

On the approximate cohomology of quasi holomorphic line bundles

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Quasi holomorphic line bundles

Let X be a compact complex manifold, and let

$$H_{BC}^{p,q}(X, \mathbb{C}) = \frac{\text{Ker } \partial \cap \text{Ker } \bar{\partial}}{\text{Im } \partial \bar{\partial}} \quad \text{in bidegree } (p, q)$$

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Given a class $\gamma \in H_{BC}^{1,1}(X, \mathbb{R})$ and a $(1, 1)$ -form u representing γ , there exists an infinite subset $S \subset \mathbb{N}$ and C^∞ Hermitian line bundles $(L_k, h_k)_{k \in S}$ equipped with Hermitian connections ∇_k ,

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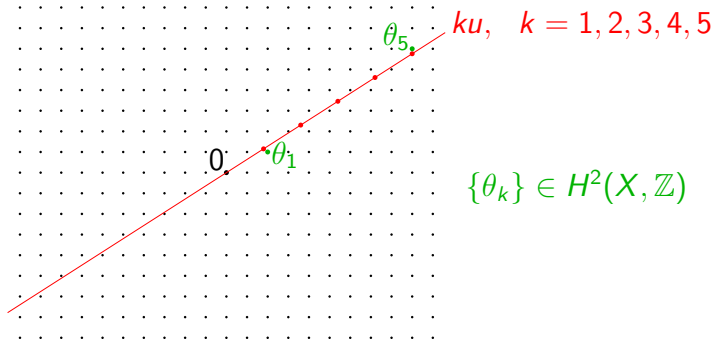
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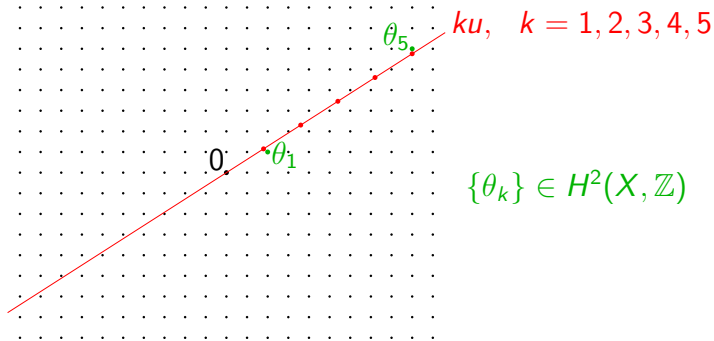
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In fact β_k can be chosen in a finite dimensional space of C^∞ closed 2-forms isomorphic to $H_{DR}^2(X, \mathbb{R})$.

Approximate holomorphic structure



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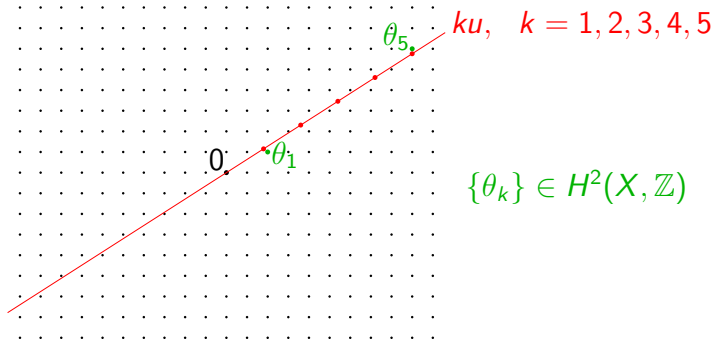


Consequence

Let $\nabla_k = \nabla_k^{1,0} + \nabla_k^{0,1}$. Then $\theta_k = ku + \beta_k$ implies

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Thus the L_k are “closer and closer” to be holomorphic as $k \rightarrow +\infty$.

Spectrum of the Laplace-Beltrami operator

Let $\bar{\square}_k = \bar{\partial}_k \bar{\partial}_k^* + \bar{\partial}_k^* \bar{\partial}_k$ be the complex Laplace-Beltrami operator of (L_k, h_k, ∇_k) with respect to some Hermitian metric ω on X .

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Let $\bar{\square}_{k,E}^{p,q}$ the operator acting on $C^\infty(X, \Lambda^{p,q} T_X^* \otimes L_k \otimes E)$, where (E, h_E) is a **holomorphic** Hermitian vector bundle of rank r .

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For a multi-index $J = \{j_1 < j_2 < \dots < j_q\} \subset \{1, \dots, n\}$, set

$$u_J(x) = \sum_{j \in J} u_j(x), \quad x \in X.$$

Fundamental spectral theory results

Consider the “spectral density functions” $\nu_u, \bar{\nu}_u$ defined by

$$\left. \begin{array}{l} \nu_u(\lambda) \\ \bar{\nu}_u(\lambda) \end{array} \right\} = \frac{2^{s-n} |u_1| \cdots |u_s|}{\Gamma(n-s+1)} \sum_{(p_1, \dots, p_s) \in \mathbb{N}^s} \left[\lambda - \sum (2p_j + 1) |u_j| \right]_+^{n-s}.$$

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Theorem ([D] 1985)

The spectrum of $\frac{1}{2\pi k} \bar{\square}_k^{p,q}$ on $C^\infty(X, \Lambda^{p,q} T_X^* \otimes L_k \otimes E)$ has an asymptotic distribution of eigenvalues such that $\forall \lambda \in \mathbb{R}$

$$\begin{aligned} r \binom{n}{p} \sum_{|J|=q} \int_X \nu_u(2\lambda + u_{\mathbb{C}J} - u_J) dV_\omega &\leq \liminf_{k \rightarrow +\infty} k^{-n} N_k^{p,q}(\lambda) \leq \\ &\leq \limsup_{k \rightarrow +\infty} k^{-n} N_k^{p,q}(\lambda) \leq r \binom{n}{p} \sum_{|J|=q} \int_X \bar{\nu}_u(2\lambda + u_{\mathbb{C}J} - u_J) dV_\omega \end{aligned}$$

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where $r = \text{rank}(E)$. By monotonicity, as $\bar{\nu}_u(\lambda) = \lim_{\lambda \rightarrow 0^+} \nu_u(\lambda)$, all four terms are equal for $\lambda \in \mathbb{R} \setminus \mathcal{D}$ with \mathcal{D} countable.

Approximate cohomology lower bounds

Proof. One first estimates the spectrum of the total Laplacian $\Delta_{k,E} = \nabla_{k,E} \nabla_{k,E}^* + \nabla_{k,E}^* \nabla_{k,E}$ (harmonic oscillator with magnetic and electric fields),

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Corollary (Laurent Jaeng, 2002)

For $\lambda_k \rightarrow 0$ slowly enough, i.e. with $k^{2+2/b_2} \lambda_k \rightarrow +\infty$, one has

$$\liminf_{k \rightarrow +\infty} k^{-n} N_{k,E}^{0,0}(\lambda_k) \geq \frac{r}{n!} \left(\int_{X(u,0)} u^n + \int_{X(u,1)} u^n \right) \quad \text{where}$$

$X(u, q) = q\text{-index set} = \{x \in X / u(x) \text{ has signature } (n-q, q)\}.$

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This implies

$$N_{k,E}^{0,1}(\delta') \geq N_{k,E}^{0,0}(\delta) - N_{k,E}^{0,0}(\lambda_k)$$

thus

$$N_{k,E}^{0,0}(\lambda_k) \geq N_{k,E}^{0,0}(\delta) - N_{k,E}^{0,1}(\delta'),$$

QED

Transcendental holomorphic Morse inequalities

Conjecture on Morse inequalities

Let $\gamma \in H_{\text{BC}}^{1,1}(X, \mathbb{R})$. Then

$$\text{Vol}(\gamma) \geq \sup_{u \in \gamma, u \in C^\infty} \int_{X(u, \leq 1)} u^n.$$

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There is however a stronger & more usual definition of the volume.

Definition

For $\gamma \in H_{BC}^{1,1}(X, \mathbb{R})$, set $\text{Vol}(\gamma) = 0$ if $\gamma \not\equiv$ any current $T \geq 0$,

and otherwise set $\text{Vol}(\gamma) = \sup_{T \in \gamma, T = u_0 + i\partial\bar{\partial}\varphi \geq 0} \int_X T_{ac}^n, u_0 \in C^\infty.$

Transcendental holomorphic Morse inequalities (2)

The conjecture on Morse inequalities is known to be true when $\gamma = c_1(L)$ is an integral class ([D-1985]). In fact, one then gets a Hermitian holomorphic line bundle (L, h) and its multiples $L^{\otimes k}$. The spectral estimates provide many holomorphic sections $\sigma_{k,\ell}$, and one gets positive currents right away by putting

$$T_k = \frac{i}{2k\pi} \partial\bar{\partial} \log \sum_{\ell} |\sigma_{k,\ell}|_h^2 + \frac{i}{2\pi} \Theta_{L,h} \geq 0$$

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Conjectural corollary (fundamental volume estimate)

Let X be compact Kähler, $\dim X = n$, and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be **nef cohomology classes**. Then

$$\text{Vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

Known results on holomorphic Morse inequalities

The conjectural corollary is derived from the main conjecture by an elementary symmetric function argument. In fact, one has a pointwise inequality of forms

$$\mathbf{1}_{X(\alpha-\beta, \leq 1)} (\alpha - \beta)^n \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

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(The proof is short, once the Calabi-Yau theorem is taken for granted).

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In the sequel of this lecture, we aim to investigate this construction and look for potential applications, especially to transcendental holomorphic Morse inequalities ...

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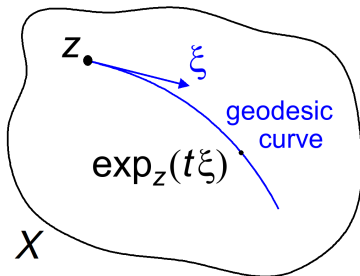
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Assume that X is equipped with a real analytic hermitian metric γ , and let $\exp : T_X \rightarrow X \times X$, $(z, \xi) \mapsto (z, \exp_z(\xi))$, $z \in X$, $\xi \in T_{X,z}$ be the associated geodesic exponential map.



Exponential map diffeomorphism and its inverse

Lemma

Denote by **exp_h** the “holomorphic” part of exp, so that for $z \in X$ and $\xi \in T_{X,z}$

$$\exp_z(\xi) = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha\beta}(z) \xi^\alpha \bar{\xi}^\beta, \quad \text{exp}_h z(\xi) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha 0}(z) \xi^\alpha.$$

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Notation

With the identification $\bar{X} \simeq_{\text{diff}} X$, let $\text{logh} : X \times \bar{X} \supset V' \rightarrow T_{\bar{X}}$ be the inverse diffeomorphism of exp and

$$U_\varepsilon = \{(z, w) \in V' \subset X \times \bar{X}; |\text{logh}_z(w)|_\gamma < \varepsilon\}, \quad \varepsilon > 0.$$

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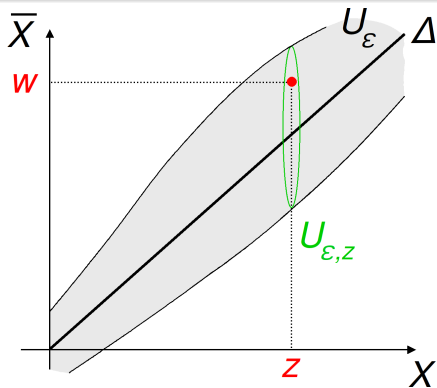
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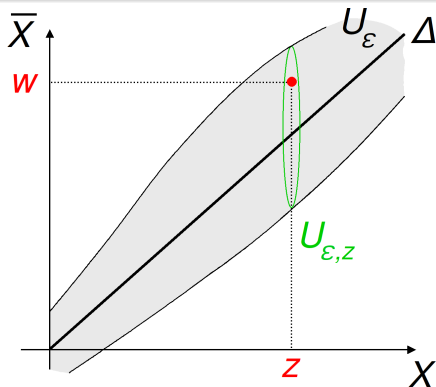
$$U_\varepsilon = \{(z, w) \in V' \subset X \times \bar{X}; |\text{logh}_z(w)|_\gamma < \varepsilon\}, \quad \varepsilon > 0.$$

Then, for $\varepsilon \ll 1$, U_ε is Stein and $\text{pr}_1 : U_\varepsilon \rightarrow X$ is a **real analytic locally trivial bundle** with fibers biholomorphic to complex balls.

Such tubular neighborhoods are Stein

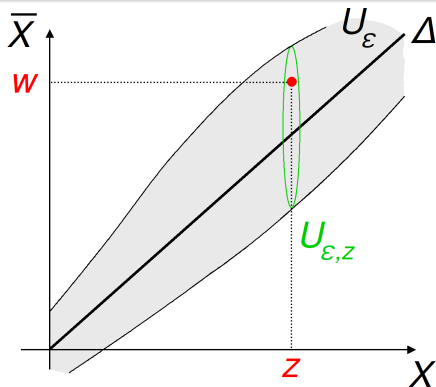


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It is of course Stein since

$$|\bar{z} - w|^2 = |z|^2 + |w|^2 - 2 \operatorname{Re} \sum z_j w_j$$

and $(z, w) \mapsto \operatorname{Re} \sum z_j w_j$ is pluriharmonic.

Bergman sheaves

Let $U_\varepsilon = U_{\gamma,\varepsilon} \subset X \times \overline{X}$ be the ball bundle as above, and

$$p = (\text{pr}_1)|_{U_\varepsilon} : U_\varepsilon \rightarrow X, \quad \bar{p} = (\text{pr}_2)|_{U_\varepsilon} : U_\varepsilon \rightarrow \overline{X}$$

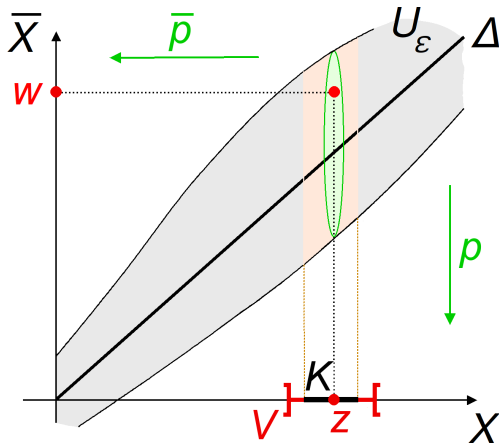
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Bergman sheaves (continued)

Definition of the Bergman sheaf \mathcal{B}_ε

The Bergman sheaf $\mathcal{B}_\varepsilon = \mathcal{B}_{\gamma,\varepsilon}$ is by definition the L^2 direct image

$$\mathcal{B}_\varepsilon = p_*^{L^2}(\bar{p}^* \mathcal{O}(K_{\bar{X}})),$$

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i.e. the space of sections over an open subset $V \subset X$ defined by $\mathcal{B}_\varepsilon(V) =$ holomorphic sections f of $\bar{p}^* \mathcal{O}(K_{\bar{X}})$ on $p^{-1}(V)$,

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that are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$:

$$\int_{p^{-1}(K)} i^{n^2} f(z, w) \wedge \overline{f(z, w)} \wedge \gamma(z)^n < +\infty, \quad \forall K \Subset V.$$

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Clearly, \mathcal{B}_ε is an \mathcal{O}_X -module over X , but since it is a space of functions in w , it is of infinite rank.

Associated Bergman bundle and holom structure

Definition of the associated Bergman bundle B_ε

We consider the vector bundle $B_\varepsilon \rightarrow X$ whose fiber B_{ε, z_0} consists of all holomorphic functions f on $p^{-1}(z_0) \subset U_\varepsilon$ such that

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Moreover, for $\varepsilon' > \varepsilon$, there is a restriction map $\mathcal{B}_{\varepsilon', z_0} \rightarrow \mathcal{B}_{\varepsilon, z_0}$ such that B_{ε, z_0} is the **L^2 completion of $\mathcal{B}_{\varepsilon', z_0} / \mathfrak{m}_{z_0} \mathcal{B}_{\varepsilon', z_0}$** .

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Question

Is there a “complex structure” on B_ε such that “ $\mathcal{B}_\varepsilon = \mathcal{O}(B_\varepsilon)$ ” ?

Bergman Dolbeault complex

For this, consider the “Bergman Dolbeault” complex $\bar{\partial} : \mathcal{F}_\varepsilon^q \rightarrow \mathcal{F}_\varepsilon^{q+1}$ over X , with $\mathcal{F}_\varepsilon^q(V) = \text{smooth } (n, q)\text{-forms}$

$$f(z, w) = \sum_{|J|=q} f_J(z, w) dw_1 \wedge \dots \wedge dw_n \wedge d\bar{z}_J, \quad (z, w) \in U_\varepsilon \cap (V \times \bar{X}),$$

such that $f_J(z, w)$ is holomorphic in w , and for all $K \Subset V$ one has

$$f(z, w) \in L^2(p^{-1}(K)) \quad \text{and} \quad \bar{\partial}_z f(z, w) \in L^2(p^{-1}(K)).$$

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An immediate consequence of this definition is:

Proposition

$\bar{\partial} = \bar{\partial}_z$ yields a complex of sheaves $(\mathcal{F}_\varepsilon^\bullet, \bar{\partial})$, and the kernel $\text{Ker } \bar{\partial} : \mathcal{F}_\varepsilon^0 \rightarrow \mathcal{F}_\varepsilon^1$ coincides with \mathcal{B}_ε .

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If we define $\mathcal{O}_{L^2}(B_\varepsilon)$ to be the sheaf of L^2_{loc} sections f of B_ε such that $\bar{\partial}f = 0$ in the sense of distributions, then we exactly have

$$\mathcal{O}_{L^2}(B_\varepsilon) = \mathcal{B}_\varepsilon \text{ as a sheaf.}$$

Bergman sheaves are “very ample”

Theorem

Assume that $\varepsilon > 0$ is taken so small that $\psi(z, w) := |\log h_z(w)|^2$ is strictly plurisubharmonic up to the boundary on the compact set $\overline{U_\varepsilon} \subset X \times \overline{X}$.

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$$H^q(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) = H^q(\Gamma(X, \mathcal{F}_\varepsilon^\bullet \otimes \mathcal{O}(E)), \overline{\partial}) = 0, \quad \forall q \geq 1.$$

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Moreover the fibers $B_{\varepsilon, z} \otimes E_z$ are always generated by global sections of $H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$.

In that sense, B_ε is a “very ample holomorphic vector bundle” (as a Hilbert bundle of infinite dimension).

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The proof is a direct consequence of Hörmander’s L^2 estimates.

Caution !!

B_ε is **NOT** a locally trivial *holomorphic* bundle.

Embedding into a Hilbert Grassmannian

Corollary of the very ampleness of Bergman sheaves

Let X be an arbitrary compact complex manifold, $E \rightarrow X$ a holomorphic vector bundle (e.g. the trivial bundle). Consider the Hilbert space $\mathbb{H} = H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$.

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$$\Psi : X \rightarrow \mathrm{Gr}(\mathbb{H}), \quad z \mapsto S_z,$$

mapping every point $z \in X$ to the infinite codimensional closed subspace S_z consisting of sections $f \in \mathbb{H}$ such that $f(z) = 0$ in $B_{\varepsilon,z}$, i.e. $f|_{p^{-1}(z)} = 0$.

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The main problem with this “holomorphic embedding” is that the holomorphicity is to be understood in a weak sense, for instance the map Ψ is not even continuous with respect to the strong metric topology of $\mathrm{Gr}(\mathbb{H})$, given by

$d(S, S') =$ Hausdorff distance of the unit balls of S, S' .

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Then one sees that a orthonormal frame of B_ε is given by

$$e_\alpha(z, w) = \pi^{-n/2} \varepsilon^{-|\alpha|-n} \sqrt{\frac{(|\alpha| + n)!}{\alpha_1! \dots \alpha_n!}} (w - \bar{z})^\alpha, \quad \alpha \in \mathbb{N}^n.$$

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This frame is non holomorphic! The $(0, 1)$ -connection $\nabla^{0,1} = \bar{\partial}$ is given by

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where $c_j = (0, \dots, 1, \dots, 0) \in \mathbb{N}^n$.

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A simple calculation of ∇^2 in the orthonormal frame (e_α) leads to:

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However there is convergence for all $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_{\varepsilon'}$, $\varepsilon' > \varepsilon$, since then $\sum_\alpha (\varepsilon'/\varepsilon)^{2|\alpha|} |\xi_\alpha|^2 < +\infty$.

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A consequence of the above formula is that B_ε is strongly Nakano positive for $\varepsilon > 0$ small enough.

Idea of proof of the asymptotic expansion

The formula is in principle a special case of a more general result proved by **Wang Xu**, expressing the curvature of **weighted Bergman bundles** \mathcal{H}_t attached to a **smooth family** $\{D_t\}$ of strongly **pseudoconvex domains**.

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which is used to compute the difference with the model case \mathbb{C}^n , for which $\log h_z(w) = w - \bar{z}$.

Back to holomorphic Morse inequalities

Idea for the general case. Let $\gamma \in H_{\text{BC}}^{1,1}(X, \mathbb{R})$ and $u \in \gamma$ a smooth form. As we have seen, one can find a sequence of Hermitian line bundles (L_k, h_k, ∇_k) such that

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- directly deal with the Hilbert Dolbeault complex of $(\text{pr}_1)_*^L(\mathcal{O}_{U_\varepsilon}(\tilde{L}_k))$, and **use Bergman estimates instead of dimension counts in Morse inequalities.**

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The conjecture is known to be true for a **projective family** $\mathcal{X} \rightarrow S$:

- Siu and Kawamata (1998) in the case of varieties of **general type**
- Siu (2000) and Păun (2004) in the arbitrary projective case

The proof is based on an iterated application of the **Ohsawa-Takegoshi L^2 extension theorem** w.r.t. an ample line bundle \mathcal{A} on \mathcal{X} :

Other potential target: invariance of plurigenera for polarized families of compact Kähler manifolds?

Conjecture

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Ohsawa-Takegoshi L^2 extension theorem w.r.t. an ample line bundle \mathcal{A} on \mathcal{X} : **replace \mathcal{A} by a Bergman bundle in the Kähler case ?**

Thank you for your attention

