definition**

Recall that a compact complex manifold is said to be rationally connected (or RC for short) if any 2 points can be joined by a chain of rational curves

Remark. $X = \mathbb{P}^2$ blown-up in $\geq 10$ points is RC but $-K_X$ not nef.



## Ex. of compact Kähler manifolds with $-K_X \geq 0$

(Recall: By Yau, $-K_X \geq 0 \Leftrightarrow \exists\omega$ Kähler with $\mathrm{Ricci}(\omega) \geq 0$.)

- Ricci flat manifolds
  - Complex tori $T = \mathbb{C}^q/\Lambda$
  - Holomorphic symplectic manifolds $S$ (also called hyperkähler): $\exists\sigma \in H^0(S, \Omega_S^2)$ symplectic
  - Calabi-Yau manifolds $Y$: $\pi_1(Y)$ finite and some multiple of $K_Y$ is trivial (may assume $\pi_1(Y) = 1$ and $K_Y$ trivial by passing to some finite étale cover)
- the rather large class of rationally connected manifolds $Z$ with $-K_Z \geq 0$
- all products $T \times \prod S_j \times \prod Y_k \times \prod Z_\ell$.

Main result. Essentially, this is a complete list !

## Structure theorem for manifolds with $-K_X \ge 0$

**Theorem [Campana, D., Peternell, 2012]**

Let $X$ be a compact Kähler manifold with $-K_X \ge 0$. Then:

(a) $\exists$ holomorphic and isometric splitting in irreducible factors

$$\widetilde{X} = \text{universal cover of } X \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times \prod Z_\ell$$

where $Y_j$ = Calabi-Yau (holonomy $\mathrm{SU}(n_j)$), $S_k$ = holomorphic symplectic (holonomy $\mathrm{Sp}(n'_k/2)$), and $Z_\ell$ = RC with $-K_{Z_\ell} \ge 0$ (holonomy $\mathrm{U}(n''_\ell)$).

(b) There exists a finite étale Galois cover $\widehat{X} \to X$ such that the Albanese map $\alpha : \widehat{X} \to \mathrm{Alb}(\widehat{X})$ is an (isometrically) locally trivial holomorphic fiber bundle whose fibers are products $\prod Y_j \times \prod S_k \times \prod Z_\ell$, as described in (a).

(c) $\pi_1(\widehat{X}) \simeq \mathbb{Z}^{2q} \rtimes \Gamma$, $\Gamma$ finite ("almost abelian" group).



## Criterion for rational connectedness

**Criterion**

Let $X$ be a projective algebraic $n$-dimensional manifold. The following properties are equivalent.

(a) $X$ is rationally connected.

(b) For every invertible subsheaf $\mathcal{F} \subset \Omega^p_X := \mathcal{O}(\Lambda^p T^*_X)$, $1 \le p \le n$, $\mathcal{F}$ is not psef.

(c) For every invertible subsheaf $\mathcal{F} \subset \mathcal{O}((T^*_X)^{\otimes p})$, $p \ge 1$, $\mathcal{F}$ is not psef.

(d) For some (resp. for any) ample line bundle $A$ on $X$, there exists a constant $C_A > 0$ such that

$$H^0(X, (T^*_X)^{\otimes m} \otimes A^{\otimes k}) = 0 \quad \forall m, k \in \mathbb{N}^* \text{ with } m \ge C_A k.$$

# Proof of the RC criterion

Proof (essentially from Peternell 2006)

(a) $\Rightarrow$ (d) is easy (RC implies there are many rational curves on which $T_X$, so $T_X^* < 0$), (d) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) are trivial.

Thus the only thing left to complete the proof is (b) $\Rightarrow$ (a).

Consider the MRC quotient $\pi : X \to Y$, given by the "equivalence relation $x \sim y$ if $x$ and $y$ can be joined by a chain of rational curves.

Then (by definition) the fibers are RC, maximal, and a result of Graber-Harris-Starr (2002) implies that $Y$ is not uniruled.

By BDPP (2004), $Y$ not uniruled $\Rightarrow K_Y$ psef. Then $\pi^* K_Y \hookrightarrow \Omega_X^p$ where $p = \dim Y$, which is a contradiction unless $p = 0$, and therefore $X$ is RC.



# Generalized holonomy principle

**Generalized holonomy principle**

Let $(E, h) \to X$ be a hermitian holomorphic vector bundle of rank $r$ over $X$ compact/$\mathbb{C}$. Assume that

$$\Theta_{E,h} \wedge \frac{\omega^{n-1}}{(n-1)!} = B\,\frac{\omega^n}{n!}, \quad B \in \mathrm{Herm}(E,E), \quad B \geq 0 \text{ on } X.$$

Let $H$ the restricted holonomy group of $(E, h)$. Then

(a) If there exists a psef invertible sheaf $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$, then $\mathcal{L}$ is flat and invariant under parallel transport by the connection of $(E^*)^{\otimes m}$ induced by the Chern connection $\nabla$ of $(E, h)$; moreover, $H$ acts trivially on $\mathcal{L}$.

(b) If $H$ satisfies $H = \mathrm{U}(r)$, then none of the invertible sheaves $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$ can be psef for $m \geq 1$.

Proof. $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$ which has trace of curvature $\leq 0$ while $\Theta_{\mathcal{L}} \geq 0$, use Bochner formula. □

# Surjectivity of the Albanese morphism

Recall that if $X$ is a compact Kähler manifold, the Albanese map

$$\alpha_X : X \to \mathrm{Alb}(X) := \mathbb{C}^q/\Lambda$$

is the holomorphic map given by

$$z \mapsto \alpha_X(z) = \Big(\int_{z_0}^{z} u_j\Big)_{1\le j\le q} \quad \text{mod period subgroup } \Lambda \subset \mathbb{C}^q,$$

where $(u_1, \ldots, u_q)$ is a basis of $H^0(X, \Omega_X^1)$.

**Theorem [Qi Zhang, 2005]**

If $X$ is projective and $-K_X$ is nef, then $\alpha_X$ is surjective.

Proof. Based on characteristic $p$ techniques.

**Theorem [M. Păun, 2012]**

If $X$ is compact Kähler and $-K_X$ is nef, then $\alpha_X$ is surjective.

Proof. Based on variation arguments for twisted Kähler-Einstein metrics.



# Approach via generically nef vector bundles (J.Cao)

**Definition**

Let $X$ compact Kähler manifold, $\mathcal{E} \to X$ torsion free sheaf.

(a) $\mathcal{E}$ is generically nef with respect to a Kähler class $\omega$ if

$$\mu_\omega(\mathcal{S}) = \omega\text{-slope of } \mathcal{S} := \frac{\int_X c_1(\mathcal{S}) \wedge \omega^{n-1}}{\mathrm{rank}\,\mathcal{S}} \ge 0$$

for all torsion free quotients $\mathcal{E} \to \mathcal{S} \to 0$.
If $\mathcal{E}$ is $\omega$-generically nef for all $\omega$, we simply say that $\mathcal{E}$ is generically nef.

(b) Let

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s = \mathcal{E}$$

be a filtration of $\mathcal{E}$ by torsion free coherent subsheaves such that the quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ are $\omega$-stable subsheaves of $\mathcal{E}/\mathcal{E}_i$ of maximal rank. We call such a sequence a refined Harder-Narasimhan (HN) filtration w.r.t. $\omega$.

# Characterization of generically nef vector bundles

It is a standard fact that refined HN-filtrations always exist, moreover

$$\mu_\omega(\mathcal{E}_i/\mathcal{E}_{i-1}) \geq \nu_\omega(\mathcal{E}_{i+1}/\mathcal{E}_i)$$

for all $i$.

**Proposition**

Let $(X, \omega)$ be a compact Kähler manifold and $\mathcal{E}$ a torsion free sehaf on $X$. Then $\mathcal{E}$ is $\omega$-generically nef if and only if

$$\mu_\omega(\mathcal{E}_{i+1}/\mathcal{E}_i) \geq 0$$

for some refined HN-filtration.

*Proof.* Easy arguments on filtrations. □



# A result of J. Cao about manifolds with $-K_X$ nef

**Theorem**

(Junyan Cao, 2013) Let $X$ be a compact Kähler manifold with $-K_X$ nef. Then the tangent bundle $T_X$ is $\omega$-generically nef for all Kähler classes $\omega$.

*Proof.* use the fact that $\forall \varepsilon > 0$, $\exists$ Kähler metric with $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\,\omega_\varepsilon$ (Yau, DPS 1995).

From this, one can deduce

**Theorem**

Let $X$ be a compact Kähler manifold with nef anticanonical bundle. Then the bundles $T_X^{\otimes m}$ are $\omega$-generically nef for all Kähler classes $\omega$ and all positive integers $m$. In particular, the bundles $S^k T_X$ and $\bigwedge^p T_X$ are $\omega$-generically nef.

# A lemma on sections of contravariant tensors

**Lemma**

Let $(X,\omega)$ be a compact Kähler manifold with $-K_X$ nef and

$$\eta \in H^0(X, (\Omega^1_X)^{\otimes m} \otimes \mathcal{L})$$

where $\mathcal{L}$ is a numerically trivial line bundle on $X$.
Then the filtered parts of $\eta$ w.r.t. the refined HN filtration are parallel w.r.t. the Bando-Siu metric in the 0 slope parts, and the $< 0$ slope parts vanish.

*Proof.* By Cao's theorem there exists a refined HN-filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s = T_X^{\otimes m}$$

with $\omega$-stable quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ such that $\mu_\omega(\mathcal{E}_{i+1}/\mathcal{E}_i) \geq 0$ for all $i$. Then we use the fact that any section in a (semi-)negative slope reflexive sheaf $\mathcal{E}_{i+1}/\mathcal{E}_i \otimes \mathcal{L}$ is parallel w.r.t. its Bando-Siu metric (resp. vanishes). □



# Smoothness of the Albanese morphism (after Cao)

**Theorem (Junyan Cao 2013)**

Non-zero holomorphic $p$-forms on a compact Kähler manifold $X$ with $-K_X$ nef vanish only on the singular locus of the refined HN filtration of $T^*X$.

This already implies the following result.

**Corollary**

Let $X$ be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map $\alpha_X : X \to \mathrm{Alb}(X)$ is a submersion on the complement of the HN filtration singular locus in X [ $\Rightarrow \alpha_X$ surjects onto $\mathrm{Alb}(X)$ ].

*Proof.* The differential $d\alpha_X$ is given by $(du_1, \ldots, du_q)$ where $(u_1, \ldots, u_q)$ is a basis of 1-forms, $q = \dim H^0(X, \Omega^1_X)$.

Cao's thm $\Rightarrow$ rank of $(du_1, \ldots, du_q)$ is $= q$ generically. □

## Isotriviality of the Albanese map

**Theorem [J. Cao, arXiv:1612.05921]**

Let $X$ be a projective manifold with nef anti-canonical bundle. Then the Albanese map $\alpha_X : X \to Y = \mathrm{Alb}(X)$ is locally isotrivial, i.e., for any small open set $U \subset Y$, $\alpha_X^{-1}(U)$ is biholomorphic to the product $U \times F$, where $F$ is the generic fiber of $\alpha_X$. Moreover $-K_F$ is again nef.

*Proof.* Let $A$ be a (large) ample line bundle on $X$ and $E = (\alpha_X)_* A$ its direct image. Then $E = (\alpha_X)_*(mK_{X/Y} + L)$ with $L = A - mK_{X/Y} = A - mK_X$ nef. By results of Berndtsson-Păun on direct images, one can show that $\det E$ is pseudoeffective. Using arguments of [DPS95], one can infer that $E' = E \otimes (\det E)^{-1/r}$, $r = \mathrm{rank}(E)$, is numerically flat, hence a locally constant coefficient system (Simpson, Deng Ya). However, if $A \gg 0$, $E$ provides equations of the fibers. □



## The simply connected case

The above results reduce the study of projective manifolds with $-K_X$ nef to the case when $\pi_1(X) = 0$.

**Theorem [Junyan Cao, Andreas Höring, 2 days ago!]**

Let $X$ be a projective manifold such that $-K_X$ is nef and $\pi_1(X) = 0$. Then $X = W \times Z$ with $K_W \sim 0$ and $Z$ is a rationally connected manifold.

**Corollary [Junyan Cao, Andreas Höring]**

Let $X$ be a projective manifold such that $-K_X$ is nef. Then after replacing $X$ with a finite étale cover, the Albanese map $\alpha_X$ is isotrivial and its fibers are of the form $\prod S_j \times \prod Y_k \times \prod Z_\ell$ with $S_j$ holomorphic symplectic, $Y_k$ Calabi-Yau and $Z_\ell$ rationally connected.

# Further problems (I)

## Partly solved questions

- Develop further the theory of singular Calabi-Yau and singular holomorphic symplectic manifolds (work of Greb-Kebekus-Peternell).
- Show that the "slope $\pm\varepsilon$" part corresponds to blown-up tori, singular Calabi-Yau and singular holomorphic symplectic manifolds (as fibers and targets).
- The rest of $T_X$ (slope $< 0$) should yield a general type orbifold quotient (according to conjectures of Campana).

## Expected more general definition

A compact Kähler manifold $X$ is a singular Calabi-Yau if $X$ has a non singular model $X'$ satisfying $\pi_1(X') = 0$ and $K_{X'} = E$ for an effective divisor $E$ of numerical dimension 0, and $H^0(X', \Omega^p_{X'}) = 0$ for $0 < p < \dim X$.



# Further problems (II)

## Definition

A compact Kähler manifold $X = X^{2p}$ is a singular hyperkähler manifold if $X$ has a non singular model $X'$ satisfying $\pi_1(X') = 0$ and possessing a section $\sigma \in H^0(X', \Omega^2_{X'})$ such that the zero divisor $E = \mathrm{div}(\sigma^p)$ has numerical dimension 0 (so that $K_{X'} = E$ again).

## Conjecture (known by BDPP for $X$ projective!)

Let $X$ be compact Kähler, and let $X \to Y$ be the MRC fibration (after taking suitable blow-ups to make it a genuine morphism). Then $K_Y$ is psef.

Proof ? Take the part of slope $> 0$ in the HN filtration of $T_X$, w.r.t. to classes in the dual of the psef cone, show that this corresponds to the MRC fibration, and apply duality.

## Further problems (III)

An interesting class of manifolds is the larger class of compact Kähler manifolds such that

$$K_X = E - D$$

where $D$ is a pseudoeffective divisor and $E$ an effective divisor of numerical dimension 0.

This class is obviously birationally invariant (while the condition $-K_X$ nef was not !).

One can hopefully expect similar decomposition theorems for varieties in this class.

They might possibly include all rationally connected varieties.



## The end

# Thank you for your attention!