Uniformization in several complex variables.

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Philippe Eyssidieux.
Institut Fourier,
Université Joseph Fourier (Grenoble 1).

The uniformization theorem in one complex variable was formulated by Riemann and completely proved by Koebe and independantly Poincaré in 1907.

Theorem 1 *Let U be a connected and simply connected Riemann Surface. Then U is isomorphic to either $\mathbf{P^1}(\mathbb{C})$ (elliptic case), $\mathbb{C}$ (parabolic case) or $\Delta = \{z \in \mathbb{C} | \quad |z| < 1\}$ (hyperbolic case).*

A Riemann surface is a paracompact complex manifold of dimension one.

The most efficent proof that a compact simply connected Riemann Surface is isomorphic to a complex projective line relies on the Riemann-Roch theorem. We shall assume in the rest of the discussion that we are in the non-compact case.

That U is isomorphic to $\mathbb{C}$ or Δ means that there is a holomorphic function $z : U \to \mathbb{C}$ which gives rise to a homeomorphism from U to $\mathbb{C}$ or Δ. Call such a z a *global uniformization parameter* for U.

Remark Global uniformization parameters are rather rigid, i.e.: determined up to an isomorphism of the model space.

In the elliptic case, given such a z every global uniformization parameter takes the form:

$$z' = az + b \qquad a \in \mathbb{C}^*, b \in C.$$

In the hyperbolic case, we have:

$$z' = \frac{pz + q}{p'z + q'} \qquad \begin{pmatrix} p & q \\ p' & q' \end{pmatrix} \in U(1, 1)$$

When U is the universal covering space of a compact Riemann surface, we have the following more precise statement:

Theorem 2 *Let S be a compact connected Riemann surface of genus g and $\pi : U \to S$ be its universal covering space.*

$g = 0$, *then S is simply connected, $U = S$ and $S \simeq \mathbf{P^1}(\mathbb{C})$.*

$g = 1$, *then $U \simeq \mathbb{C}$, $S \simeq \Lambda \backslash \mathbb{C}$ where $\Lambda \simeq \mathbb{Z}^2$ is a rank 2 discrete subgroup of $\mathbb{C}$.*

$g \geq 2$ *then $U \simeq \Delta$, $S \simeq \Gamma \backslash \Delta$ where $\Gamma \subset PU(1,1)$ is a torsion-free cocompact discrete subgroup isomorphic to*

$$< a_1, b_1, \ldots, a_g, b_g \quad | \quad a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 >$$

For the proof of the uniformization theorem, we need to construct a global uniformization parameter. There are various ways of doing this, some of them working only a posteriori:

- (in the compact elliptic case) elliptic integrals.
- Solve Dirichlet problems to manufacture a Green's function.
- (in the hyperbolic case) Find an extremal disk in S for the Kobayashi metric.
- (in the hyperbolic case) Find an extremal function on U for the Caratheodory metric.
- (in the compact hyperbolic case) Find a conformal hyperbolic metric, solving
$$1 + \Delta_g \phi = e^{\phi} v$$
where g is a conformal metric and $v > 0$ is a suitable smooth function.
- .....

Question: What happens in several complex variables?

Structure of the universal covering space

The Shafarevich conjecture -actually this was not formulated as a conjecture by Shafarevich, just as a guess-predicts that the universal covering space of a complex projective manifold (compact and embeddable in $\mathbf{P}^N(\mathbb{C})$) should be holomorphically convex.

This problem is open, in spite of recent positive results. These results are expected to hold for compact Kähler manifolds.

Evidence comes from examples of projective (or compact Kähler) manifolds with Stein universal covering. They include abelian varieties (or compact complex tori) quotient of bounded symmetric examples, Mostow-Siu-Deraux examples.... Furthermore the conjecture is invariant by product, birational transformations and also enjoys some heredity.

Structure of the fundamental group

Any finitely generated group can arise as the fundamental group of a compact complex manifold of dimension 3 (Taubes).

On the other (Non-abelian) Hodge Theory strongly restricts the class of Kähler groups, i.e.: finitely generated groups arising as the fundamental group of a compact Kähler manifold.

Characterizing the class of Kähler groups in terms of their algebraic or combinatorial properties seems to be out of reach.

From now on, we will restrict to the Kähler case and introduce the basic ideas of Non abelian Hodge Theory focusing on their applications to the Shafarevich conjecture.

Reformulations of the Shafarevich conjecture

Let X be a compact Kähler manifold and $H \subset \pi_1(X)$ be a normal subgroup.

Say (X, H) satisfies (HC) iff $H\backslash\widetilde{X^{univ}}$ is holomorphically convex.

Example If (X, H) satisfies (HC) and $f : Y \to X$ is an holomorphic map from a compact Kähler manifold then $(Y, f_*^{-1}H)$ satisfies (HC).

If (X, H) satisfies (HC), then there is a proper holomorphic mapping with connected fibers

$$\widetilde{s^H} : H\backslash\widetilde{X^{univ}} \to \widetilde{S_H(X)}$$

which contracts precisely the compact connected analytic subspaces of $H\backslash\widetilde{X^{univ}}$. The mapping s^H is equivariant under the Galois group $G = H\backslash\pi_1(X)$ which acts properly and cocompactly on $\widetilde{S_H(X)}$.

The quotient map $s^H : X \to G\backslash\widetilde{S_H(X)}$ is called the H-Shafarevich morphism.

Constructing the H-Shafarevich morphism is the first step to settle when trying to prove (HC).

The second step is to prove that the normal complex space $\widetilde{S_H(X)}$ is Stein.

The best general result on the first step of the Shafarevich conjecture is:

Theorem 3 *(Campana 1993, independant work by Kollár in the projective case) One can construct a meromorphic map $\widetilde{s^H} : H\backslash\widetilde{X^{univ}} \to \widetilde{S_H(X)}$ which is proper and holomorphic outside $\pi^{-1}Z$ where $Z \subset X$ is a proper complex analytic subvariety whose general fiber is a maximal compact connected analytic subvariety of $H\backslash\widetilde{X^{univ}}$.*

Cycle-theoretic methods do not give anything on the second step. Indeed they are blind to the specific nature of H. To get (HC), one must take H large enough in view of the Cousin example.

Example Let A be a simple abelian surface $A = \Lambda\backslash\mathbb{C}^2$ with $\Lambda \simeq \mathbb{Z}^4$. Let $\rho : \Lambda \to \mathbb{Z}$ be a surjective morphism. Then $(X, H) = (A, \ker(\rho))$ is non compact and has no nonconstant holomorphic functions. On the other hand, the first half of (HC) does hold since $\widetilde{s^H} = id$ satisfies the above property since $\ker(\gamma)\backslash\mathbb{C}^2$ has no positive dimensional compact complex submanifold.

Crash course in (Abelian) Hodge theory

Definition 4 *Let X be a complex manifold. A symplectic (real) 2-form ω is a Kähler form if it can be expressed in local complex coordinates as:*

$$\omega = \frac{\sqrt{-1}}{2} \sum_{1 \le a,b \le n} g_{a\bar{b}} dz^a \wedge d\bar{z}^b$$

with $(g_{a\bar{b}})$ a smooth positive definite hermitian matrix.

A Kähler manifold is a complex manifold which carries a Kähler form.

Example $\mathbf{P}^N(\mathbb{C})$ is Kähler. More generally every complex projective manifold is a compact Kähler manifold.

Proposition 5 *Let X be a compact Kähler manifold.*

Every holomorphic 1-form is closed and harmonic. Hence, a holomorphic 1-form is exact iff it is zero.

Conversely a complex harmonic one form η uniquely decomposes as $\eta = \phi + \bar{\psi}$ where ϕ, ψ are holomorphic one forms.

In particular, $b_1(X) \equiv 0[2]$ and we have a Hodge decomposition:

$$H^1_{DR}(X, \mathbb{C}) = H^{1,0} \oplus \overline{H^{1,0}} \qquad H^{1,0} = H^0(X, \Omega^1_X).$$

This decomposition is orthonormal and definite for the non degenerate form of signature $(h^0(\Omega^1), h^0(\Omega^1))$ defined by:

$$S(\eta, \bar{\eta}) = \sqrt{-1} \int_X \eta \wedge \bar{\eta} \wedge \omega^{n-1}.$$

Using this we show (HC) for $(X, [\pi_1, \pi_1])$ using the proper holomorphic mapping constructed from abelian integrals:

$$\left(\begin{array}{ccc} [\pi_1, \pi_1] \backslash \widetilde{X^{univ}} & \longrightarrow & H^0(\Omega^1_X)^* \\ p & \mapsto & (\alpha \mapsto \int_{\tilde{x}}^{p} \alpha) \end{array} \right).$$

Crash course in (Non-Abelian) Hodge theory

The non abelian cohomology set $H^1(X, GL_N(\mathbb{C}))$ can be interpreted as the set of conjugacy classes of representations $\rho : \pi_1(X) \to GL_N(\mathbb{C})$.

If it exists a harmonic ρ-equivariant mapping h_ρ:

$$h_\rho : \tilde{X}^{univ} \to GL_N(\mathbb{C})/U(N)$$

is well defined up to the action of the normalizer of $\rho(\pi_1(X))$ in $GL_N(\mathbb{C})$. This might be seen as the harmonic representative of the cohomology class defined by ρ.

Theorem 6 *(Corlette-Labourie) Let (M, g) be a compact Riemannian manifold and $\rho : \pi_1(M) \to GL_N(\mathbb{C})$ be a representation. Then ρ has a harmonic representative iff ρ is semisimple.*

(Eells-Sampson) If X is a compact Kähler manifold then the harmonic representative of a semisimple representation does not depend on the Kähler metric and if $f : Y \to X$ is a holomorphic mapping then $h_\rho \circ \tilde{f}$ is a harmonic representative for $f^*\rho$.

Higgs bundles, according to C. Simpson.

The information on h_ρ can be repackaged in a rank N Higgs bundle.

Definition 7 *A Higgs bundle (E, θ) on a compact Kähler manifold X is:*

- *a holomorphic bundle E on X,*
- *$\theta : E \to E \otimes \Omega^1_X$ a holomorphic $End(E)$-valued 1-form,*
- *such that $\theta \wedge \theta = 0$.*

Definition 8 *A Higgs subsheaf $(F, \theta) \subset (E, \theta)$ is a coherent analytic subsheaf $F \subset E$ which is θ-stable.*

A Higgs bundle is stable iff for every non trivial Higgs subsheaf $(F, \theta) \subset (E, \theta)$ we have:

$$\frac{\int_X c_1(F).\omega^{n-1}}{rk(F)} < \frac{\int_X c_1(E).\omega^{n-1}}{rk(E)}.$$

A Higgs bundle is polystable iff it is a direct sum of stable Higgs bundles.

Theorem 9 *(Simpson) A polystable Higgs bundle with $\int_X c_1(E)\omega^{n-1} = \int_X c_2(E)\omega^{n-2} = 0$ comes from a semisimple representation of the fundamental group.*

Example If $\rho : \pi_1(X) \to \mathbb{C}^*$ is an abelian representation then we may construct $\log|\rho| : \pi_1(X) \to \mathbb{R}$ and $\rho/|\rho| : \pi_1(X) \to U(1)$.

The representation $\log|\rho|$ can be interpreted as a degree 1 real cohomology class whose harmonic representative takes the form $\Re(\alpha)$ with $\alpha \in H^0(\Omega^1_X)$.

A holomorphic line bundle is numerically trivial iff it its the holomorphic line bundle underlying a flat $U(1)$ line bundle. Hence there is a well defined numerically trivial line bundle L unambiguously attached to $\rho/|\rho|$.

The Higgs bundle attached to ρ is (L, α).

Conversly if L is numerically trivial and $\alpha \in H^0(\Omega^1_X)$ L carries a flat unitary connection d and $d + \alpha$ is a flat connection on L since $d\alpha = 0$.

An ubiquitous class of examples

Definition 10 *A $\mathbb{C}$-VHS (polarized complex variation of Hodge structures) on X of weight $w \in \mathbb{Z}$ is a 5-tuple $(X, \mathbb{V}, \mathcal{F}^\bullet, \overline{\mathcal{G}}^\bullet, S)$ where:*

1. *$\mathbb{V}$ is a local system of finite dimensional $\mathbb{C}$-vector spaces,*
2. *S a non degenerate flat sesquilinear pairing on $\mathbb{V}$,*
3. *$\mathcal{F}^\bullet = (\mathcal{F}^p)_{p\in\mathbb{Z}}$ a biregular decreasing filtration of $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_X$ by holomorphic subbundles such that $d'\mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_X$,*
4. *$\overline{\mathcal{G}}^\bullet = (\overline{\mathcal{G}}^q)_{q\in\mathbb{Z}}$ a biregular decreasing filtration of $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_{\bar{X}}$ by antiholomorphic subbundles such that $d''\overline{\mathcal{G}}^p \subset \overline{\mathcal{G}}^{p-1} \otimes \Omega^1_{\bar{X}}$,*
5. *for every point $x \in X$ the fiber at x $(\mathbb{V}_x, \mathcal{F}^\bullet_x, \overline{\mathcal{G}}^\bullet_x)$ is a $\mathbb{C}$-HS polarized by S_x.*

These conditions can be expressed in saying that there is a Hodge decomposition:

$$\mathbb{V}_x = \oplus_{p+q=w} H^{p,q}_x$$

where $H^{p,q}_x$ is a C^∞-subbundle, the decomposition being S-orthogonal and $(-1)^p.S_{H^{p,q}} > 0$. The link is given by the formulae:

$$\mathcal{F}^P = \oplus_{p\geq P} H^{p,q} \quad \overline{\mathcal{G}^Q} = \oplus_{q\geq Q} H^{p,q}.$$

Proposition 11 *We can endow $H^{p,q}$ with a holomorphic structure by indentifying it with $\mathcal{F}^P/\mathcal{F}^{P+1}$. The d'-fundamental form then gives a holomorphic map: $\nabla' : H^{p,q} \to H^{p-1,q+1} \otimes \Omega^1_X$.*

Then, $(\oplus_p H^{p,q}, \oplus_p \nabla')$ is the Higgs bundle attached to $\mathbb{V}$.

Theorem 12 *(Simpson) If a numerically trivial polystable Higgs bundle satisfies $(E, \theta) \simeq (E, t\theta)$ for all $t \in \mathbb{C}^*$ then it underlies a polarizable $\mathbb{C}$-VHS.*

The uniformization theorem in the compact hyperbolic case

Let us use Theorem 12.

Let C be a Riemann surface of genus $g \geq 2$. Then Ω^1_X had degree $2g-2 \equiv 0[2]$. Hence there is a holomorphic line bundle $\mathcal{S}$ such that $\mathcal{S}^{\otimes 2} \simeq \Omega^1_X$. This gives $\eta : \mathcal{S} \to \mathcal{S}^{-1} \otimes \Omega^1_X$.

The Higgs bundle $(E,\theta) = (\mathcal{S} \oplus \mathcal{S}^{-1}, \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix})$ is numerically trivial and stable. It satisfies Simpson's condition for a $\mathbb{C}$-VHS with $(h^{1,0}, h^{0,1}) = (1,1)$.

The period mapping of this $\mathbb{C}$-VHS is actually a holomorphic mapping, equivariant by a reprentation $\rho : \pi_1(C) \to U(1,1)$

$$z : \widetilde{C^{univ}} \to U(1,1)/U(1) \times U(1) = \Delta$$

whose derivative is (essentially) η hence never vanishes. In particular $z^* ds^2_\Delta$ is a hyperbolic conformal metric and the uniformization theorem for C follows.

The period mapping for this VHS is thus a global uniformization parameter

Reductive Shafarevich Conjecture

Theorem 13 *(E., 2004)*
Let X be a connected complex projective manifold and

$$H = \bigcap_{\rho:\pi_1(X)\to GL_N(\mathbb{C}) \ \textit{reductive}} \ker(\rho).$$

Then (X, H) satisfies (HC).

In fact, this is a consequence of a hereditary statement I will not describe.

This also implies that if $\pi_1(X)$ has a complex linear representation with infinite image then there are non constant holomorphic functions on $\tilde{X}^{univ}$.

The period mappings of the $\mathbb{C}$-VHS constructed via Theorem 12 are major ingredients in the proof of this statement.

Linear Shafarevich Conjecture

Theorem 14 *(E.- Katzarkov-Pantev-Ramachandran, 2009, arxiv:0904.0693)*
Let X be a connected complex projective manifold and

$$H = \bigcap_{\rho:\pi_1(X)\to GL_N(A) \ \ A \text{ a } \mathbb{C}\text{-algebra.}} \ker(\rho).$$

Then (X, H) satisfies (HC).

We have not been able to develop a hereditary statement implying this. Hopefully, this can be done.

The compact Kähler case is expected to hold true.

Classical Abelian integrals are actually period mapping for unipotent VMHS (variations of Mixed Hodge structures, a generalization of VHS). The period mappings of the universal $\mathbb{C}$-VMHS constructed in (E.- Simpson 2009, arxiv:0902.2626) play actually the central role in this case and turn out to be the right non-abelian analogue of abelian integrals.