Algebraic structure of the ring of jet differential operators and hyperbolic varieties

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Kobayashi metric / hyperbolic manifolds

- Let X be a complex manifold, $n = \dim_{\mathbb{C}} X$.
- X is said to be hyperbolic in the sense of Brody if there are no non-constant entire holomorphic curves $f : \mathbb{C} \to X$.
- Brody has shown that for X compact, hyperbolicity is equivalent to the non degeneracy of the Kobayashi pseudo-metric : $x \in X$, $\xi \in T_X$

$$k_x(\xi) = \inf\{\lambda > 0\,;\ \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f_*(0) = \xi\}$$

- Hyperbolic varieties are especially interesting for their expected diophantine properties :
  Conjecture (S. Lang) *If a projective variety X defined over $\mathbb{Q}$ is hyperbolic, then $X(\mathbb{Q})$ is finite.*

Hyperbolicity and curvature

- Case $n = 1$ (compact Riemann surfaces):

$$\begin{array}{ll} X = \mathbb{P}^1 & (g = 0, \quad T_X > 0) \\ X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) & (g = 1, \quad T_X = 0) \end{array}$$

obviously non hyperbolic : $\exists f : \mathbb{C} \to X$.
- If $g \geq 2$, $X \simeq \mathbb{D}/\Gamma$ $(T_X < 0)$, then X hyperbolic.
- **The n-dimensional case** (Kobayashi)
If T_X is negatively curved ($T_X^* > 0$, i.e. ample), then X is hyperbolic.
- **Examples :** $X = \Omega/\Gamma$, Ω bounded symmetric domain.
- **Conjecture GT.** *Conversely, if a compact manifold X is hyperbolic, then it should be of general type, i.e. $K_X = \Lambda^n T_X^*$ should be big and nef (Ricci < 0, possibly with some degeneration).*

Conjectural characterizations of hyperbolicity

- **Theorem.** *Let X be projective algebraic. Consider the following properties :*
(P1) *X is hyperbolic*
(P2) *Every subvariety Y of X is of general type.*
(P3) *$\exists \varepsilon > 0$, $\forall C \subset X$ algebraic curve*

$$2g(\bar{C}) - 2 \geq \varepsilon \deg(C).$$

(X *"algebraically hyperbolic"*)
(P4) *X possesses a jet-metric with negative curvature on its k-jet bundle X_k [to be defined later], for $k \geq k_0 \gg 1$.*

Then (P4) $\Rightarrow$ (P1), (P2), (P3),
(P1) $\Rightarrow$ (P3),
and if Conjecture GT holds, (P1) $\Rightarrow$ (P2).
- It is expected that all 4 properties (P1), (P2), (P3), (P4) are equivalent for projective varieties.

Green-Griffiths-Lang conjecture

- **Conjecture** (Green-Griffith-Lang = GGL) *Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.*
- Combining the above conjectures, we get :
  Expected consequence (of GT + GGL)
  (P1) *X is hyperbolic*
  (P2) *Every subvariety Y of X is of general type*
  are equivalent.
- The main idea in order to attack GGL is to use differential equations. Let

$$\mathbb{C} \to X, \quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$$

be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X.

Definition of algebraic differential operators

- Consider algebraic differential operators which can be written locally in multi-index notation

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k} \end{aligned}$$

where $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are holomorphic coefficients on X and $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet.
Obvious $\mathbb{C}^*$-action :

$$\lambda \cdot f(t) = f(\lambda t), \quad (\lambda \cdot f)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$$

$\Rightarrow$ weighted degree $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$.
- **Definition.** $E^{\mathrm{GG}}_{k,m}$ *is the sheaf (bundle) of algebraic differential operators of order k and weighted degree m.*

Vanishing theorem for differential operators

- **Fundamental vanishing theorem**
(Green-Griffiths '78, Demailly '95, Siu '96)
Let $P \in H^0(X, E^{\mathrm{GG}}_{k,m} \otimes \mathcal{O}(-A))$ be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then for any $f : \mathbb{C} \to X$, $P(f_{[k]}) \equiv 0$.
- *Proof.* One can assume that A is very ample and intersects $f(\mathbb{C})$. Also assume f' bounded (this is not so restrictive by Brody !). Then all $f^{(k)}$ are bounded by Cauchy inequality. Hence
$$\mathbb{C} \ni t \mapsto P(f', f'', \ldots, f^{(k)})(t)$$
is a bounded holomorphic function on $\mathbb{C}$ which vanishes at some point. Apply Liouville's theorem ! $\square$

Geometric interpretation of vanishing theorem

- Let $X_k^{\mathrm{GG}} = J_k(X)^*/\mathbb{C}^*$ be the projectivized k-jet bundle of X = quotient of non constant k-jets by $\mathbb{C}^*$-action.
Fibers are weighted projective spaces.
Observation. *If $\pi_k : X_k^{\mathrm{GG}} \to X$ is canonical projection and $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is the tautological line bundle, then*
$$E^{\mathrm{GG}}_{k,m} = (\pi_k)_* \mathcal{O}_{X_k^{\mathrm{GG}}}(m)$$
- Saying that $f : \mathbb{C} \to X$ satisfies the differential equation $P(f_{[k]}) = 0$ means that
$$f_{[k]}(\mathbb{C}) \subset Z_P$$
where Z_P is the zero divisor of the section
$$\sigma_P \in H^0(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* \mathcal{O}(-A))$$
associated with P.

Consequence of fundamental vanishing theorem

- **Consequence of fundamental vanishing theorem.** *If $P_j \in H^0(X, E^{\mathrm{GG}}_{k,m} \otimes \mathcal{O}(-A))$ is a basis of sections then the image $f(\mathbb{C})$ lies in $Y = \pi_k(\bigcap Z_{P_j})$, hence property asserted by the GGL conjecture holds true if there are "enough independent differential equations" so that*

$$Y = \pi_k(\bigcap_j Z_{P_j}) \subsetneq X.$$

- However, some differential equations are useless. On a surface with coordinates (z_1, z_2), a Wronskian equation $f_1' f_2'' - f_2' f_1'' = 0$ tells us that $f(\mathbb{C})$ sits on a line, but $f_2''(t) = 0$ says that the second component is linear affine in time, an essentially meaningless information which is lost by a change of parameter $t \mapsto \varphi(t)$.

Invariant differential operators

- The k-th order Wronskian operator

$$W_k(f) = f' \wedge f'' \wedge \ldots \wedge f^{(k)}$$

(locally defined in coordinates) has degree $m = \frac{k(k+1)}{2}$ and

$$W_k(f \circ \varphi) = \varphi'^m W_k(f) \circ \varphi.$$

- **Definition.** *A differential operator P of order k and degree m is said to be invariant by reparametrization if*

$$P(f \circ \varphi) = \varphi'^m P(f) \circ \varphi$$

for any parameter change $t \mapsto \varphi(t)$. Consider their set

$$E_{k,m} \subset E^{\mathrm{GG}}_{k,m} \quad (\textit{a subbundle})$$

(Any polynomial $Q(W_1, W_2, \ldots W_k)$ is invariant, but for $k \geq 3$ there are other invariant operators.)

Category of directed manifolds

- **Definition.** *Category of directed manifolds :*
  - Objects are pairs (X, V) where X is a complex manifold and $V \subset T_X$ (subbundle or subsheaf)
  - Arrows $\psi : (X, V) \to (Y, W)$ are holomorphic maps with $\psi_* V \subset W$
  - "Absolute case" (X, T_X)
  - "Relative case" $(X, T_{X/S})$ where $X \to S$
  - "Integrable case" when $[V, V] \subset V$ (foliations)
- **Fonctor "1-jet" :** $(X, V) \mapsto (\tilde X, \tilde V)$ where :

$$\tilde X = P(V) = \text{bundle of projective spaces of lines in } V$$
$$\pi : \tilde X = P(V) \to X, \quad (x, [v]) \mapsto x, \quad v \in V_x$$
$$\tilde V_{(x,[v])} = \{\xi \in T_{\tilde X,(x,[v])}\,;\ \pi_*\xi \in \mathbb{C}v \subset T_{X,x}\}$$

Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde X$$
$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde X, \tilde V) \quad \text{(projectivized 1st-jet)}$$

- **Definition.** *Semple jet bundles :*
  - $(X_k, V_k) = k$-th iteration of fonctor $(X, V) \mapsto (\tilde X, \tilde V)$
  - $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized k-jet of f.
- **Basic exact sequences**

$$0 \to T_{\tilde X/X} \to \tilde V \xrightarrow{\pi_\star} \mathcal{O}_{\tilde X}(-1) \to 0 \qquad \Rightarrow \text{rk}\,\tilde V = r = \text{rk}\,V$$
$$0 \to \mathcal{O}_{\tilde X} \to \pi^\star V \otimes \mathcal{O}_{\tilde X}(1) \to T_{\tilde X/X} \to 0 \quad \text{(Euler)}$$
$$0 \to T_{X_k/X_{k-1}} \to V_k \xrightarrow{(\pi_k)_\star} \mathcal{O}_{X_k}(-1) \to 0 \qquad \Rightarrow \text{rk}\,V_k = r$$
$$0 \to \mathcal{O}_{X_k} \to \pi_k^\star V_{k-1} \otimes \mathcal{O}_{X_k}(1) \to T_{X_k/X_{k-1}} \to 0 \quad \text{(Euler)}$$

Direct image formula

- For $n = \dim X$ and $r = \mathrm{rk}\, V$, get a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

  with $\dim X_k = n + k(r-1)$, $\mathrm{rk}\, V_k = r$,
  and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.
- **Theorem.** *X_k is a smooth compactification of*

$$X_k^{\mathrm{GG,reg}}/G_k = J_k^{\mathrm{GG,reg}}/G_k$$

  where G_k is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization: $(f, \varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k-jets of regular curves.
- **Direct image formula.** $(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = E_{k,m} V^* =$ *invariant algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.*

Results obtained so far

- Using this technology and deep results of McQuillan for curve foliations on surfaces, D. – El Goul proved in 1998
  Theorem. (solution of Kobayashi conjecture)
  A very generic surface $X \subset \mathbb{P}^3$ of degree ≥ 21 is hyperbolic.
  (McQuillan got independently degree ≥ 35).
- The result was improved in 2004 by M. Păun, degree ≥ 18 is enough, with "generic" instead of "very generic". Paun's technique exploits a new idea of Y.T. Siu based on C. Voisin's work, which consists of studying vector fields on the the universal jet space of the universal family of hypersurfaces of $\mathbb{P}^{n+1}$ (with $n = 2$ here).
- **Dimension 3 case.** (Erwan Rousseau, 2006–2007)
  If $X \subset \mathbb{P}^4$ is a generic 3-fold of degree d, then
  – for $d \geq 97$, every $f : \mathbb{C} \to X$ satisfies a diff. equation.
  – for $d \geq 593$, every $f : \mathbb{C} \to X$ is algebraically degenerate.

Algebraic structure of differential rings

- Although very interesting, results are currently limited by lack of knowledge on jet bundles and differential operators
- **Unknown !** *Is the ring of germs of invariant differential operators on* $(\mathbb{C}^n, T_{\mathbb{C}^n})$ *at the origin*
  $$\mathcal{A}_{k,n} = \bigoplus_m E_{k,m} T^*_{\mathbb{C}^n} \quad \textit{finitely generated ?}$$
- At least this is OK for $\forall n$, $k \le 2$ and $n = 2$, $k \le 4$:

$$\mathcal{A}_{1,n} = \mathcal{O}[f'_1, \ldots, f'_n]$$
$$\mathcal{A}_{2,n} = \mathcal{O}[f'_1, \ldots, f'_n, W^{[ij]}], \quad W^{[ij]} = f'_i f''_j - f'_j f''_i$$
$$\mathcal{A}_{3,2} = \mathcal{O}[f'_1, f'_2, W_1, W_2][W]^2, \quad W_i = f'_i DW - 3f''_i W$$
$$\mathcal{A}_{4,2} = \mathcal{O}[f'_1, f'_2, W_{11}, W_{22}, S][W]^6, \quad W_{ii} = f'_i DW_i - 5f''_i W_i$$

where $W = f'_1 f''_2 - f'_2 f''_1$ is 2-dim Wronskian and $S = (W_1 DW_2 - W_2 DW_1)/W$. Also known: $\mathcal{A}_{3,3}$ (E. Rousseau, 2004), $\mathcal{A}_{5,2}$ (J. Merker, 2007)

Strategy : evaluate growth of differential operators

- The strategy of the proofs is that the algebraic structure of $\mathcal{A}_{k,n}$ allows to compute the Euler characteristic $\chi(X, E_{k,m} \otimes A^{-1})$, e.g. on surfaces

$$\chi(X, E_{k,m} \otimes A^{-1}) = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3).$$

- Hence for $13c_1^2 - 9c_2 > 0$, using Bogomolov's vanishing theorem for $H^2(X, (T^*_X)^{\otimes m} \otimes A^{-1})$ for $m \gg 0$, one gets

$$h^0(X, E_{k,m} \otimes A^{-1}) \ge \chi = h^0 - h^1 = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3)$$

- Therefore many global differential operators exist for surfaces with $13c_1^2 - 9c_2 > 0$, e.g. surfaces of degree large enough in $\mathbb{P}^3$, $d \ge 15$ (end of proof uses stability)

Trouble / more general perspectives

- Trouble is, in higher dimensions n, intermediate coho- mology groups $H^q(X, E_{k,m}T_X^*)$, $0 < q < n$, don't vanish !!
- **Main conjecture** (Generalized GGL)
  If (X, V) is directed manifold of general type, i.e. $\det V^*$ big, then $\exists Y \subsetneq X$ such that every non-constant $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ is contained in Y.
- **Strategy.** OK by Ahlfors-Schwarz lemma if $r = \mathrm{rk}\, V = 1$.
  First try to get differential equations $f_{[k]}(\mathbb{C}) \subset Z \subsetneq X_k$.
  Take minimal such k. If $k = 0$, we are done! Otherwise $k \geq 1$ and $\pi_{k,k-1}(Z) = X_{k-1}$, thus $W = V_k \cap T_Z$ has rank $< \mathrm{rk}\, V_k = r$ and should have again $\det W^*$ big (unless some degeneration occurs ?). Use induction on r !
- **Needed induction step.** *If (X, V) has $\det V^*$ big and $Z \subset X_k$ irreducible with $\pi_{k,k-1}(Z) = X_{k-1}$, then (Z, W), $W = V_k \cap T_Z$ has $\mathcal{O}_{Z_\ell}(1)$ big on (Z_ℓ, W_ℓ), $\ell \gg 0$.*

Use holomorphic Morse inequalities !

- **Simple case of Morse inequalities**
  (Demailly, Siu, Catanese, Trapani)
  If $L = \mathcal{O}(A - B)$ is a difference of big nef divisors A, B, then L is big as soon as
  $$A^n - nA^{n-1} \cdot B > 0.$$
- My PhD student S. Diverio has recently worked out this strategy for hypersurfaces $X \subset \mathbb{P}^{n+1}$, with
  $$L = \bigotimes_{1 \leq j < k} \pi_{k,j}^* \mathcal{O}_{X_j}(2 \cdot 3^{k-j-1}) \otimes \mathcal{O}_{X_k}(1),$$
  $$B = \pi_{k,0}^* \mathcal{O}_X(2 \cdot 3^{k-1}), \quad A = L + B \Rightarrow L = A - B.$$
  In this way, one obtains equations of order $k = n$, when $d \geq d_n$ and $n \leq 6$ (although the method might work also for $n > 6$). One can check that
  $$d_2 = 15, \quad d_3 = 82, \quad d_4 = 329, \quad d_5 = 1222, \quad d_6 \text{ exists.}$$

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