



On the computational complexity of mathematical functions

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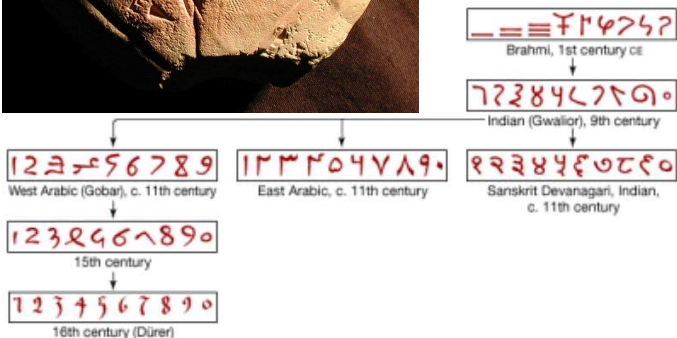
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Computing, a very old concern



Babylonian mathematical tablet allowing the computation of $\sqrt{2}$ (1800 – 1600 BC)

Decimal numeral system invented in India (~ 500 BC ?) :



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The next progress was the discovery of the first **infinite series** formula by Madhava (circa 1350 – 1450), a prominent mathematician-astronomer from Kerala (formula rediscovered in the XVIIe century by Leibniz and Gregory) :

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots + \frac{(-1)^n}{2n+1} + \dots$$

Convergence is unfortunately very slow, but Madhava was able to improve convergence and reached in this way 11 decimal places.

Ramanujan's formula for π



Srinivasa Ramanujan (1887 – 1920),
a self-taught mathematical prodigee.
His work dealt mainly with
arithmetics and function theory

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{+\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}} \quad (1910).$$

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Each term is approximately **10^8 times smaller** than the preceding one, so the convergence is very fast.

Computational complexity theory

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 - tries to classify problems according to their difficulty
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 - **linear complexity** when **$\# \text{ steps} \leq C N$**
 - **quadratic complexity** when **$\# \text{ steps} \leq C N^2$**
 - **quasi-linear complexity** when **$\# \text{ steps} \leq C_\epsilon N^{1+\epsilon}$** , $\forall \epsilon > 0$.

First observations about complexity

- Addition has **linear complexity**:

consider decimal numbers of the form $0.a_1a_2a_3 \dots a_N$,
 $0.b_1b_2b_3 \dots b_N$, we have

$$\sum_{1 \leq n \leq N} a_n 10^{-n} + \sum_{1 \leq n \leq N} b_n 10^{-n} = \sum_{1 \leq n \leq N} (a_n + b_n) 10^{-n},$$

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- What about multiplication ?

$$\sum_{1 \leq k \leq N} a_k 10^{-k} \times \sum_{1 \leq \ell \leq N} b_\ell 10^{-\ell} = \sum_{1 \leq n \leq N} c_n 10^{-n}, \quad c_n = \sum_{k+\ell=n} a_k b_\ell.$$

Calculation of each c_n requires at most N elementary multiplications and $N - 1$ additions and corresponding carries, thus the algorithm requires less than $N \times 3N$ steps.

Thus multiplication has at most **quadratic complexity**.

The Karatsuba algorithm

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Yes !! It was discovered by Karatsuba around 1960 that multiplication has complexity less than $C N^{\log_2 3} \simeq C N^{1.585}$

Karatsuba's idea: for $N = 2q$ even, split $x = 0.a_1a_2 \dots a_N$ as

$$x = x' + 10^{-q}x'', \quad x' = 0.a_1a_2 \dots a_q, \quad x'' = 0.a_{q+1}a_{q+2} \dots a_{2q}$$

and similarly $y = 0.b_1b_2 \dots b_N = y' + 10^{-q}y''$. To calculate xy , one would normally need $x'y'$, $x''y''$ and $x'y'' + x''y'$ which take 4 multiplications and 1 addition of q -digit numbers.

However, one can use only 3 multiplications by calculating

$$x'y', \quad x''y'', \quad x'y'' + x''y' = x'y' + x''y'' - (x' - x'')(y' - y'')$$

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(at the expense of 4 additions). One then proceeds inductively to conclude that the time $T(N)$ needed for $N = 2^s$ satisfies

$$T(2^s) \leq 3 T(2^{s-1}) + 4 \cdot 2^{s-1}.$$

Optimal complexity of multiplication

It is an easy exercise to conclude by induction that $T(2^s) \leq 6 \cdot 3^s - 4 \cdot 2^s$ if one assumes $T(1) = 1$, and so

$$T(2^s) \leq 6 \cdot 3^s \Rightarrow T(N) \leq C N^{\log_2 3}.$$

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The Schönage-Strassen algorithm is based on the use of **discrete Fourier transforms**. The theory comes from **Joseph Fourier**, the founder of my university in 1810...

Joseph Fourier



Joseph Fourier (1768 – 1830)
in his suit of member of
Académie des Sciences,
of which he became
“Secrétaire Perpétuel”
(Head) in 1822.

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In 1824, he predicts the **green house effect** !

Heat equation and Fourier series

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In fact all periodic phenomena can be described in this way. This is the basis of the modern theory of signal processing and electromagnetism.

Discrete Fourier transform

Let $(a_n)_{0 \leq n < N}$ be a finite sequence of numbers and let u be a primitive N -th root of unity, i.e.

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Main formulas of Fourier theory

Fourier transform of a convolution:

For $a = (a_n)$ and $b = (b_n)$ define $c = a * b$ to be the sequence

$$c_n = \sum_{p+q=n \pmod N} a_p b_q \quad \text{“convolution of } a \text{ and } b\text{.”}$$

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Proof.
$$\sum_s c_s u^{sn} = \sum_s \left(\sum_{k+l=s} a_k b_l \right) u^{sn} = \sum_{k,l} a_k u^{kn} b_l u^{ln} = \widehat{a}_n \widehat{b}_n.$$

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$$\widehat{\widehat{a}}_n = \sum_k \left(\sum_\ell a_\ell u^{k\ell} \right) u^{kn} = \sum_\ell a_\ell \left(\sum_k u^{k(n+\ell)} \right) \text{ and}$$

$$\sum_k u^{k(n+\ell)} = 0 \text{ if } \ell \neq -n \text{ and } \sum_k u^{k(n+\ell)} = N \text{ if } \ell = -n.$$

Fast Fourier Transform (FFT)

Consequence: To calculate the convolution $c = a * b$ (which is what we need to calculate $\sum a_k 10^{-k} \sum b_\ell 10^{-\ell}$), one calculates the Fourier transforms (\hat{a}_n) , (\hat{b}_n) , then $\hat{c}_n = \hat{a}_n \hat{b}_n$, which gives back $(-c_{-n})$ and thus (c_n) by **Fourier inversion**.

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FFT algorithm: assume that $N = 2^s$ (in our example $N = 65536 = 2^{16}$) and define inductively $\alpha_{n,0} = a_n$ and

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By considering the binary decomposition $n = \sum n_k 2^k$, $0 \leq k < s$, of any integer $n = 0 \dots N - 1$, one sees that $\alpha_{n,s} = \hat{a}_n$. The calculation requires only s steps, each of which requires N additions and $2N$ multiplications (using $u^{2^{k+1}n} = (u^{2^k n})^2$), so in total we consume only $3sN = 3N \log_2 N$ operations !

Other mathematical functions

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Approximate division can be obtained solely from multiplication!

If x_0 is a rough approximation of $1/a$, then the sequence

$$x_{n+1} = 2x_n - ax_n^2$$

satisfies $1 - ax_{n+1} = (1 - ax_n)^2$, and so inductively $1 - ax_n = (1 - ax_0)^{2^n}$ will converge extremely fast to 0. In fact if $|1 - ax_0| < 1/10$ and $n \sim \log_2 N$, we get already N correct digits. Hence we need iterating only $\log_2 N$ times the sequence, and so **division is also quasi-linear in time**.

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Similarly, square roots can be approximated by using only multiplications and divisions, thanks to the “Babylonian algorithm” :

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right), \quad x_0 > 0$$

What about π ?



In fact Carl-Friedrich Gauss (another mathematical prodigee...) discovered around 1797 the following formula for the **arithmetic-geometric mean**:

start from real numbers $a, b > 0$ and define inductively $a_0 = a$, $b_0 = b$ and

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.$$

What about π ?



In fact Carl-Friedrich Gauss (another mathematical prodigee...) discovered around 1797 the following formula for the **arithmetic-geometric mean**:

start from real numbers $a, b > 0$ and define inductively $a_0 = a$, $b_0 = b$ and

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.$$

Then (a_n) and (b_n) converge (extremely fast, only $\sim \log_2 N$ steps to get N correct digits) towards

$$M(a, b) = \frac{2\pi}{I(a, b)} \quad \text{where } I(a, b) = \int_0^{2\pi} \frac{dx}{\sqrt{a^2 \cos^2 x + b^2 \sin^2 x}}$$

(an “elliptic integral”).

The Brent-Salamin formula

Using this and another formula due to Legendre (1752 – 1833), Brent and Salamin found in 1976 a remarkable formula for π . Define

$$c_n = \sqrt{a_n^2 - b_n^2}$$

in the arithmetic-geometric sequence. Then

$$\pi = \frac{4 M(1, 1/\sqrt{2})^2}{1 - \sum_{n=1}^{+\infty} 2^{n+1} c_n^2}.$$

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This formula has been used several times to break the world record, which seems to be 5 trillions digits since 2010 (however, there exist so efficient quadratic complexity formulas that they are still competitive at that level...)

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leads to calculate n^2 coefficients, each of which requires n multiplications and $(n - 1)$ additions, so in total $n^2(2n - 1) \sim 2n^3$ operations.

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The fastest known algorithm, due to Coppersmith and Winograd in 1987 has $\#steps \leq C n^{2.38}$ (quite complicated!)