



L^2 extension theorem for sections defined on non reduced analytic subvarieties

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References

This is a joint work with [Junyan Cao & Shin-ichi Matsumura](#)

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The extension problem

Let (X, ω) be a **possibly noncompact** n -dimensional Kähler manifold, $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf, $Y = V(\mathcal{I})$ its zero variety and

$$\mathcal{O}_Y = \mathcal{O}_X / \mathcal{I}.$$

Here Y may be non reduced, i.e. \mathcal{O}_Y may have **nilpotent elements**.

Also, let (L, h_L) be a hermitian holomorphic line bundle on X , and

$$\Theta_{L, h_L} = i \partial \bar{\partial} \log h_L^{-1}$$

its curvature current (we allow singular metrics, $h_L = e^{-\varphi}$, $\varphi \in L^1_{\text{loc}}$, Θ_{L, h_L} being computed in the sense of currents).

Question

Under which conditions on X , $Y = V(\mathcal{I})$, (L, h_L) is

$$H^q(X, K_X \otimes L) \rightarrow H^q(Y, (K_X \otimes L)|_Y) = H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{O}_X / \mathcal{I})$$

a surjective restriction morphism?

Assumptions (1)

We assume X to be **holomorphically convex**. By the Cartan-Remmert theorem, this is the case iff X admits a **proper holomorphic map** $p : X \rightarrow S$ only a Stein complex space S .

Observation : cohomology is then always Hausdorff

Let X be a holomorphically convex complex space and \mathcal{F} a coherent analytic sheaf over X . Then all cohomology groups $H^q(X, \mathcal{F})$ are **Hausdorff** with respect to their natural topology (local uniform convergence of holomorphic Čech cochains)

Proof. $H^q(X, \mathcal{F}) \simeq H^0(S, R^q p_* \mathcal{F})$ is a Fréchet space.

Corollary

To solve an equation $\bar{\partial} u = v$ on a holomorphically convex manifold X , it is enough to solve it approximately:

$$\bar{\partial} u_\varepsilon = v + w_\varepsilon, \quad w_\varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

Assumptions (2)

We assume that the subvariety $Y \subset X$ is defined by

$$Y = V(\mathcal{I}(e^{-\psi})), \quad \mathcal{O}_Y := \mathcal{O}_X / \mathcal{I}(e^{-\psi})$$

where ψ is a quasi-psh function with *analytic singularities*, i.e. locally on a neighborhood V of an arbitrary point $x_0 \in X$ we have

$$\psi(z) = c \log \sum |g_j(z)|^2 + v(z), \quad g_j \in \mathcal{O}_X(V), \quad c > 0, \quad v \in C^\infty(V),$$

and $\mathcal{I}(e^{-\psi}) \subset \mathcal{O}_X$ is the **multiplier ideal sheaf**

$$\mathcal{I}(e^{-\psi})_{x_0} = \{f \in \mathcal{O}_{X,x_0}; \exists U \ni x_0, \int_U |f|^2 e^{-\psi} d\lambda < +\infty\}$$

Claim (Nadel)

$\mathcal{I}(e^{-\psi})$ is a coherent ideal sheaf.

Moreover $\mathcal{I}(e^{-\psi})$ is always an integrally closed ideal.

Typical choice: $\psi(z) = c \log |s(z)|_{h_E}^2$, $c > 0$, $s \in H^0(X, E)$.

Log resolution / reduction to the divisorial case

The simplest case is when $Y = \sum m_j Y_j$ is an effective simple normal crossing divisor and $\mathcal{O}_Y = \mathcal{O}_X / \mathcal{O}_X(-Y)$. We can then take

$$\psi(z) = \sum c_j \log |\sigma_{Y_j}|_{h_j}^2, \quad c_j > 0, \quad [c_j] = m_j,$$

for some smooth hermitian metric h_j on $\mathcal{O}_X(Y_j)$. Then

$$\mathcal{I}(e^{-\psi}) = \mathcal{O}_X(-\sum m_j Y_j), \quad i\partial\bar{\partial}\psi = \sum c_j(2\pi[Y_j] - \Theta_{\mathcal{O}(Y_j), h_j})$$

The case of a higher codimensional multiplier ideal scheme $\mathcal{I}(e^{-\psi})$ can easily be reduced to the divisorial case by using a suitable log resolution (a composition of blow ups, thanks to Hironaka's desingularization theorem).

Main results

Theorem (JY. Cao, D- , S-i. Matsumura, January 2017)

Take (X, ω) to be **Kähler and holomorphically convex**, and let (L, h_L) be a hermitian line bundle such that

$$(**) \quad \Theta_{L, h_L} + (1 + \alpha\delta)i\partial\bar{\partial}\psi \geq 0 \quad \text{in the sense of currents}$$

for some $\delta(x) > 0$ continuous and $\alpha = 0, 1$. Then:

the morphism induced by the natural inclusion $\mathcal{I}(h_L e^{-\psi}) \rightarrow \mathcal{I}(h_L)$

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-\psi})) \rightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L))$$

is injective for every $q \geq 0$, in other words, the sheaf morphism $\mathcal{I}(h) \rightarrow \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})$ yields a surjection

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)) \rightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})).$$

Corollary (take h_L smooth $\Rightarrow \mathcal{I}(h_L) = \mathcal{O}_X$, and $Y = V(\mathcal{I}(e^{-\psi}))$)

If h_L is smooth, $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}(e^{-\psi})$ and h_L, ψ satisfy **(**)**, then $H^q(X, K_X \otimes L) \rightarrow H^q(Y, (K_X \otimes L)|_Y)$ is surjective.

Comments / algebraic consequences

The exact sequence $0 \rightarrow \mathcal{I}(h_L e^{-\psi}) \rightarrow \mathcal{I}(h_L) \rightarrow \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi}) \rightarrow 0$ implies that both injectivity and surjectivity hold when

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-\psi})) = 0,$$

and for this it is enough to have a **strict curvature assumption**

$$(***) \quad \Theta_{L, h_L} + i\partial\bar{\partial}\psi \geq \delta\omega > 0 \quad \text{in the sense of currents.}$$

Corollary (purely algebraic)

Assume that X is projective (or that one has a projective morphism $X \rightarrow S$ over an affine algebraic base S). Let $Y = \sum m_j Y_j$ be an effective divisor and $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{O}_X(-Y)$. If (as \mathbb{Q} -divisors)

$$(**) \quad L - (1 + \delta) \sum c_j Y_j = G_\delta + U_\delta, \quad [c_j] = m_j$$

with $\delta = 0$ or $\delta_0 \in \mathbb{Q}_+^*$, G_δ semiample and $U_\delta \in \text{Pic}^0(X)$, then

$$H^q(X, K_X \otimes L) \rightarrow H^q(Y, (K_X \otimes L)|_Y)$$

is surjective.

Possible use for abundance by induction on $\dim X$?

For a line bundle L , one defines the Kodaira-Iitaka dimension $\kappa(L) = \limsup_{m \rightarrow +\infty} \log \dim H^0(X, L^{\otimes m}) / \log m$ and the numerical dimension $\text{nd}(L) = \text{maximum power of non zero positive intersection of a positive current } T \in c_1(L)$, if L is pseudo-effective, and $\text{nd}(L) = -\infty$ otherwise. They always satisfy

$$-\infty \leq \kappa(L) \leq \text{nd}(L) \leq n = \dim X.$$

One says that L is **abundant** if $\kappa(L) = \text{nd}(L)$. The **abundance conjecture** states that K_X is always abundant if X is nonsingular and projective (or even compact Kähler). More generally, it is expected that $K_X + \Delta$ is abundant for every effective klt \mathbb{Q} -divisor Δ .

When X is not uniruled, i.e. K_X is pseudoeffective, one can ask whether the following **generalized abundance property** holds true: let L be a line bundle such that $L - \varepsilon K_X$ is pseudoeffective, $0 < \varepsilon \ll 1$; does there exist $G \in \text{Pic}^0(X)$ such that $L + G$ is abundant ?

Twisted Bochner-Kodaira-Nakano inequality (Ohsawa-Takegoshi)

Let (X, ω) be a Kähler manifold and let $\eta, \lambda > 0$ be smooth functions on X .

For every compactly supported section $u \in C_c^\infty(X, \Lambda^{p,q} T_X^* \otimes L)$ with values in a hermitian line bundle (L, h_L) , one has

$$\begin{aligned} & \|(\eta + \lambda)^{\frac{1}{2}} \bar{\partial}^* u\|^2 + \|\eta^{\frac{1}{2}} \bar{\partial} u\|^2 + \|\lambda^{\frac{1}{2}} \partial u\|^2 + 2\|\lambda^{-\frac{1}{2}} \partial \eta \wedge u\|^2 \\ & \geq \int_X \langle B_{L, h_L, \omega, \eta, \lambda}^{p,q} u, u \rangle dV_{X, \omega} \end{aligned}$$

where $dV_{X, \omega} = \frac{1}{n!} \omega^n$ is the Kähler volume element and $B_{L, h_L, \omega, \eta, \lambda}^{p,q}$ is the Hermitian operator on $\Lambda^{p,q} T_X^* \otimes L$ such that

$$B_{L, h_L, \omega, \eta, \lambda}^{p,q} = [\eta i\Theta_L - i\partial\bar{\partial}\eta - i\lambda^{-1}\partial\eta \wedge \bar{\partial}\eta, \Lambda_\omega].$$

Approximate solutions to $\bar{\partial}$ -equations

Main L^2 estimate

Let (X, ω) be a Kähler manifold possessing a complete Kähler metric let (E, h_E) be a Hermitian vector bundle over X . Assume that $B = B_{E, h, \omega, \eta, \lambda}^{n, q}$ satisfies $B + \varepsilon \text{Id} > 0$ for some $\varepsilon > 0$ (so that B can be just semi-positive or even slightly negative).

Take a section $v \in L^2(X, \Lambda^{n, q} T_X^* \otimes E)$ such that $\bar{\partial}v = 0$ and

$$M(\varepsilon) := \int_X \langle (B + \varepsilon \text{Id})^{-1} v, v \rangle dV_{X, \omega} < +\infty.$$

Then there exists an approximate solution $u_\varepsilon \in L^2(X, \Lambda^{n, q-1} T_X^* \otimes E)$ and a **correction term** $w_\varepsilon \in L^2(X, \Lambda^{n, q} T_X^* \otimes E)$ such that

$$\bar{\partial}u_\varepsilon = v + w_\varepsilon \quad \text{and}$$

$$\int_X (\eta + \lambda)^{-1} |u_\varepsilon|^2 dV_{X, \omega} + \frac{1}{\varepsilon} \int_X |w_\varepsilon|^2 dV_{X, \omega} \leq M(\varepsilon).$$

Proof: setting up the relevant $\bar{\partial}$ equation (1)

Every cohomology class in

$$H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi}))$$

is represented by a holomorphic Čech q -cocycle with respect to a Stein covering $\mathcal{U} = (U_i)$, say $(c_{i_0 \dots i_q})$,

$$c_{i_0 \dots i_q} \in H^0(U_{i_0} \cap \dots \cap U_{i_q}, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})).$$

By the standard sheaf theoretic isomorphism with Dolbeault cohomology, this class is represented by a smooth (n, q) -form

$$f = \sum_{i_0, \dots, i_q} c_{i_0 \dots i_q} \rho_{i_0} \bar{\partial} \rho_{i_1} \wedge \dots \wedge \bar{\partial} \rho_{i_q}$$

by means of a partition of unity (ρ_i) subordinate to (U_i) . This form is to be interpreted as a form on the (non reduced) analytic subvariety Y associated with the colon ideal sheaf $\mathcal{J} = \mathcal{I}(h e^{-\psi}) : \mathcal{I}(h)$ and the structure sheaf $\mathcal{O}_Y = \mathcal{O}_X / \mathcal{J}$.

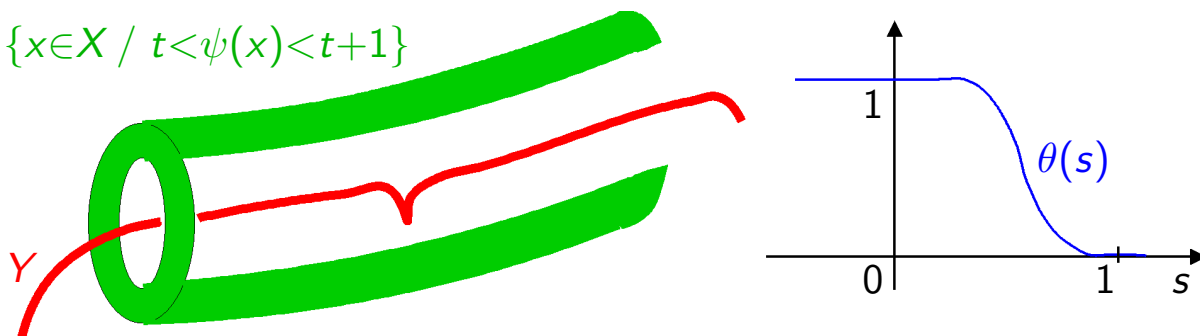
Proof: setting up the relevant $\bar{\partial}$ equation (2)

We get an extension of f as a smooth (no longer $\bar{\partial}$ -closed) (n, q) -form on X by taking

$$\tilde{f} = \sum_{i_0, \dots, i_q} \tilde{c}_{i_0 \dots i_q} \rho_{i_0} \bar{\partial} \rho_{i_1} \wedge \dots \wedge \bar{\partial} \rho_{i_q}$$

where $\tilde{c}_{i_0 \dots i_q} =$ extension of $c_{i_0 \dots i_q}$ from $U_{i_0} \cap \dots \cap U_{i_q} \cap Y$ to $U_{i_0} \cap \dots \cap U_{i_q}$

$\{x \in X / t < \psi(x) < t+1\}$



Now, truncate \tilde{f} as $\theta(\psi - t) \cdot \tilde{f}$ on the green hollow tubular neighborhood, and solve an approximate $\bar{\partial}$ -equation

$$(*) \quad \bar{\partial} u_{t, \varepsilon} = \bar{\partial}(\theta(\psi - t) \cdot \tilde{f}) + w_{t, \varepsilon}$$

Proof: setting up the relevant $\bar{\partial}$ equation (3)

Here we have

$$\bar{\partial}(\theta(\psi - t) \cdot \tilde{f}) = \theta'(\psi - t) \bar{\partial} \psi \wedge \tilde{f} + \theta(\psi - t) \cdot \bar{\partial} \tilde{f}$$

where the first term vanishes near Y and the second one is L^2 with respect to $h_L e^{-\psi}$ (as the image of $\bar{\partial} \tilde{f}$ in $\mathcal{I}(h_L) / \mathcal{I}(h_L e^{-\psi})$ is $\bar{\partial} f = 0$).

With ad hoc “twisting functions” $\eta = \eta_t := 1 - \delta_{\chi_t}(\psi)$, $\lambda := \pi(1 + \delta^2 \psi^2)$ and a suitable adjustment $\varepsilon = e^{(1+\beta)t}$, $\beta \ll 1$, we can find approximate L^2 solutions of the $\bar{\partial}$ -equation such that

$$\bar{\partial} u_{t, \varepsilon} = \bar{\partial}(\theta(\psi - t) \cdot \tilde{f}) + w_{t, \varepsilon}, \quad \int_X |u_{t, \varepsilon}|_{\omega, h_L}^2 e^{-\psi} dV_{X, \omega} < +\infty$$

and

$$\lim_{t \rightarrow -\infty} \int_X |w_{t, \varepsilon}|_{\omega, h_L}^2 e^{-\psi} dV_{X, \omega} = 0.$$

The estimate on $u_{t, \varepsilon}$ with respect to the weight $h_L e^{-\psi}$ shows that $\theta(\psi - t) \cdot \tilde{f} - u_{t, \varepsilon}$ is an approximate extension of f . \square

Can one get estimates for the extension ?

The answer is **yes if ψ is log canonical**, namely $\mathcal{I}(e^{-(1-\varepsilon)\psi}) = \mathcal{O}_X$ for all $\varepsilon > 0$.

Ohsawa's residue measure

If ψ is log canonical, one can also associate in a natural way a measure $dV_{Y^\circ, \omega}[\psi]$ on the set Y° of regular points of Y as follows. If $g \in \mathcal{C}_c(Y^\circ)$ is a compactly supported continuous function on Y° and \tilde{g} a compactly supported extension of g to X , one sets

$$\int_{Y^\circ} g dV_{Y^\circ, \omega}[\psi] = \lim_{t \rightarrow -\infty} \int_{\{x \in X, t < \psi(x) < t+1\}} \tilde{g} e^{-\psi} dV_{X, \omega}$$

Theorem

If ψ is lc and the curvature hypothesis is satisfied, for any f in $H^0(Y, K_X \otimes L \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi}))$ s.t. $\int_{Y^\circ} |f|_{\omega, h_L}^2 dV_{Y^\circ, \omega}[\psi] < +\infty$, there exists $\tilde{f} \in H^0(X, K_X \otimes L \otimes \mathcal{I}(h_L))$ which extends f , such that

$$\int_X (1 + \delta^2 \psi^2)^{-1} e^{-\psi} |\tilde{f}|_{\omega, h_L}^2 dV_{X, \omega} \leq \frac{34}{\delta} \int_{Y^\circ} |f|_{\omega, h_L}^2 dV_{Y^\circ, \omega}[\psi].$$

Can one get estimates for the extension ? (sequel)

If ψ is not log canonical, consider the "last jumps" $m_{p-1} < m_p \leq 1$ such that $\mathcal{I}(h_L e^{-m_{p-1}\psi}) \supsetneq \mathcal{I}(h_L e^{-m_p\psi}) = \mathcal{I}(h_L e^{-\psi})$ and assume

$$f \in H^0(Y, K_X \otimes L \otimes \mathcal{I}(h_L e^{-m_{p-1}\psi})/\mathcal{I}(h_L e^{-m_p\psi})),$$

i.e., f vanishes just a little bit less than prescribed by the sheaf $\mathcal{I}(h_L e^{-\psi})$. Then there is still a corresponding residue measure:

Higher multiplicity residue measure

If f is as above, and \tilde{f} is a local extension, one can associate a higher multiplicity residue measure $|f|^2 dV_{Y^\circ, \omega}[\psi]$ (formal notation) as follows. If $g \in \mathcal{C}_c(Y^\circ)$ and \tilde{g} a compactly supported extension of g to X , one sets

$$\int_{Y^\circ} g |f|^2 dV_{Y^\circ, \omega}[\psi] = \lim_{t \rightarrow -\infty} \int_{\{x \in X, t < \psi(x) < t+1\}} \tilde{g} |\tilde{f}|^2 e^{-m_p\psi} dV_{X, \omega}$$

Then a global extension $\tilde{f} \in H^0(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-m_{p-1}\psi}))$ exists, that satisfies the expected L^2 estimate.

Special case / limitations of the L^2 estimates

In the special case when ψ is given by $\psi(z) = r \log |s(z)|_{h_E}^2$ for a section $s \in H^0(X, E)$ generically transverse to the zero section of a rank r vector bundle E on X , the subvariety $Y = s^{-1}(0)$ has codimension r , and one can check easily that

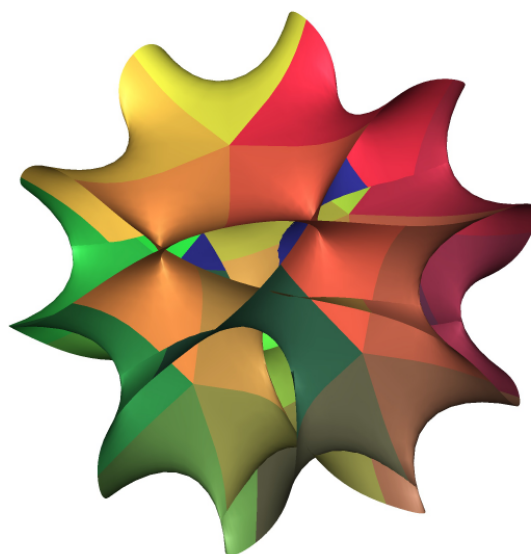
$$dV_{Y^\circ, \omega}[\psi] = \frac{dV_{Y^\circ, \omega}}{|\Lambda^r(ds)|_{\omega, h_E}^2}.$$

Thus one sees that the residue measure takes into account in a very precise manner the singularities of Y . It may happen that $dV_{Y^\circ, \omega}[\psi]$ has infinite mass near the singularities of Y , as is the case when Y is a simple normal crossing divisor.

Therefore, sections $s \in H^0(Y, (K_X \otimes L)|_Y)$ may not be L^2 with respect to $dV_{Y^\circ, \omega}[\psi]$, and the L^2 estimate of the approximate extension can blow up as $\varepsilon \rightarrow 0$. The surprising fact is this is however sufficient to prove the qualitative extension theorem, but without any effective L^2 estimate in the limit.

The end

Thank you!



Tentative image of a Calabi-Yau manifold ($K_X = \mathcal{O}_X$). It would be important to know about the generalized abundance property in that case.