Entire curves and algebraic differential equations

Jean-Pierre Demailly

Institut Fourier, Université de Grenoble I, France

April 16 2009, Saint-Martin d'Hères
IF - IMPA Conference

Entire curves

- **Definition.** By an entire curve we mean a non constant holomorphic map $f : \mathbb{C} \to X$ into a complex n-dimensional manifold.

Entire curves

- **Definition.** By an entire curve we mean a non constant holomorphic map $f : \mathbb{C} \to X$ into a complex n-dimensional manifold.
- If X is a bounded open subset $\Omega \subset \mathbb{C}^n$, then there are no entire curves $f : \mathbb{C} \to \Omega$ (Liouville's theorem)

Entire curves

- **Definition.** By an entire curve we mean a non constant holomorphic map $f : \mathbb{C} \to X$ into a complex n-dimensional manifold.
- If X is a bounded open subset $\Omega \subset \mathbb{C}^n$, then there are no entire curves $f : \mathbb{C} \to \Omega$ (Liouville's theorem)
- $X = \overline{\mathbb{C}} \smallsetminus \{0, 1, \infty\} = \mathbb{C} \smallsetminus \{0, 1\}$ has no entire curves (Picard's theorem)

Entire curves

- **Definition.** By an entire curve we mean a non constant holomorphic map $f : \mathbb{C} \to X$ into a complex n-dimensional manifold.
- If X is a bounded open subset $\Omega \subset \mathbb{C}^n$, then there are no entire curves $f : \mathbb{C} \to \Omega$ (Liouville's theorem)
- $X = \overline{\mathbb{C}} \smallsetminus \{0, 1, \infty\} = \mathbb{C} \smallsetminus \{0, 1\}$ has no entire curves (Picard's theorem)
- A complex torus $X = \mathbb{C}^n/\Lambda$ (Λ lattice) has a lot of entire curves. As $\mathbb{C}$ simply connected, every $f : \mathbb{C} \to X = \mathbb{C}^n/\Lambda$ lifts as $\tilde{f} : \mathbb{C} \to \mathbb{C}^n$,

$$\tilde{f}(t) = (\tilde{f}_1(t), \ldots, \tilde{f}_n(t))$$

and $\tilde{f}_j : \mathbb{C} \to \mathbb{C}$ can be arbitrary entire functions.

Projective algebraic varieties

- Consider now the complex projective n-space

$$\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} \smallsetminus \{0\})/\mathbb{C}^*, \qquad [z] = [z_0 : z_1 : \ldots : z_n].$$

Projective algebraic varieties

- Consider now the complex projective n-space

$$\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} \smallsetminus \{0\})/\mathbb{C}^*, \qquad [z] = [z_0 : z_1 : \ldots : z_n].$$

- An entire curve $f : \mathbb{C} \to \mathbb{P}^n$ is given by a map

$$t \longmapsto [f_0(t) : f_1(t) : \ldots : f_n(t)]$$

where $f_j : \mathbb{C} \to \mathbb{C}$ are holomorphic functions without common zeroes (so there are a lot of them).

Projective algebraic varieties

- Consider now the complex projective n-space

$$\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} \smallsetminus \{0\})/\mathbb{C}^*, \qquad [z] = [z_0 : z_1 : \ldots : z_n].$$

- An entire curve $f : \mathbb{C} \to \mathbb{P}^n$ is given by a map

$$t \longmapsto [f_0(t) : f_1(t) : \ldots : f_n(t)]$$

  where $f_j : \mathbb{C} \to \mathbb{C}$ are holomorphic functions without common zeroes (so there are a lot of them).

- More generally, look at a (complex) projective manifold, i.e.

$$X^n \subset \mathbb{P}^N, \qquad X = \{[z]\,;\ P_1(z) = ... = P_k(z) = 0\}$$

  where $P_j(z) = P_j(z_0, z_1, \ldots, z_N)$ are homogeneous polynomials (of some degree d_j), such that X is non singular.

Kobayashi metric / hyperbolic manifolds

- For a complex manifold, $n = \dim_{\mathbb{C}} X$, one defines the Kobayashi pseudo-metric : $x \in X$, $\xi \in T_X$

$$\kappa_x(\xi) = \inf\{\lambda > 0\,;\ \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f_*(0) = \xi\}$$

On $\mathbb{C}^n$, $\mathbb{P}^n$ or complex tori $X = \mathbb{C}^n/\Lambda$, one has $\kappa_X \equiv 0$.

Kobayashi metric / hyperbolic manifolds

- For a complex manifold, $n = \dim_{\mathbb{C}} X$, one defines the Kobayashi pseudo-metric : $x \in X$, $\xi \in T_X$

  $$\kappa_x(\xi) = \inf\{\lambda > 0\,;\ \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f_*(0) = \xi\}$$

  On $\mathbb{C}^n$, $\mathbb{P}^n$ or complex tori $X = \mathbb{C}^n/\Lambda$, one has $\kappa_X \equiv 0$.
- X is said to be hyperbolic (in the sense of Kobayashi) if the associated integrated pseudo-distance is a distance (i.e. it separates points – Hausdorff topology),

Kobayashi metric / hyperbolic manifolds

- For a complex manifold, $n = \dim_{\mathbb{C}} X$, one defines the Kobayashi pseudo-metric : $x \in X$, $\xi \in T_X$

$$\kappa_x(\xi) = \inf\{\lambda > 0\,;\ \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f_*(0) = \xi\}$$

  On $\mathbb{C}^n$, $\mathbb{P}^n$ or complex tori $X = \mathbb{C}^n/\Lambda$, one has $\kappa_X \equiv 0$.
- X is said to be hyperbolic (in the sense of Kobayashi) if the associated integrated pseudo-distance is a distance (i.e. it separates points – Hausdorff topology),
- **Theorem.** (Brody) If X is compact then X is Kobayashi hyperbolic if and only if there are no entire holomorphic curves $f : \mathbb{C} \to X$ (Brody hyperbolicity).

Kobayashi metric / hyperbolic manifolds

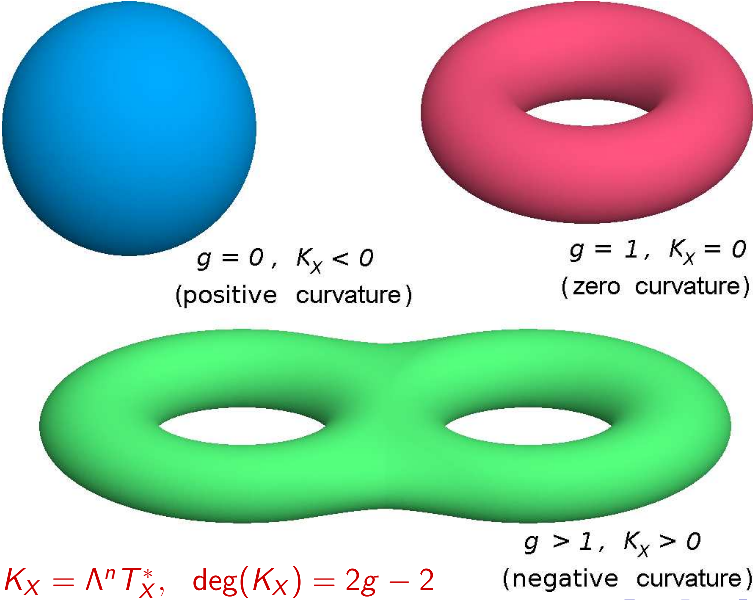
- For a complex manifold, $n = \dim_{\mathbb{C}} X$, one defines the Kobayashi pseudo-metric : $x \in X$, $\xi \in T_X$

$$\kappa_x(\xi) = \inf\{\lambda > 0\,;\ \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f_*(0) = \xi\}$$

  On $\mathbb{C}^n$, $\mathbb{P}^n$ or complex tori $X = \mathbb{C}^n/\Lambda$, one has $\kappa_X \equiv 0$.
- X is said to be hyperbolic (in the sense of Kobayashi) if the associated integrated pseudo-distance is a distance (i.e. it separates points – Hausdorff topology),
- **Theorem.** (Brody) If X is compact then X is Kobayashi hyperbolic if and only if there are no entire holomorphic curves $f : \mathbb{C} \to X$ (Brody hyperbolicity).
- Hyperbolic varieties are especially interesting for their expected diophantine properties :

  Conjecture (S. Lang) *If a projective variety X defined over $\mathbb{Q}$ is hyperbolic, then $X(\mathbb{Q})$ is finite.*

Complex curves ($n = 1$) : genus and curvature



$K_X = \Lambda^n T_X^*, \quad \deg(K_X) = 2g - 2$

Curves : hyperbolicity and curvature

- Case $n = 1$ (compact Riemann surfaces):

$$\begin{array}{ll} X = \mathbb{P}^1 & (g = 0, \quad T_X > 0) \\ X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) & (g = 1, \quad T_X = 0) \end{array}$$

obviously non hyperbolic : $\exists f : \mathbb{C} \to X$.

Curves : hyperbolicity and curvature

- Case $n = 1$ (compact Riemann surfaces):

$$\begin{array}{ll} X = \mathbb{P}^1 & (g = 0, \quad T_X > 0) \\ X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) & (g = 1, \quad T_X = 0) \end{array}$$

  obviously non hyperbolic : $\exists f : \mathbb{C} \to X$.

- If $g \geq 2$, $\;X \simeq \mathbb{D}/\Gamma \;\; (T_X < 0)$, then X hyperbolic.

- Case $n = 1$ (compact Riemann surfaces):

$$\begin{array}{ll} X = \mathbb{P}^1 & (g = 0, \quad T_X > 0) \\ X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) & (g = 1, \quad T_X = 0) \end{array}$$

  obviously non hyperbolic : $\exists f : \mathbb{C} \to X$.

- If $g \geq 2$, $\quad X \simeq \mathbb{D}/\Gamma \quad (T_X < 0)$, then X hyperbolic.
- **The *n*-dimensional case** (Kobayashi)
  If T_X is negatively curved ($T_X^* > 0$, i.e. ample), then X is hyperbolic.
  Recall that a holomorphic vector bundle E is ample iff its symmetric powers $S^m E$ have global sections which generate 1-jets of (germs of) sections at any point $x \in X$.
- **Examples :** $X = \Omega/\Gamma$, Ω bounded symmetric domain.

Varieties of general type

- **Definition** *A non singular projective variety X is said to be of general type if the growth of pluricanonical sections*

$$\dim H^0(X, K_X^{\otimes m}) \sim cm^n, \qquad K_X = \Lambda^n T_X^*$$

is maximal.

(sections locally of the form $f(z)\,(dz_1 \wedge \ldots \wedge dz_n)^{\otimes m}$)

Example: A non singular hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree d satisfies $K_X = \mathcal{O}(d-n-2)$, it is of general type iff $d > n+2$.

Varieties of general type

- **Definition** *A non singular projective variety X is said to be of general type if the growth of pluricanonical sections*

$$\dim H^0(X, K_X^{\otimes m}) \sim cm^n, \qquad K_X = \Lambda^n T_X^*$$

is maximal.

(sections locally of the form $f(z)\,(dz_1 \wedge \ldots \wedge dz_n)^{\otimes m}$)

Example: A non singular hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree d satisfies $K_X = \mathcal{O}(d - n - 2)$,
it is of general type iff $d > n + 2$.

- **Conjecture GT.** *If a compact manifold X is hyperbolic, then it should be of general type, i.e. $K_X = \Lambda^n T_X^*$ should be of positive curvature (Ricci < 0, possibly with some degeneration).*

Conjectural characterizations of hyperbolicity

- **Theorem.** *Let X be projective algebraic. Consider the following properties :*
  (P1) *X is hyperbolic*
  (P2) *Every subvariety Y of X is of general type.*
  (P3) $\exists\varepsilon > 0$, $\forall C \subset X$ *algebraic curve*
  $$2g(\bar{C}) - 2 \geq \varepsilon \deg(C).$$
  (X *"algebraically hyperbolic"*)
  (P4) *X possesses a jet-metric with negative curvature on its k-jet bundle X_k [to be defined later], for $k \geq k_0 \gg 1$.*
  Then (P4) $\Rightarrow$ (P1), (P2), (P3),
  (P1) $\Rightarrow$ (P3),
  and if Conjecture GT holds, (P1) $\Rightarrow$ (P2).

Conjectural characterizations of hyperbolicity

- **Theorem.** *Let X be projective algebraic. Consider the following properties :*
  $(\mathrm{P1})$ *X is hyperbolic*
  $(\mathrm{P2})$ *Every subvariety Y of X is of general type.*
  $(\mathrm{P3})$ $\exists\varepsilon > 0$, $\forall C \subset X$ *algebraic curve*
  $$2g(\bar{C}) - 2 \geq \varepsilon \deg(C).$$
  (X *"algebraically hyperbolic"*)
  $(\mathrm{P4})$ *X possesses a jet-metric with negative curvature on its k-jet bundle X_k [to be defined later], for $k \geq k_0 \gg 1$.*
  Then $(\mathrm{P4}) \Rightarrow (\mathrm{P1}),\ (\mathrm{P2}),\ (\mathrm{P3}),$
  $(\mathrm{P1}) \Rightarrow (\mathrm{P3}),$
  and if Conjecture GT holds, $(\mathrm{P1}) \Rightarrow (\mathrm{P2}).$
- It is expected that all 4 properties (P1), (P2), (P3), (P4) are equivalent for projective varieties.

Green-Griffiths-Lang conjecture

Conjecture (Green-Griffith-Lang = GGL) *Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.*

Green-Griffiths-Lang conjecture

Conjecture (Green-Griffith-Lang = GGL) *Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.* Combining the above conjectures, we get :

- **Expected consequence** (of GT + GGL)
  (P1) *X is hyperbolic*
  (P2) *Every subvariety Y of X is of general type*
  are equivalent.

Green-Griffiths-Lang conjecture

Conjecture (Green-Griffith-Lang = GGL) *Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.* Combining the above conjectures, we get :

- **Expected consequence** (of GT + GGL)
  (P1) *X is hyperbolic*
  (P2) *Every subvariety Y of X is of general type*
  are equivalent.
- The main idea in order to attack GGL is to use differential equations. Let

$$\mathbb{C} \to X, \quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$$

be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X.

Definition of algebraic differential operators

- Consider algebraic differential operators which can be written locally in multi-index notation

$$
\begin{aligned}
P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\
&= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k}
\end{aligned}
$$

where $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are holomorphic coefficients on X and $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet.

Definition of algebraic differential operators

- Consider algebraic differential operators which can be written locally in multi-index notation

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\ f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k} \end{aligned}$$

where $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are holomorphic coefficients on X and $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet. Obvious $\mathbb{C}^*$-action :

$$\lambda \cdot f(t) = f(\lambda t), \quad (\lambda \cdot f)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$$

$\Rightarrow$ weighted degree $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$.

Definition of algebraic differential operators

- Consider algebraic differential operators which can be written locally in multi-index notation

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k} \end{aligned}$$

  where $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are holomorphic coefficients on X and $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet. Obvious $\mathbb{C}^*$-action :

$$\lambda \cdot f(t) = f(\lambda t), \quad (\lambda \cdot f)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$$

  $\Rightarrow$ weighted degree $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$.

- **Definition.** $E^{\mathrm{GG}}_{k,m}$ *is the sheaf (bundle) of algebraic differential operators of order k and weighted degree m.*

Vanishing theorem for differential operators

- **Fundamental vanishing theorem**
  ([Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]
  Let $P \in H^0(X, E^{\rm GG}_{k,m} \otimes \mathcal{O}(-A))$ *be a global algebraic differential operator whose coefficients vanish on some ample divisor* A*. Then for any* $f : \mathbb{C} \to X$, $P(f_{[k]}) \equiv 0$.

Vanishing theorem for differential operators

- **Fundamental vanishing theorem**
  ([Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]
  Let $P \in H^0(X, E^{\mathrm{GG}}_{k,m} \otimes \mathcal{O}(-A))$ be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then for any $f : \mathbb{C} \to X$, $P(f_{[k]}) \equiv 0$.
- *Proof*. One can assume that A is very ample and intersects $f(\mathbb{C})$. Also assume f' bounded (this is not so restrictive by Brody !). Then all $f^{(k)}$ are bounded by Cauchy inequality. Hence

$$\mathbb{C} \ni t \mapsto P(f', f'', \ldots, f^{(k)})(t)$$

is a bounded holomorphic function on $\mathbb{C}$ which vanishes at some point. Apply Liouville's theorem ! $\square$

Geometric interpretation of vanishing theorem

- Let $X_k^{\rm GG} = J_k(X)^*/\mathbb{C}^*$ be the projectivized k-jet bundle of X = quotient of non constant k-jets by $\mathbb{C}^*$-action. Fibers are weighted projective spaces.
  Observation. *If $\pi_k : X_k^{\rm GG} \to X$ is canonical projection and $\mathcal{O}_{X_k^{\rm GG}}(1)$ is the tautological line bundle, then*

$$E_{k,m}^{\rm GG} = (\pi_k)_* \mathcal{O}_{X_k^{\rm GG}}(m)$$

Geometric interpretation of vanishing theorem

- Let $X_k^{\mathrm{GG}} = J_k(X)^*/\mathbb{C}^*$ be the projectivized k-jet bundle of X = quotient of non constant k-jets by $\mathbb{C}^*$-action. Fibers are weighted projective spaces.

  Observation. *If $\pi_k : X_k^{\mathrm{GG}} \to X$ is canonical projection and $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is the tautological line bundle, then*

$$E_{k,m}^{\mathrm{GG}} = (\pi_k)_* \mathcal{O}_{X_k^{\mathrm{GG}}}(m)$$

- Saying that $f : \mathbb{C} \to X$ satisfies the differential equation $P(f_{[k]}) = 0$ means that

$$f_{[k]}(\mathbb{C}) \subset Z_P$$

  where Z_P is the zero divisor of the section

$$\sigma_P \in H^0(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* \mathcal{O}(-A))$$

  associated with P.

- **Consequence of fundamental vanishing theorem.** *If $P_j \in H^0(X, E_{k,m}^{\mathrm{GG}} \otimes \mathcal{O}(-A))$ is a basis of sections then the image $f(\mathbb{C})$ lies in $Y = \pi_k(\bigcap Z_{P_j})$, hence property asserted by the GGL conjecture holds true if there are "enough independent differential equations" so that*

$$Y = \pi_k(\bigcap_j Z_{P_j}) \subsetneq X.$$

Consequence of fundamental vanishing theorem

- **Consequence of fundamental vanishing theorem.** *If $P_j \in H^0(X, E_{k,m}^{\mathrm{GG}} \otimes \mathcal{O}(-A))$ is a basis of sections then the image $f(\mathbb{C})$ lies in $Y = \pi_k(\bigcap Z_{P_j})$, hence property asserted by the GGL conjecture holds true if there are "enough independent differential equations" so that*

$$Y = \pi_k(\bigcap_j Z_{P_j}) \subsetneq X.$$

- However, some differential equations are useless. On a surface with coordinates (z_1, z_2), a Wronskian equation $f_1' f_2'' - f_2' f_1'' = 0$ tells us that $f(\mathbb{C})$ sits on a line, but $f_2''(t) = 0$ says that the second component is linear affine in time, an essentially meaningless information which is lost by a change of parameter $t \mapsto \varphi(t)$.

Invariant differential operators

- The k-th order Wronskian operator

$$W_k(f) = f' \wedge f'' \wedge \ldots \wedge f^{(k)}$$

(locally defined in coordinates) has degree $m = \frac{k(k+1)}{2}$ and

$$W_k(f \circ \varphi) = \varphi'^m W_k(f) \circ \varphi.$$

Invariant differential operators

- The k-th order Wronskian operator

$$W_k(f) = f' \wedge f'' \wedge \ldots \wedge f^{(k)}$$

(locally defined in coordinates) has degree $m = \frac{k(k+1)}{2}$ and

$$W_k(f \circ \varphi) = \varphi'^m W_k(f) \circ \varphi.$$

- **Definition.** *A differential operator P of order k and degree m is said to be invariant by reparametrization if*

$$P(f \circ \varphi) = \varphi'^m P(f) \circ \varphi$$

for any parameter change $t \mapsto \varphi(t)$. Consider their set

$$E_{k,m} \subset E_{k,m}^{\mathrm{GG}} \quad (\textit{a subbundle})$$

(Any polynomial $Q(W_1, W_2, \ldots W_k)$ is invariant, but for $k \geq 3$ there are other invariant operators.)

Category of directed manifolds

- **Definition.** *Category of directed manifolds :*
  - Objects are pairs (X, V) where X is a complex manifold and $V \subset T_X$ (subbundle or subsheaf)
  - Arrows $\psi : (X, V) \to (Y, W)$ are holomorphic maps with $\psi_* V \subset W$

- **Definition.** *Category of directed manifolds :*
  - Objects are pairs (X, V) where X is a complex manifold and $V \subset T_X$ (subbundle or subsheaf)
  - Arrows $\psi : (X, V) \to (Y, W)$ are holomorphic maps with $\psi_* V \subset W$
  - "Absolute case" (X, T_X)
  - "Relative case" $(X, T_{X/S})$ where $X \to S$
  - "Integrable case" when $[V, V] \subset V$ (foliations)

Category of directed manifolds

- **Definition.** *Category of directed manifolds :*
  - Objects are pairs (X, V) where X is a complex manifold and $V \subset T_X$ (subbundle or subsheaf)
  - Arrows $\psi : (X, V) \to (Y, W)$ are holomorphic maps with $\psi_* V \subset W$
  - "Absolute case" (X, T_X)
  - "Relative case" $(X, T_{X/S})$ where $X \to S$
  - "Integrable case" when $[V, V] \subset V$ (foliations)
- **Fonctor "1-jet" :** $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :

$$\tilde{X} = P(V) = \text{bundle of projective spaces of lines in } V$$
$$\pi : \tilde{X} = P(V) \to X, \quad (x, [v]) \mapsto x, \quad v \in V_x$$
$$\tilde{V}_{(x,[v])} = \left\{\xi \in T_{\tilde{X},(x,[v])}\,;\ \pi_*\xi \in \mathbb{C}v \subset T_{X,x}\right\}$$

Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \quad \text{(projectivized 1st-jet)}$$

Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \quad \text{(projectivized 1st-jet)}$$

- **Definition.** *Semple jet bundles :*
  – $(X_k, V_k) =$ k-th iteration of fonctor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
  – $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized k-jet of f.

Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$
$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \quad \text{(projectivized 1st-jet)}$$

- **Definition.** *Semple jet bundles :*

  – $(X_k, V_k) = k$-th iteration of fonctor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
  – $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized k-jet of f.

- **Basic exact sequences**

$$0 \to T_{\tilde{X}/X} \to \tilde{V} \xrightarrow{\pi_\star} \mathcal{O}_{\tilde{X}}(-1) \to 0 \qquad \Rightarrow \text{rk}\,\tilde{V} = r = \text{rk}\,V$$
$$0 \to \mathcal{O}_{\tilde{X}} \to \pi^\star V \otimes \mathcal{O}_{\tilde{X}}(1) \to T_{\tilde{X}/X} \to 0 \quad \text{(Euler)}$$

Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \quad \text{(projectivized 1st-jet)}$$

- **Definition.** *Semple jet bundles :*
  - $(X_k, V_k) = k$-th iteration of fonctor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
  - $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized k-jet of f.

- **Basic exact sequences**

$$0 \to T_{\tilde{X}/X} \to \tilde{V} \xrightarrow{\pi_\star} \mathcal{O}_{\tilde{X}}(-1) \to 0 \qquad \Rightarrow \text{rk}\, \tilde{V} = r = \text{rk}\, V$$

$$0 \to \mathcal{O}_{\tilde{X}} \to \pi^\star V \otimes \mathcal{O}_{\tilde{X}}(1) \to T_{\tilde{X}/X} \to 0 \quad \text{(Euler)}$$

$$0 \to T_{X_k/X_{k-1}} \to V_k \xrightarrow{(\pi_k)_\star} \mathcal{O}_{X_k}(-1) \to 0 \qquad \Rightarrow \text{rk}\, V_k = r$$

$$0 \to \mathcal{O}_{X_k} \to \pi_k^\star V_{k-1} \otimes \mathcal{O}_{X_k}(1) \to T_{X_k/X_{k-1}} \to 0 \quad \text{(Euler)}$$

Direct image formula

- For $n = \dim X$ and $r = \operatorname{rk} V$, get a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\operatorname{rk} V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

Direct image formula

- For $n = \dim X$ and $r = \operatorname{rk} V$, get a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\operatorname{rk} V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

- **Theorem.** *X_k is a smooth compactification of*

$$X_k^{\mathrm{GG,reg}}/G_k = J_k^{\mathrm{GG,reg}}/G_k$$

where G_k is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization: $(f, \varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k-jets of regular curves.

Direct image formula

- For $n = \dim X$ and $r = \mathrm{rk}\, V$, get a tower of $\mathbb{P}^{r-1}$-bundles
$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$
with $\dim X_k = n + k(r-1)$, $\mathrm{rk}\, V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.
- **Theorem.** *X_k is a smooth compactification of*
$$X_k^{\mathrm{GG,reg}}/G_k = J_k^{\mathrm{GG,reg}}/G_k$$
where G_k is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization: $(f, \varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k-jets of regular curves.
- **Direct image formula.** $(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = E_{k,m} V^* =$ *invariant algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.*

Results obtained so far

- Using this technology and deep results of McQuillan for curve foliations on surfaces, D. – El Goul proved in 1998
  Theorem. (solution of Kobayashi conjecture)
  A very generic surface $X \subset \mathbb{P}^3$ of degree ≥ 21 is hyperbolic. (McQuillan got independently degree ≥ 35).

Results obtained so far

- Using this technology and deep results of McQuillan for curve foliations on surfaces, D. – El Goul proved in 1998
  Theorem. (solution of Kobayashi conjecture)
  A very generic surface $X \subset \mathbb{P}^3$ *of degree* ≥ 21 *is hyperbolic.* (McQuillan got independently degree ≥ 35).
- This result was improved by [Paun 2008], degree ≥ 18 is enough, with “generic” instead of “very generic”. Paun’s technique exploits a new idea of [Siu 2004] based on C. Voisin’s work (1996), which consists of studying vector fields on the the universal jet space of the universal family of hypersurfaces of $\mathbb{P}^{n+1}$.

Results obtained so far

- Using this technology and deep results of McQuillan for curve foliations on surfaces, D. – El Goul proved in 1998
  Theorem. (solution of Kobayashi conjecture)
  A very generic surface $X \subset \mathbb{P}^3$ of degree ≥ 21 is hyperbolic. (McQuillan got independently degree ≥ 35).
- This result was improved by [Paun 2008], degree ≥ 18 is enough, with "generic" instead of "very generic". Paun's technique exploits a new idea of [Siu 2004] based on C. Voisin's work (1996), which consists of studying vector fields on the the universal jet space of the universal family of hypersurfaces of $\mathbb{P}^{n+1}$.
- **Higher dimensions.** Combining these ideas, E. Rousseau - J. Merker - S. Diverio (2006-2008, see [DRM08]) just proved the Green-Griffiths conjecture for generic hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n$ large.

Algebraic structure of differential rings

- Although very interesting, results are currently limited by lack of knowledge on jet bundles and differential operators
- **Unknown !** *Is the ring of germs of invariant differential operators on* $(\mathbb{C}^n, T_{\mathbb{C}^n})$ *at the origin*
$$\mathcal{A}_{k,n} = \bigoplus_m E_{k,m} T^*_{\mathbb{C}^n} \quad \textit{finitely generated ?}$$

Algebraic structure of differential rings

- Although very interesting, results are currently limited by lack of knowledge on jet bundles and differential operators
- **Unknown !** *Is the ring of germs of invariant differential operators on* $(\mathbb{C}^n, T_{\mathbb{C}^n})$ *at the origin*
$$\mathcal{A}_{k,n} = \bigoplus_m E_{k,m} T^*_{\mathbb{C}^n} \quad \textit{finitely generated ?}$$
- At least this is OK for $\forall n$, $k \leq 2$ and $n = 2$, $k \leq 4$:

$$\mathcal{A}_{1,n} = \mathcal{O}[f'_1, \ldots, f'_n]$$
$$\mathcal{A}_{2,n} = \mathcal{O}[f'_1, \ldots, f'_n, W^{[ij]}], \quad W^{[ij]} = f'_i f''_j - f'_j f''_i$$
$$\mathcal{A}_{3,2} = \mathcal{O}[f'_1, f'_2, W_1, W_2][W]^2, \quad W_i = f'_i DW - 3f''_i W$$
$$\mathcal{A}_{4,2} = \mathcal{O}[f'_1, f'_2, W_{11}, W_{22}, S][W]^6, \quad W_{ii} = f'_i DW_i - 5f''_i W_i$$

where $W = f'_1 f''_2 - f'_2 f''_1$ is 2-dim Wronskian and
$S = (W_1 DW_2 - W_2 DW_1)/W$. Also known:
$\mathcal{A}_{3,3}$ (E. Rousseau, 2004), $\mathcal{A}_{5,2}$ (J. Merker, 2007)

Strategy : evaluate growth of differential operators

- The strategy of the proofs is that the algebraic structure of $\mathcal{A}_{k,n}$ allows to compute the Euler characteristic $\chi(X, E_{k,m} \otimes \mathcal{O}(-A))$, e.g. on surfaces

$$\chi(X, E_{k,m} \otimes \mathcal{O}(-A)) = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3).$$

Strategy : evaluate growth of differential operators

- The strategy of the proofs is that the algebraic structure of $\mathcal{A}_{k,n}$ allows to compute the Euler characteristic $\chi(X, E_{k,m} \otimes \mathcal{O}(-A))$, e.g. on surfaces

$$\chi(X, E_{k,m} \otimes \mathcal{O}(-A)) = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3).$$

- Hence for $13c_1^2 - 9c_2 > 0$, using Bogomolov's vanishing theorem for $H^2(X, (T_X^*)^{\otimes m} \otimes \mathcal{O}(-A))$ for $m \gg 0$, one gets

$$h^0(X, E_{k,m} \otimes \mathcal{O}(-A)) \geq \chi = h^0 - h^1 = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3)$$

Strategy : evaluate growth of differential operators

- The strategy of the proofs is that the algebraic structure of $\mathcal{A}_{k,n}$ allows to compute the Euler characteristic $\chi(X, E_{k,m} \otimes \mathcal{O}(-A))$, e.g. on surfaces
$$\chi(X, E_{k,m} \otimes \mathcal{O}(-A)) = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3).$$

- Hence for $13c_1^2 - 9c_2 > 0$, using Bogomolov's vanishing theorem for $H^2(X, (T_X^*)^{\otimes m} \otimes \mathcal{O}(-A))$ for $m \gg 0$, one gets
$$h^0(X, E_{k,m} \otimes \mathcal{O}(-A)) \geq \chi = h^0 - h^1 = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3)$$

- Therefore many global differential operators exist for surfaces with $13c_1^2 - 9c_2 > 0$, e.g. surfaces of degree large enough in $\mathbb{P}^3$, $d \geq 15$ (end of proof uses stability)

Trouble / more general perspectives

- Trouble is, in higher dimensions n, intermediate cohomology groups $H^q(X, E_{k,m}T^*_X)$, $0 < q < n$, don't vanish !!

Trouble / more general perspectives

- Trouble is, in higher dimensions n, intermediate cohomology groups $H^q(X, E_{k,m}T_X^*)$, $0 < q < n$, don't vanish !!
- **Main conjecture** (Generalized GGL)
  If (X, V) is directed manifold of general type, i.e. $\det V^*$ *big, then* $\exists Y \subsetneq X$ *such that every non-constant* $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ *is contained in* Y.

Trouble / more general perspectives

- Trouble is, in higher dimensions n, intermediate cohomology groups $H^q(X, E_{k,m}T^*_X)$, $0 < q < n$, don't vanish !!
- **Main conjecture** (Generalized GGL)
  If (X, V) is directed manifold of general type, i.e. $\det V^*$ big, then $\exists Y \subsetneq X$ such that every non-constant $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ is contained in Y.
- **Strategy.** OK by Ahlfors-Schwarz lemma if $r = \mathrm{rk}\, V = 1$. First try to get differential equations $f_{[k]}(\mathbb{C}) \subset Z \subsetneq X_k$. Take minimal such k. If $k = 0$, we are done! Otherwise $k \geq 1$ and $\pi_{k,k-1}(Z) = X_{k-1}$, thus $W = V_k \cap T_Z$ has rank $< \mathrm{rk}\, V_k = r$ and should have again $\det W^*$ big (unless some degeneration occurs ?). Use induction on r !

Trouble / more general perspectives

- Trouble is, in higher dimensions n, intermediate cohomology groups $H^q(X, E_{k,m}T^*_X)$, $0 < q < n$, don't vanish !!
- **Main conjecture** (Generalized GGL)
  If (X, V) is directed manifold of general type, i.e. $\det V^*$ *big, then* $\exists Y \subsetneq X$ *such that every non-constant* $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ *is contained in* Y.
- **Strategy.** OK by Ahlfors-Schwarz lemma if $r = \operatorname{rk} V = 1$. First try to get differential equations $f_{[k]}(\mathbb{C}) \subset Z \subsetneq X_k$. Take minimal such k. If $k = 0$, we are done! Otherwise $k \geq 1$ and $\pi_{k,k-1}(Z) = X_{k-1}$, thus $W = V_k \cap T_Z$ has rank $< \operatorname{rk} V_k = r$ and should have again $\det W^*$ big (unless some degeneration occurs ?). Use induction on r !
- **Needed induction step.** *If (X, V) has* $\det V^*$ *big and* $Z \subset X_k$ *irreducible with* $\pi_{k,k-1}(Z) = X_{k-1}$, *then* (Z, W), $W = V_k \cap T_Z$ *has* $\mathcal{O}_{Z_\ell}(1)$ *big on* (Z_ℓ, W_ℓ), $\ell \gg 0$.

Use holomorphic Morse inequalities !

- **Simple case of Morse inequalities**
  ([Demailly 1985, 1994], [Trapani 1995])
  If $L = \mathcal{O}(A - B)$ is a difference of big nef divisors A, B, then L is big as soon as

$$A^n - nA^{n-1} \cdot B > 0.$$

Use holomorphic Morse inequalities !

- **Simple case of Morse inequalities** ([Demailly 1985, 1994], [Trapani 1995])
  If $L = \mathcal{O}(A - B)$ is a difference of big nef divisors A, B, then L is big as soon as
  $$A^n - nA^{n-1} \cdot B > 0.$$
- My PhD student S. Diverio has recently (2008) worked out this strategy for hypersurfaces $X \subset \mathbb{P}^{n+1}$, with
  $$L = \bigotimes_{1 \le j < k} \pi_{k,j}^* \mathcal{O}_{X_j}(2 \cdot 3^{k-j-1}) \otimes \mathcal{O}_{X_k}(1),$$
  $$B = \pi_{k,0}^* \mathcal{O}_X(2 \cdot 3^{k-1}), \quad A = L + B \Rightarrow L = A - B.$$
  In this way, one obtains differential equations of order $k = n$, when $d \ge d_n$, e.g. for $d_n = n^{5n^4}$. One can check
  $$d_2 = 15, \quad d_3 = 82, \quad d_4 = 329, \quad d_5 = 1222, \quad \ldots.$$

Combine with vector fields on universal jet spaces !

- This approach produces a priori only one differential equation P such that $P(f_{[k]}) = 0$, which is not enough for the GGL conjecture.

Combine with vector fields on universal jet spaces !

- This approach produces a priori only one differential equation P such that $P(f_{[k]}) = 0$, which is not enough for the GGL conjecture.
- If there are global meromorphic vector fields $\xi_1, \ldots, \xi_m$ on X_k with poles along a divisor B, then $D_{\xi_1} D_{\xi_2} \ldots D_{\xi_m} P$, $m = \deg P$ might define a suitable new global differential operator in $H^0(X, E_{k,m}^{\mathrm{GG}} \otimes \mathcal{O}(-A + mB))$.

Combine with vector fields on universal jet spaces !

- This approach produces a priori only one differential equation P such that $P(f_{[k]}) = 0$, which is not enough for the GGL conjecture.
- If there are global meromorphic vector fields $\xi_1, \ldots, \xi_m$ on X_k with poles along a divisor B, then $D_{\xi_1} D_{\xi_2} \ldots D_{\xi_m} P$, $m = \deg P$ might define a suitable new global differential operator in $H^0(X, E_{k,m}^{\mathrm{GG}} \otimes \mathcal{O}(-A + mB))$.
- Trouble is, one has to choose $A > mB$ and these vector fields exist only with B large !

Combine with vector fields on universal jet spaces !

- This approach produces a priori only one differential equation P such that $P(f_{[k]}) = 0$, which is not enough for the GGL conjecture.
- If there are global meromorphic vector fields $\xi_1, \ldots, \xi_m$ on X_k with poles along a divisor B, then $D_{\xi_1} D_{\xi_2} \ldots D_{\xi_m} P$, $m = \deg P$ might define a suitable new global differential operator in $H^0(X, E_{k,m}^{\mathrm{GG}} \otimes \mathcal{O}(-A + mB))$.
- Trouble is, one has to choose $A > mB$ and these vector fields exist only with B large !
- However (observation by Voisin & Siu), on the universal family $\mathcal{X}_k$, suitable vector fields exist (with B small).

Combine with vector fields on universal jet spaces !

- This approach produces a priori only one differential equation P such that $P(f_{[k]}) = 0$, which is not enough for the GGL conjecture.
- If there are global meromorphic vector fields $\xi_1, \ldots, \xi_m$ on X_k with poles along a divisor B, then $D_{\xi_1} D_{\xi_2} \ldots D_{\xi_m} P$, $m = \deg P$ might define a suitable new global differential operator in $H^0(X, E_{k,m}^{\mathrm{GG}} \otimes \mathcal{O}(-A + mB))$.
- Trouble is, one has to choose $A > mB$ and these vector fields exist only with B large !
- However (observation by Voisin & Siu), on the universal family $\mathcal{X}_k$, suitable vector fields exist (with B small).
- End of 2008, Diverio-Merker-Rousseau [DRM08] proved the Green-Griffiths-Lang conjecture for $X \subset \mathbb{P}^{n+1}$ generic of degree $d \geq d_n \geq n^{(n+1)^{n+5}}$.

[Brody78] Brody, R.: *Compact manifolds and hyperbolicity*. Trans. Amer. Math. Soc. **235** (1978), 213–219

[Demailly85] Demailly, J.-P.: *Champs Magnétiques et Inégalités de Morse pour la d"-cohomologie*. Ann. Inst. Fourier (Grenoble) **35** (1985), no. 4, 189–229.

[Demailly94] Demailly, J.-P.: *L^2 vanishing theorems for positive line bundles and adjunction theory*. arXiv:alg-geom/9410022 ; Lecture Notes of the CIME Session "Transcendental methods in Algebraic Geometry", Cetraro, Italy, July 1994, Ed. F. Catanese, C. Ciliberto, Lecture Notes in Math., Vol. 1646, 1–97.

[Demailly95] Demailly, J.-P.: *Algebraic Criteria for Kobayashi Hyperbolic Projective Varieties and Jet Differentials*. Algebraic geometry – Santa Cruz 1995, 285–360, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[D-EG00] Demailly, J.-P., El Goul, J.: *Hyperbolicity of Generic Surfaces of High Degree in Projective 3-Space*. Amer. J. Math. **122** (2000), no. 3, 515–546.

[Diverio08] Diverio, S.: *Existence of global invariant jet differentials on projective hypersurfaces of high degree* . arXiv:0802.0045, [math.AG].

[DRM08] Diverio, S., Merker, J., Rousseau, E.: *Effective algebraic degeneracy*. arXiv:0811.2346, [math.AG].

[F-H91] Fulton, W., Harris, J.: *Representation Theory: A First Course*. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991, xvi+551 pp.

[G-G79] Green, M., Griffiths, P.: *Two Applications of Algebraic Geometry to Entire Holomorphic Mappings*. The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), pp. 41–74, Springer, New York-Berlin, 1980.

[Kobayashi70] Kobayashi S.: *Hyperbolic Manifolds and Holomorphic Mappings*. Marcel Dekker, Inc., New York 1970 ix+148 pp.

[Lang86] Lang S.: *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. **14** (1986), no. 2, 159–205.

[Merker08] Merker, J.: *Low pole order frames on vertical jets of the universal hypersurface*. arXiv:0805.3987 [math.AG].

[Paun08] Păun, M.: *Vector fields on the total space of hypersurfaces in the projective space and hyperbolicity*. Math. Ann. **340** (2008), 875–892.

[Rousseau05] Rousseau, E: *Weak Analytic Hyperbolicity of Generic Hypersurfaces of High Degree in the Complex Projective Space of Dimension* 4. arXiv:math/0510285v1 [math.AG].

[Rousseau06a] Rousseau, E.: *Étude des Jets de Demailly-Semple en Dimension* 3. Ann. Inst. Fourier (Grenoble) **56** (2006), no. 2, 397–421.

[Rousseau06b] Rousseau, E: *Équations Diffi¿$\frac{1}{2}$rentielles sur les Hypersurfaces de* $\mathbb{P}^4$. J. Math. Pures Appl. (9) **86** (2006), no. 4, 322–341.

[SY97] Siu, Y.-T., Yeung, S.-K.: *Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees*. Amer. J. Math. **119** (1997), 1139–1172.

[Siu04] Siu, Y.-T.: *Hyperbolicity in Complex Geometry*. The legacy of Niel Henrik Abel, 543–566, Springer, Berlin, 2004.

[Trapani95] Trapani, S.: *Numerical criteria for the positivity of the difference of ample divisors*, Math. Z. **219** (1995), no. 3, 387–401.

[Voisin96] Voisin, C.: *On a conjecture of Clemens on rational curves on hypersurfaces*. J. Diff. Geom. **44** (1996), 200213. A correction: *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Diff. Geom. **49** (1998), 601611.

[Vojta87] Vojta, P.: *Diophantine Approximations and Value Distribution Theory*, Springer-Verlag, Lecture Notes in Mathematics no. 1239, 1987.