

Entire curves and algebraic differential equations

Jean-Pierre Demailly

Institut Fourier, Université de Grenoble I, France

April 16 2009, Saint-Martin d'Hères
IF - IMPA Conference

Entire curves

- **Definition.** By an **entire curve** we mean a non constant holomorphic map $f : \mathbb{C} \rightarrow X$ into a complex n -dimensional manifold.
- If X is a **bounded** open subset $\Omega \subset \mathbb{C}^n$, then there are no entire curves $f : \mathbb{C} \rightarrow \Omega$ (**Liouville's theorem**)
- $X = \overline{\mathbb{C}} \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$ has no entire curves (**Picard's theorem**)
- A complex torus $X = \mathbb{C}^n / \Lambda$ (Λ lattice) has a lot of entire curves. As \mathbb{C} simply connected, every $f : \mathbb{C} \rightarrow X = \mathbb{C}^n / \Lambda$ lifts as $\tilde{f} : \mathbb{C} \rightarrow \mathbb{C}^n$,

$$\tilde{f}(t) = (\tilde{f}_1(t), \dots, \tilde{f}_n(t))$$

and $\tilde{f}_j : \mathbb{C} \rightarrow \mathbb{C}$ can be arbitrary entire functions.

Projective algebraic varieties

- Consider now the complex projective n -space

$$\mathbb{P}^n = \mathbb{P}_{\mathbb{C}}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*, \quad [z] = [z_0 : z_1 : \dots : z_n].$$

- An entire curve $f : \mathbb{C} \rightarrow \mathbb{P}^n$ is given by a map

$$t \longmapsto [f_0(t) : f_1(t) : \dots : f_n(t)]$$

where $f_j : \mathbb{C} \rightarrow \mathbb{C}$ are holomorphic functions without common zeroes (so there are a lot of them).

- More generally, look at a (complex) **projective manifold**, i.e.

$$X^n \subset \mathbb{P}^N, \quad X = \{[z]; P_1(z) = \dots = P_k(z) = 0\}$$

where $P_j(z) = P_j(z_0, z_1, \dots, z_N)$ are homogeneous polynomials (of some degree d_j), such that X is **non singular**.

Kobayashi metric / hyperbolic manifolds

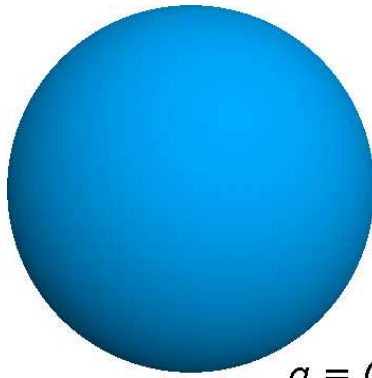
- For a complex manifold, $n = \dim_{\mathbb{C}} X$, one defines **the Kobayashi pseudo-metric** : $x \in X, \xi \in T_x$

$$\kappa_x(\xi) = \inf\{\lambda > 0; \exists f : \mathbb{D} \rightarrow X, f(0) = x, \lambda f_*(0) = \xi\}$$

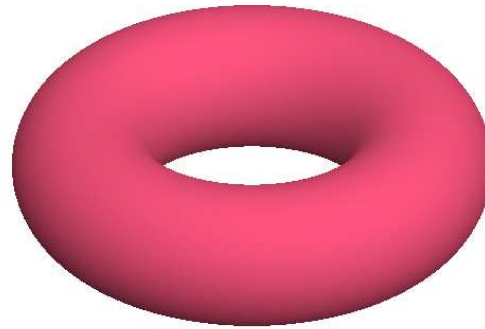
On $\mathbb{C}^n, \mathbb{P}^n$ or complex tori $X = \mathbb{C}^n/\Lambda$, one has $\kappa_X \equiv 0$.

- X is said to be **hyperbolic (in the sense of Kobayashi)** if the associated integrated pseudo-distance is a distance (i.e. it separates points – Hausdorff topology),
- **Theorem. (Brody)** If X is **compact** then X is Kobayashi hyperbolic if and only if there are no entire holomorphic curves $f : \mathbb{C} \rightarrow X$ (**Brody hyperbolicity**).
- Hyperbolic varieties are especially interesting for their expected diophantine properties :
Conjecture (S. Lang) *If a projective variety X defined over \mathbb{Q} is hyperbolic, then $X(\mathbb{Q})$ is finite.*

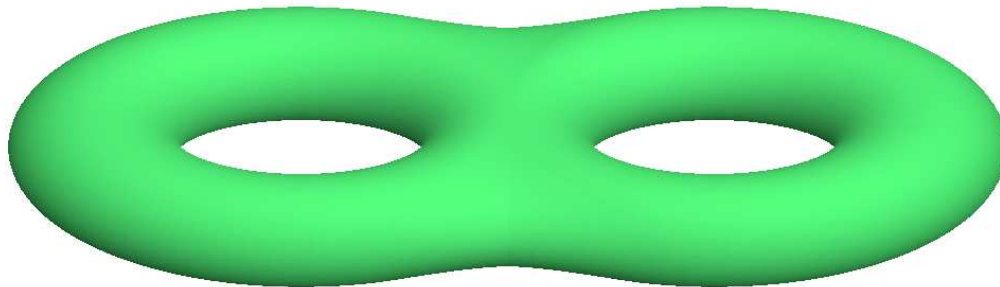
Complex curves ($n = 1$) : genus and curvature



$g = 0, K_X < 0$
(positive curvature)



$g = 1, K_X = 0$
(zero curvature)



$K_X = \Lambda^n T_X^*, \deg(K_X) = 2g - 2$

$g > 1, K_X > 0$
(negative curvature)

Curves : hyperbolicity and curvature

- Case $n = 1$ (compact Riemann surfaces):

$$\begin{aligned} X = \mathbb{P}^1 & \quad (g = 0, T_X > 0) \\ X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) & \quad (g = 1, T_X = 0) \end{aligned}$$

obviously non hyperbolic : $\exists f : \mathbb{C} \rightarrow X$.

- If $g \geq 2$, $X \simeq \mathbb{D}/\Gamma$ ($T_X < 0$), then X hyperbolic.

- **The n -dimensional case (Kobayashi)**

If T_X is negatively curved ($T_X^* > 0$, i.e. ample), then X is hyperbolic.

Recall that a holomorphic vector bundle E is **ample** iff its symmetric powers $S^m E$ have global sections which generate 1-jets of (germs of) sections at any point $x \in X$.

- **Examples** : $X = \Omega/\Gamma$, Ω bounded symmetric domain.

- **Definition** A non singular projective variety X is said to be of *general type* if the growth of pluricanonical sections

$$\dim H^0(X, K_X^{\otimes m}) \sim cm^n, \quad K_X = \Lambda^n T_X^*$$

is maximal.

(sections locally of the form $f(z) (dz_1 \wedge \dots \wedge dz_n)^{\otimes m}$)

Example: A non singular hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree d satisfies $K_X = \mathcal{O}(d - n - 2)$,
it is of general type iff $d > n + 2$.

- **Conjecture GT.** If a compact manifold X is hyperbolic, then it should be of general type, i.e. $K_X = \Lambda^n T_X^*$ should be of positive curvature (Ricci < 0 , possibly with some degeneration).

Conjectural characterizations of hyperbolicity

- **Theorem.** Let X be projective algebraic. Consider the following properties :
(P1) X is *hyperbolic*
(P2) Every subvariety Y of X is of *general type*.
(P3) $\exists \varepsilon > 0, \forall C \subset X$ algebraic curve

$$2g(\bar{C}) - 2 \geq \varepsilon \deg(C).$$

(X “*algebraically hyperbolic*”)

(P4) X possesses a *jet-metric with negative curvature* on its k -jet bundle X_k [to be defined later], for $k \geq k_0 \gg 1$.

Then (P4) \Rightarrow (P1), (P2), (P3),

(P1) \Rightarrow (P3),

and if Conjecture GT holds, (P1) \Rightarrow (P2).

- It is expected that all 4 properties (P1), (P2), (P3), (P4) are equivalent for projective varieties.

Conjecture (Green-Griffiths-Lang = GGL) *Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \rightarrow X$ one has $f(\mathbb{C}) \subset Y$.* Combining the above conjectures, we get :

- **Expected consequence** (of GT + GGL)
 - (P1) X is *hyperbolic*
 - (P2) Every subvariety Y of X is of *general type*
 are equivalent.
- The main idea in order to attack GGL is to use differential equations. Let

$$\mathbb{C} \rightarrow X, \quad t \mapsto f(t) = (f_1(t), \dots, f_n(t))$$

be a curve written in some local holomorphic coordinates (z_1, \dots, z_n) on X .

Definition of algebraic differential operators

- Consider **algebraic differential operators** which can be written locally in multi-index notation

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \dots, f^{(k)}) \\ &= \sum a_{\alpha_1 \alpha_2 \dots \alpha_k} (f(t)) f'(t)^{\alpha_1} f''(t)^{\alpha_2} \dots f^{(k)}(t)^{\alpha_k} \end{aligned}$$

where $a_{\alpha_1 \alpha_2 \dots \alpha_k}(z)$ are holomorphic coefficients on X and $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \dots, f^{(k)})$ its **k -jet**.
Obvious \mathbb{C}^* -action :

$$\lambda \cdot f(t) = f(\lambda t), \quad (\lambda \cdot f)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$$

\Rightarrow **weighted degree** $m = |\alpha_1| + 2|\alpha_2| + \dots + k|\alpha_k|$.

- **Definition.** $E_{k,m}^{\text{GG}}$ is the sheaf (bundle) of algebraic differential operators of order k and weighted degree m .

- **Fundamental vanishing theorem**

([Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996])

Let $P \in H^0(X, E_{k,m}^{GG} \otimes \mathcal{O}(-A))$ be a global algebraic differential operator whose coefficients vanish on some ample divisor A . Then for any $f : \mathbb{C} \rightarrow X$, $P(f_{[k]}) \equiv 0$.

- *Proof.* One can assume that A is very ample and intersects $f(\mathbb{C})$. Also assume f' bounded (this is not so restrictive by Brody !). Then all $f^{(k)}$ are bounded by Cauchy inequality. Hence

$$\mathbb{C} \ni t \mapsto P(f', f'', \dots, f^{(k)})(t)$$

is a bounded holomorphic function on \mathbb{C} which vanishes at some point. Apply Liouville's theorem ! □

Geometric interpretation of vanishing theorem

- Let $X_k^{GG} = J_k(X)^*/\mathbb{C}^*$ be the **projectivized k -jet bundle** of X = quotient of non constant k -jets by \mathbb{C}^* -action. Fibers are weighted projective spaces.

Observation. If $\pi_k : X_k^{GG} \rightarrow X$ is canonical projection and $\mathcal{O}_{X_k^{GG}}(1)$ is the **tautological line bundle**, then

$$E_{k,m}^{GG} = (\pi_k)_* \mathcal{O}_{X_k^{GG}}(m)$$

- Saying that $f : \mathbb{C} \rightarrow X$ satisfies the differential equation $P(f_{[k]}) = 0$ means that

$$f_{[k]}(\mathbb{C}) \subset Z_P$$

where Z_P is the zero divisor of the section

$$\sigma_P \in H^0(X_k^{GG}, \mathcal{O}_{X_k^{GG}}(m) \otimes \pi_k^* \mathcal{O}(-A))$$

associated with P .

- **Consequence of fundamental vanishing theorem.**

If $P_j \in H^0(X, E_{k,m}^{\text{GG}} \otimes \mathcal{O}(-A))$ is a basis of sections then the image $f(\mathbb{C})$ lies in $Y = \pi_k(\bigcap_j Z_{P_j})$, hence property asserted by the GGL conjecture holds true if there are “enough independent differential equations” so that

$$Y = \pi_k(\bigcap_j Z_{P_j}) \subsetneq X.$$

- However, **some differential equations are useless**. On a surface with coordinates (z_1, z_2) , a Wronskian equation $f_1' f_2'' - f_2' f_1'' = 0$ tells us that $f(\mathbb{C})$ sits on a line, but $f_2''(t) = 0$ says that the second component is linear affine in time, an essentially **meaningless information** which is lost by a change of parameter $t \mapsto \varphi(t)$.

Invariant differential operators

- The k -th order Wronskian operator

$$W_k(f) = f' \wedge f'' \wedge \dots \wedge f^{(k)}$$

(locally defined in coordinates) has degree $m = \frac{k(k+1)}{2}$ and

$$W_k(f \circ \varphi) = \varphi'^m W_k(f) \circ \varphi.$$

- **Definition.** A differential operator P of order k and degree m is said to be invariant by reparametrization if

$$P(f \circ \varphi) = \varphi'^m P(f) \circ \varphi$$

for any parameter change $t \mapsto \varphi(t)$. Consider their set

$$E_{k,m} \subset E_{k,m}^{\text{GG}} \quad (\text{a subbundle})$$

(Any polynomial $Q(W_1, W_2, \dots, W_k)$ is invariant, but for $k \geq 3$ there are other invariant operators.)

- **Definition.** *Category of directed manifolds* :
 - **Objects** are pairs (X, V) where X is a complex manifold and $V \subset T_X$ (subbundle or subsheaf)
 - **Arrows** $\psi : (X, V) \rightarrow (Y, W)$ are holomorphic maps with $\psi_* V \subset W$
 - “**Absolute case**” (X, T_X)
 - “**Relative case**” $(X, T_{X/S})$ where $X \rightarrow S$
 - “**Integrable case**” when $[V, V] \subset V$ (foliations)
- **Fonctor “1-jet”** : $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :

$$\begin{aligned} \tilde{X} &= P(V) = \text{bundle of projective spaces of lines in } V \\ \pi : \tilde{X} = P(V) &\rightarrow X, \quad (x, [v]) \mapsto x, \quad v \in V_x \\ \tilde{V}_{(x, [v])} &= \{ \xi \in T_{\tilde{X}, (x, [v])}; \pi_* \xi \in \mathbb{C}v \subset T_{X, x} \} \end{aligned}$$

Simple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ tangent to V

$$\begin{aligned} f_{[1]}(t) &:= (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X} \\ f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) &\rightarrow (\tilde{X}, \tilde{V}) \quad (\text{projectivized 1st-jet}) \end{aligned}$$

- **Definition.** *Simple jet bundles* :
 - $(X_k, V_k) = k$ -th iteration of fonctor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
 - $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X_k, V_k)$ is the **projectivized k -jet of f** .

- **Basic exact sequences**

$$0 \rightarrow T_{\tilde{X}/X} \rightarrow \tilde{V} \xrightarrow{\pi^*} \mathcal{O}_{\tilde{X}}(-1) \rightarrow 0 \quad \Rightarrow \text{rk } \tilde{V} = r = \text{rk } V$$

$$0 \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \pi^* V \otimes \mathcal{O}_{\tilde{X}}(1) \rightarrow T_{\tilde{X}/X} \rightarrow 0 \quad (\text{Euler})$$

$$0 \rightarrow T_{X_k/X_{k-1}} \rightarrow V_k \xrightarrow{(\pi_k)^*} \mathcal{O}_{X_k}(-1) \rightarrow 0 \quad \Rightarrow \text{rk } V_k = r$$

$$0 \rightarrow \mathcal{O}_{X_k} \rightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \rightarrow T_{X_k/X_{k-1}} \rightarrow 0 \quad (\text{Euler})$$

Direct image formula

- For $n = \dim X$ and $r = \operatorname{rk} V$, get a **tower of \mathbb{P}^{r-1} -bundles**

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r - 1)$, $\operatorname{rk} V_k = r$,
and **tautological line bundles $\mathcal{O}_{X_k}(1)$** on $X_k = P(V_{k-1})$.

- **Theorem.** X_k is a smooth compactification of

$$X_k^{\operatorname{GG},\operatorname{reg}} / G_k = J_k^{\operatorname{GG},\operatorname{reg}} / G_k$$

where G_k is the group of k -jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization:
 $(f, \varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k -jets of regular curves.

- **Direct image formula.** $(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = E_{k,m} V^* =$
invariant algebraic differential operators $f \mapsto P(f_{[k]})$
acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$.

Results obtained so far

- Using this technology and **deep results of McQuillan** for curve foliations on surfaces, D. – El Goul proved in 1998
Theorem. (solution of Kobayashi conjecture)
A very generic surface $X \subset \mathbb{P}^3$ of degree ≥ 21 is hyperbolic.
(McQuillan got independently degree ≥ 35).
- This result was improved by [Paun 2008], **degree ≥ 18** is enough, with “generic” instead of “very generic”. Paun’s technique exploits a new idea of [Siu 2004] based on C. Voisin’s work (1996), which consists of studying vector fields on the the **universal jet space of the universal family of hypersurfaces of \mathbb{P}^{n+1}** .
- **Higher dimensions.** Combining these ideas, E. Rousseau - J. Merker - S. Diverio (2006-2008, see [DRM08]) just proved the Green-Griffiths conjecture for **generic** hypersurfaces $X \subset \mathbb{P}^{n+1}$ of **degree $d \geq d_n$ large**.

Algebraic structure of differential rings

- Although very interesting, results are currently limited by **lack of knowledge on jet bundles and differential operators**
- **Unknown !** *Is the ring of germs of invariant differential operators on $(\mathbb{C}^n, T_{\mathbb{C}^n})$ at the origin*

$$\mathcal{A}_{k,n} = \bigoplus_m E_{k,m} T_{\mathbb{C}^n}^* \quad \text{finitely generated ?}$$

- At least this is OK for $\forall n, k \leq 2$ and $n = 2, k \leq 4$:

$$\mathcal{A}_{1,n} = \mathcal{O}[f'_1, \dots, f'_n]$$

$$\mathcal{A}_{2,n} = \mathcal{O}[f'_1, \dots, f'_n, W^{[ij]}], \quad W^{[ij]} = f'_i f''_j - f'_j f''_i$$

$$\mathcal{A}_{3,2} = \mathcal{O}[f'_1, f'_2, W_1, W_2][W]^2, \quad W_i = f'_i DW - 3f''_i W$$

$$\mathcal{A}_{4,2} = \mathcal{O}[f'_1, f'_2, W_{11}, W_{22}, S][W]^6, \quad W_{ii} = f'_i DW_i - 5f''_i W_i$$

where $W = f'_1 f''_2 - f'_2 f''_1$ is 2-dim Wronskian and
 $S = (W_1 DW_2 - W_2 DW_1)/W$. Also known:

$$\mathcal{A}_{3,3} \text{ (E. Rousseau, 2004), } \mathcal{A}_{5,2} \text{ (J. Merker, 2007)}$$

Strategy : evaluate growth of differential operators

- The strategy of the proofs is that the algebraic structure of $\mathcal{A}_{k,n}$ allows to compute the Euler characteristic $\chi(X, E_{k,m} \otimes \mathcal{O}(-A))$, e.g. on surfaces

$$\chi(X, E_{k,m} \otimes \mathcal{O}(-A)) = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3).$$

- Hence for $13c_1^2 - 9c_2 > 0$, using **Bogomolov's vanishing theorem for $H^2(X, (T_X^*)^{\otimes m} \otimes \mathcal{O}(-A))$** for $m \gg 0$, one gets

$$h^0(X, E_{k,m} \otimes \mathcal{O}(-A)) \geq \chi = h^0 - h^1 = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3)$$

- Therefore many global differential operators exist for surfaces with $13c_1^2 - 9c_2 > 0$, e.g. surfaces of degree large enough in \mathbb{P}^3 , $d \geq 15$ (**end of proof uses stability**)

- **Trouble is**, in higher dimensions n , intermediate cohomology groups $H^q(X, E_{k,m} T_X^*)$, $0 < q < n$, don't vanish !!
- **Main conjecture** (Generalized GGL)
If (X, V) is directed manifold of general type, i.e. $\det V^$ big, then $\exists Y \subsetneq X$ such that every non-constant $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ is contained in Y .*
- **Strategy.** OK by Ahlfors-Schwarz lemma if $r = \text{rk } V = 1$.
 First try to get **differential equations** $f_{[k]}(\mathbb{C}) \subset Z \subsetneq X_k$.
 Take **minimal such k** . If $k = 0$, we are done! Otherwise $k \geq 1$ and $\pi_{k,k-1}(Z) = X_{k-1}$, thus $W = V_k \cap T_Z$ has $\text{rank} < \text{rk } V_k = r$ and should have again $\det W^*$ big (unless some degeneration occurs ?). **Use induction on r !**
- **Needed induction step.** *If (X, V) has $\det V^*$ big and $Z \subset X_k$ irreducible with $\pi_{k,k-1}(Z) = X_{k-1}$, then (Z, W) , $W = V_k \cap T_Z$ has $\mathcal{O}_{Z_\ell}(1)$ big on (Z_ℓ, W_ℓ) , $\ell \gg 0$.*

Use holomorphic Morse inequalities !

- **Simple case of Morse inequalities**
 ([Demailly 1985, 1994], [Trapani 1995])
If $L = \mathcal{O}(A - B)$ is a difference of big nef divisors A, B , then L is big as soon as

$$A^n - nA^{n-1} \cdot B > 0.$$

- My PhD student S. Diverio has recently (2008) worked out this strategy for **hypersurfaces** $X \subset \mathbb{P}^{n+1}$, with

$$L = \bigotimes_{1 \leq j < k} \pi_{k,j}^* \mathcal{O}_{X_j}(2 \cdot 3^{k-j-1}) \otimes \mathcal{O}_{X_k}(1),$$

$$B = \pi_{k,0}^* \mathcal{O}_X(2 \cdot 3^{k-1}), \quad A = L + B \Rightarrow L = A - B.$$

In this way, one obtains differential equations of order $k = n$, when $d \geq d_n$, e.g. for $d_n = n^{5n^4}$. One can check

$$d_2 = 15, \quad d_3 = 82, \quad d_4 = 329, \quad d_5 = 1222, \quad \dots$$

- This approach produces a priori only **one differential equation** P such that $P(f_{[k]}) = 0$, which is not enough for the GGL conjecture.
- If there are global meromorphic vector fields ξ_1, \dots, ξ_m on X_k with poles along a divisor B , then $D_{\xi_1} D_{\xi_2} \dots D_{\xi_m} P$, $m = \deg P$ might define a suitable new global differential operator in $H^0(X, E_{k,m}^{\text{GG}} \otimes \mathcal{O}(-A + mB))$.
- Trouble is, one has to choose $A > mB$ and these vector fields exist only with **B large** !
- However (observation by Voisin & Siu), on the universal family \mathcal{X}_k , suitable vector fields exist (**with B small**).
- End of 2008, Diverio-Merker-Rousseau [DRM08] proved the Green-Griffiths-Lang conjecture for $X \subset \mathbb{P}^{n+1}$ **generic of degree $d \geq d_n \geq n^{(n+1)^{n+5}}$** .

[Brody78] Brody, R.: *Compact manifolds and hyperbolicity*.
Trans. Amer. Math. Soc. **235** (1978), 213–219

[Demailly85] Demailly, J.-P.: *Champs Magnétiques et Inégalités de Morse pour la d'' -cohomologie*. Ann. Inst. Fourier (Grenoble) **35** (1985), no. 4, 189–229.

[Demailly94] Demailly, J.-P.: *L^2 vanishing theorems for positive line bundles and adjunction theory*. arXiv:alg-geom/9410022;
Lecture Notes of the CIME Session “Transcendental methods in Algebraic Geometry”, Cetraro, Italy, July 1994, Ed. F. Catanese, C. Ciliberto, Lecture Notes in Math., Vol. 1646, 1–97.

[Demailly95] Demailly, J.-P.: *Algebraic Criteria for Kobayashi Hyperbolic Projective Varieties and Jet Differentials*. Algebraic geometry – Santa Cruz 1995, 285–360, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[D-EG00] Demailly, J.-P., El Goul, J.: *Hyperbolicity of Generic Surfaces of High Degree in Projective 3-Space*. Amer. J. Math. **122** (2000), no. 3, 515–546.

[Diverio08] Diverio, S.: *Existence of global invariant jet differentials on projective hypersurfaces of high degree*. arXiv:0802.0045, [math.AG].

[DRM08] Diverio, S., Merker, J., Rousseau, E.: *Effective algebraic degeneracy*. arXiv:0811.2346, [math.AG].

[F-H91] Fulton, W., Harris, J.: *Representation Theory: A First Course*. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991, xvi+551 pp.

[G-G79] Green, M., Griffiths, P.: *Two Applications of Algebraic Geometry to Entire Holomorphic Mappings*. The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), pp. 41–74, Springer, New York-Berlin, 1980.

[Kobayashi70] Kobayashi S.: *Hyperbolic Manifolds and Holomorphic Mappings*. Marcel Dekker, Inc., New York 1970 ix+148 pp.

[Lang86] Lang S.: *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. **14** (1986), no. 2, 159–205.

[Merker08] Merker, J.: *Low pole order frames on vertical jets of the universal hypersurface*. arXiv:0805.3987 [math.AG].

[Paun08] Păun, M.: *Vector fields on the total space of hypersurfaces in the projective space and hyperbolicity*. Math. Ann. **340** (2008), 875–892.

[Rousseau05] Rousseau, E.: *Weak Analytic Hyperbolicity of Generic Hypersurfaces of High Degree in the Complex Projective Space of Dimension 4*. arXiv:math/0510285v1 [math.AG].

[Rousseau06a] Rousseau, E.: *Étude des Jets de Demailly-Semple en Dimension 3*. Ann. Inst. Fourier (Grenoble) **56** (2006), no. 2, 397–421.

[Rousseau06b] Rousseau, E.: *Équations Différentielles sur les Hypersurfaces de \mathbb{P}^4* . J. Math. Pures Appl. (9) **86** (2006), no. 4, 322–341.

[SY97] Siu, Y.-T., Yeung, S.-K.: *Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees*. Amer. J. Math. **119** (1997), 1139–1172.

[Siu04] Siu, Y.-T.: *Hyperbolicity in Complex Geometry*. The legacy of Niel Henrik Abel, 543–566, Springer, Berlin, 2004.

[Trapani95] Trapani, S.: *Numerical criteria for the positivity of the difference of ample divisors*, Math. Z. **219** (1995), no. 3, 387–401.

[Voisin96] Voisin, C.: *On a conjecture of Clemens on rational curves on hypersurfaces*. J. Diff. Geom. **44** (1996), 200213. A correction: *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Diff. Geom. **49** (1998), 601611.

[Vojta87] Vojta, P.: *Diophantine Approximations and Value Distribution Theory*, Springer-Verlag, Lecture Notes in Mathematics no. 1239, 1987.