UNIVERSITÉ Grenoble Alpes

INSTITUT DE FRANCE
Académie des sciences

On the approximate cohomology of quasi holomorphic line bundles

Jean-Pierre Demailly

Institut Fourier, Université Grenoble Alpes & Académie des Sciences de Paris

Virtual Conference
Geometry and TACoS
hosted at Università di Firenze
July 7 – 21, 2020

Quasi holomorphic line bundles

Let X be a compact complex manifold, and let

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) = \frac{\mathrm{Ker}\,\partial \cap \mathrm{Ker}\,\overline{\partial}}{\mathrm{Im}\,\partial\overline{\partial}} \qquad \text{in bidegree } (p,q)$$

be the corresponding Bott-Chern cohomology groups.

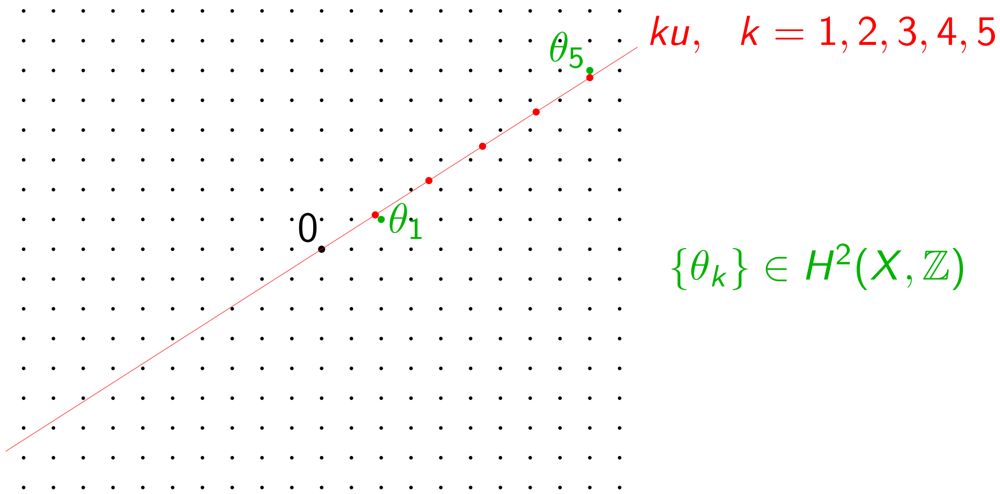
Basic observation (cf. Laurent Laeng, PhD thesis 2002)

Given a class $\gamma \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ and a $(1,1)$-form u representing γ, there exists an infinite subset $S \subset \mathbb{N}$ and C^∞ Hermitian line bundles $(L_k,h_k)_{k\in S}$ equipped with Hermitian connections ∇_k, such that the curvature 2-forms $\theta_k = \frac{i}{2\pi}\nabla_k^2$ satisfy $\theta_k = ku + \beta_k$ and

$$\beta_k = O(k^{-1/b_2}), \qquad b_2 = b_2(X).$$

Proof. This is a consequence of Kronecker's approximation theorem applied to the lattice $H^2(X,\mathbb{Z}) \hookrightarrow H^2_{\mathrm{DR}}(X,\mathbb{R})$.
In fact β_k can be chosen in a finite dimensional space of C^∞ closed 2-forms isomorphic to $H^2_{\mathrm{DR}}(X,\mathbb{R})$.

Approximate holomorphic structure



Consequence

Let $\nabla_k = \nabla_k^{1,0} + \nabla_k^{0,1}$. Then $\theta_k = ku + \beta_k$ implies

$$(\nabla_k^{0,1})^2 = \theta_k^{0,2} = \beta_k^{0,2} = O(k^{-1/b_2}).$$

Thus the L_k are "closer and closer" to be holomorphic as $k \to +\infty$.

Spectrum of the Laplace-Beltrami operator

Let $\overline{\Box}_k = \overline{\partial}_k \overline{\partial}_k^* + \overline{\partial}_k^* \overline{\partial}_k$ be the complex Laplace-Beltrami operator of (L_k, h_k, ∇_k) with respect to some Hermitian metric ω on X.
Let $\overline{\Box}_{k,E}^{p,q}$ the operator acting on $C^\infty(X, \Lambda^{p,q} T_X^* \otimes L_k \otimes E)$, where (E, h_E) is a holomorphic Hermitian vector bundle of rank r.

We are interested in analyzing the (discrete) spectrum of the elliptic operator $\overline{\Box}_{k,E}^{p,q}$. Since the curvature is $\theta_k \simeq ku$, it is better to renormalize and to consider instead $\frac{1}{2\pi k}\overline{\Box}_{k,E}^{p,q}$. For $\lambda \in \mathbb{R}$, we define

$$N_k^{p,q}(\lambda) = \dim \bigoplus \text{eigenspaces of } \frac{1}{2\pi k}\overline{\Box}_{k,E}^{p,q} \text{ of eigenvalues} \le \lambda.$$

Let $u_j(x)$, $1 \le j \le n$, be the eigenvalues of $u(x)$ with respect to $\omega(x)$ at any point $x \in X$, ordered so that if $s = \mathrm{rank}(u(x))$, then
$|u_1(x)| \ge \cdots \ge |u_s(x)| > |u_{s+1}(x)| = \cdots = |u_n(x)| = 0$.

For a multi-index $J = \{j_1 < j_2 < \ldots < j_q\} \subset \{1, \ldots, n\}$, set

$$u_J(x) = \sum\nolimits_{j \in J} u_j(x), \quad x \in X.$$

Fundamental spectral theory results

Consider the "spectral density functions" ν_u, $\overline{\nu}_u$ defined by

$$\left.\begin{array}{r}\nu_u(\lambda)\\ \overline{\nu}_u(\lambda)\end{array}\right\} = \frac{2^{s-n}\,|u_1|\cdots|u_s|}{\Gamma(n-s+1)} \sum_{(p_1,\ldots,p_s)\in\mathbb{N}^s} \Big[\lambda - \sum(2p_j+1)|u_j|\Big]_+^{n-s}.$$

(where $0^0=0$ for ν_u, resp. $0^0=1$ for $\overline{\nu}_u$).

Theorem ([D] 1985)

The spectrum of $\frac{1}{2\pi k}\overline{\Box}_k^{p,q}$ on $C^\infty(X,\Lambda^{p,q}T_X^*\otimes L_k\otimes E)$ has an asymptotic distribution of eigenvalues such that $\forall\lambda\in\mathbb{R}$

$$r\binom{n}{p}\sum_{|J|=q}\int_X \nu_u(2\lambda+u_{\complement J}-u_J)\,dV_\omega \le \liminf_{k\to+\infty} k^{-n}N_k^{p,q}(\lambda) \le$$

$$\le \limsup_{k\to+\infty} k^{-n}N_k^{p,q}(\lambda) \le r\binom{n}{p}\sum_{|J|=q}\int_X \overline{\nu}_u(2\lambda+u_{\complement J}-u_J)\,dV_\omega$$

where $r=\mathrm{rank}(E)$. By monotonicity, as $\overline{\nu}_u(\lambda)=\lim_{\lambda\to 0_+}\nu_u(\lambda)$, all four terms are equal for $\lambda\in\mathbb{R}\smallsetminus\mathcal{D}$ with $\mathcal{D}$ countable.

Approximate cohomology lower bounds

Proof. One first estimates the spectrum of the total Laplacian $\Delta_{k,E}=\nabla_{k,E}\nabla_{k,E}^*+\nabla_{k,E}^*\nabla_{k,E}$ (harmonic oscillator with magnetic and electric fields), and then one uses a Bochner formula to relate $\overline{\Box}_{k,E}$ and $\Delta_{k,E}$ ($\overline{\Box}_{k,E}\simeq\frac{1}{2}\Delta_{k,E}+$ curvature terms) for each (p,q).

Important special case $\lambda=0$ (harmonic forms)

$$\sum_{|J|=q}\overline{\nu}_u(u_{\complement J}-u_J)\,dV_\omega=(-1)^q\frac{u^n}{n!}\,.$$

Corollary (Laurent laeng, 2002)

For $\lambda_k\to 0$ slowly enough, i.e. with $k^{2+2/b_2}\lambda_k\to+\infty$, one has

$$\liminf_{k\to+\infty} k^{-n}N_{k,E}^{0,0}(\lambda_k)\ge\frac{r}{n!}\Big(\int_{X(u,0)}u^n+\int_{X(u,1)}u^n\Big)\quad\text{where}$$

$X(u,q)=q$-index set $=\big\{x\in X/\ u(x)$ has signature $(n-q,q)\big\}$.

Proof of the lower bound

Proof. One uses the fact that for $\delta' > \delta > 0$ and $k \gg 1$, the composition $\Pi \circ \overline{\partial}_k$ with an eigenspace projection yields an injection

$$\bigoplus_{\lambda \in]\lambda_k, \delta]} \mathrm{eigenspace}^{0,0}_\lambda \hookrightarrow \bigoplus_{\lambda \in]0,\delta']} \mathrm{eigenspace}^{0,1}_\lambda .$$

In fact, in the holomorphic case $\overline{\partial}_k^2 = 0$ implies $\overline{\partial}_k \Box^{0,0}_k = \Box^{0,1}_k \overline{\partial}_k$, hence $\overline{\partial}_k$ maps the $(0,0)$-eigenspaces to the $(0,1)$-eigenspaces for the same eigenvalues, and one can even take $\lambda_k = 0$, $\delta' = \delta$.

In the quasi holomorphic case $\overline{\partial}_k^2 = O(k^{-1/b_2})$, one can show that $\Box^{0,1}_k \overline{\partial}_k - \overline{\partial}_k \Box^{0,0}_k = \overline{\partial}_k^* \overline{\partial}_k^2$ yields a small "deviation" of the eigenvalues to $[\lambda_k - \varepsilon, \delta + \varepsilon]$ with $\varepsilon < \min(\lambda_k, \delta' - \delta)$, whence the injectivity.

This implies

$$N^{0,1}_{k,E}(\delta') \geq N^{0,0}_{k,E}(\delta) - N^{0,0}_{k,E}(\lambda_k)$$

thus

$$N^{0,0}_{k,E}(\lambda_k) \geq N^{0,0}_{k,E}(\delta) - N^{0,1}_{k,E}(\delta'), \qquad \text{QED}$$

Transcendental holomorphic Morse inequalities

Conjecture on Morse inequalities

Let $\gamma \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$. Then

$$\mathrm{Vol}(\gamma) \geq \sup_{u \in \gamma,\, u \in C^\infty} \int_{X(u, \leq 1)} u^n .$$

(One could even suspect equality, an even stronger conjecture !).

If one sets by definition

$$\mathrm{Vol}(\gamma) = \sup_{u \in \gamma} \lim_{\lambda \to 0_+} \liminf_{k \to +\infty} N^{0,0}_k(\lambda)$$

for the eigenspaces of the sequence (L_k, h_k, ∇_k) approximating ku, then the above expected lower bound is a theorem!

There is however a stronger & more usual definition of the volume.

Definition

For $\gamma \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$, set $\mathrm{Vol}(\gamma) = 0$ if $\gamma \not\ni$ any current $T \geq 0$,

and otherwise set $\mathrm{Vol}(\gamma) = \sup_{T \in \gamma,\, T = u_0 + i\partial\overline{\partial}\varphi \geq 0} \int_X T^n_{\mathrm{ac}}, \quad u_0 \in C^\infty.$

Transcendental holomorphic Morse inequalities (2)

The conjecture on Morse inequalities is known to be true when $\gamma = c_1(L)$ is an integral class ([D-1985]). In fact, one then gets a Hermitian holomorphic line bundle (L, h) and its multiples $L^{\otimes k}$. The spectral estimates provide many holomorphic sections $\sigma_{k,\ell}$, and one gets positive currents right away by putting

$$T_k = \frac{i}{2k\pi}\partial\overline{\partial}\log\sum_{\ell}|\sigma_{k,\ell}|_h^2 + \frac{i}{2\pi}\Theta_{L,h} \geq 0$$

(the volume estimate can be derived from there by Fujita).

In the "quasi-holomorphic" case, one only gets eigenfunctions $\sigma_{k,\ell}$ with small eigenvalues, and the positivity of T_k is a priori lost.

Conjectural corollary (fundamental volume estimate)

Let X be compact Kähler, $\dim X = n$, and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be nef cohomology classes. Then

$$\mathrm{Vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

Known results on holomorphic Morse inequalities

The conjectural corollary is derived from the main conjecture by an elementary symmetric function argument. In fact, one has a pointwise inequality of forms

$$\mathbf{1}_{X(\alpha-\beta,\leq 1)}(\alpha - \beta)^n \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

Again, the corollary is known for $\gamma = \alpha - \beta$ when α, β are integral classes (by [D-1993] and independently [Trapani, 1993]).

Recently (2016), the volume estimate for $\gamma = \alpha - \beta$ transcendental has been established by D. Witt-Nyström when X is projective, using deep facts on Monge-Ampère operators and upper envelopes.

Xiao and Popovici also proved in the Kähler case that

$$\alpha^n - n\alpha^{n-1} \cdot \beta > 0 \quad \Rightarrow \quad \mathrm{Vol}(\alpha - \beta) > 0$$

and $\alpha - \beta$ contains a Kähler current.

(The proof is short, once the Calabi-Yau theorem is taken for granted).

Projective vs Kähler vs non Kähler varieties

Problem. Investigate positivity for general compact manifolds/$\mathbb{C}$.

Obviously, non projective varieties do not carry any ample line bundle.

In the Kähler case, a Kähler class $\{\omega\} \in H^{1,1}(X, \mathbb{R})$, $\omega > 0$, may sometimes be used as a substitute for a polarization.

What for non Kähler compact complex manifolds?

Surprising facts (?)

– Every compact complex manifold X carries a "very ample" complex Hilbert bundle, produced by means of a natural Bergman space construction.

– The curvature of this bundle is strongly positive in the sense of Nakano, and is given by a universal formula.

In the sequel of this lecture, we aim to investigate this construction and look for potential applications, especially to transcendental holomorphic Morse inequalities ...

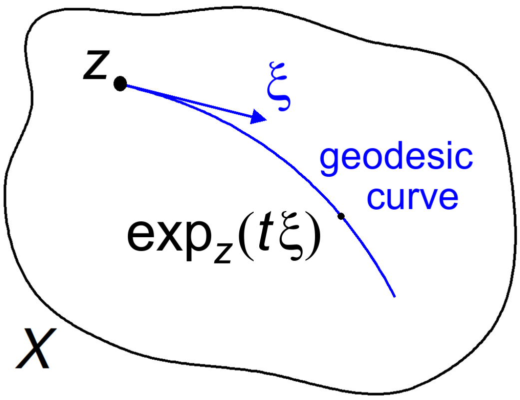
Tubular neighborhoods (thanks to Grauert)

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$.

Denote by $\overline{X}$ its complex conjugate $(X, -J)$, so that $\mathcal{O}_{\overline{X}} = \overline{\mathcal{O}_X}$.

The diagonal of $X \times \overline{X}$ is totally real, and by Grauert, we know that it possesses a fundamental system of Stein tubular neighborhoods.

Assume that X is equipped with a real analytic hermitian metric γ, and let $\exp : T_X \to X \times X$, $(z, \xi) \mapsto (z, \exp_z(\xi))$, $z \in X$, $\xi \in T_{X,z}$ be the associated geodesic exponential map.



Exponential map diffeomorphism and its inverse

Lemma

Denote by exph the "holomorphic" part of exp, so that for $z \in X$ and $\xi \in T_{X,z}$

$$\exp_z(\xi) = \sum_{\alpha,\beta\in\mathbb{N}^n} a_{\alpha\,\beta}(z)\xi^\alpha\overline{\xi}^\beta, \quad \mathrm{exph}_z(\xi) = \sum_{\alpha\in\mathbb{N}^n} a_{\alpha\,0}(z)\xi^\alpha.$$

Then $d_\xi \exp_z(\xi)_{\xi=0} = d_\xi\,\mathrm{exph}_z(\xi)_{\xi=0} = \mathrm{Id}_{T_X}$, and so exph is a diffeomorphism from a neighborhood V of the 0 section of T_X to a neighborhood V' of the diagonal in $X \times X$.

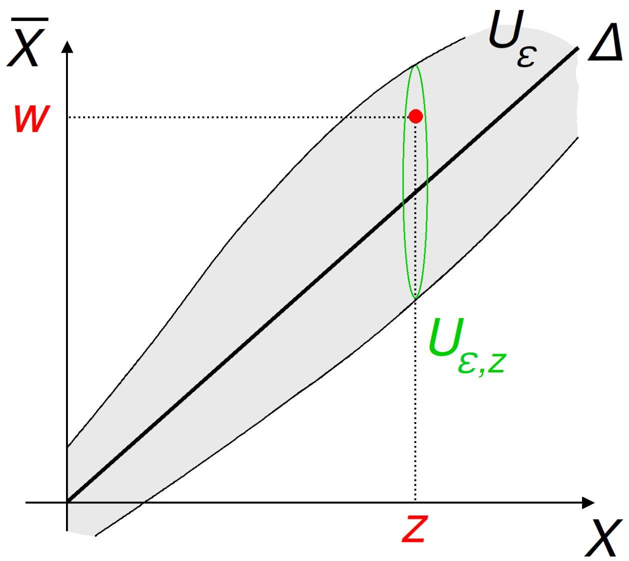
Notation

With the identification $\overline{X} \simeq_{\mathrm{diff}} X$, let $\mathrm{logh} : X \times \overline{X} \supset V' \to T_{\overline{X}}$ be the inverse diffeomorphism of exph and

$$U_\varepsilon = \{(z,w) \in V' \subset X \times \overline{X}\,;\ |\mathrm{logh}_z(w)|_\gamma < \varepsilon\}, \quad \varepsilon > 0.$$

Then, for $\varepsilon \ll 1$, U_ε is Stein and $\mathrm{pr}_1 : U_\varepsilon \to X$ is a real analytic locally trivial bundle with fibers biholomorphic to complex balls.

Such tubular neighborhoods are Stein



In the special case $X = \mathbb{C}^n$, $U_\varepsilon = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n\,;\ |\overline{z} - w| < \varepsilon\}$.
It is of course Stein since

$$|\overline{z} - w|^2 = |z|^2 + |w|^2 - 2\,\mathrm{Re}\sum z_j w_j$$

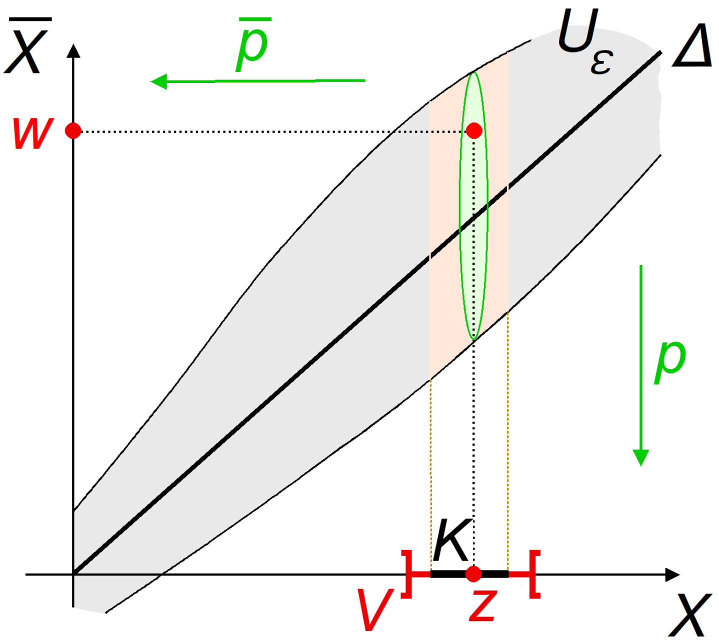
and $(z,w) \mapsto \mathrm{Re}\sum z_j w_j$ is pluriharmonic.

Bergman sheaves

Let $U_\varepsilon = U_{\gamma,\varepsilon} \subset X \times \overline{X}$ be the ball bundle as above, and

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X, \qquad \overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to \overline{X}$$

the natural projections.



Bergman sheaves (continued)

Definition of the Bergman sheaf $\mathcal{B}_\varepsilon$

The Bergman sheaf $\mathcal{B}_\varepsilon = \mathcal{B}_{\gamma,\varepsilon}$ is by definition the L^2 direct image

$$\mathcal{B}_\varepsilon = p_*^{L^2}(\overline{p}^*\mathcal{O}(K_{\overline{X}})),$$

i.e. the space of sections over an open subset $V \subset X$ defined by $\mathcal{B}_\varepsilon(V) =$ holomorphic sections f of $\overline{p}^*\mathcal{O}(K_{\overline{X}})$ on $p^{-1}(V)$,

$$f(z,w) = f_1(z,w)\, dw_1 \wedge \ldots \wedge dw_n, \quad z \in V,$$

that are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$:

$$\int_{p^{-1}(K)} i^{n^2} f(z,w) \wedge \overline{f(z,w)} \wedge \gamma(z)^n < +\infty, \quad \forall K \Subset V.$$

(This L^2 condition is the reason we speak of "L^2 direct image").

Clearly, $\mathcal{B}_\varepsilon$ is an $\mathcal{O}_X$-module over X, but since it is a space of functions in w, it is of infinite rank.

Associated Bergman bundle and holom structure

Definition of the associated Bergman bundle B_ε

We consider the vector bundle $B_\varepsilon \to X$ whose fiber B_{ε,z_0} consists of all holomorphic functions f on $p^{-1}(z_0) \subset U_\varepsilon$ such that

$$\|f(z_0)\|^2 = \int_{p^{-1}(z_0)} i^{n^2} f(z_0,w) \wedge \overline{f(z_0,w)} < +\infty.$$

Then B_ε is a real analytic locally trivial Hilbert bundle whose fiber B_{ε,z_0} is isomorphic to the Hardy-Bergman space $\mathcal{H}^2(B(0,\varepsilon))$ of L^2 holomorphic n-forms on $p^{-1}(z_0) \simeq B(0,\varepsilon) \subset \mathbb{C}^n$.
The Ohsawa-Takegoshi extension theorem implies that every $f \in B_{\varepsilon,z_0}$ can be extended as a germ $\tilde f$ in the sheaf $\mathcal{B}_{\varepsilon,z_0}$.
Moreover, for $\varepsilon' > \varepsilon$, there is a restriction map $\mathcal{B}_{\varepsilon',z_0} \to B_{\varepsilon,z_0}$ such that B_{ε,z_0} is the L^2 completion of $\mathcal{B}_{\varepsilon',z_0}/\mathfrak{m}_{z_0}\mathcal{B}_{\varepsilon',z_0}$.

Question

Is there a "complex structure" on B_ε such that "$\mathcal{B}_\varepsilon = \mathcal{O}(B_\varepsilon)$" ?

Bergman Dolbeault complex

For this, consider the "Bergman Dolbeault" complex $\overline\partial : \mathcal{F}^q_\varepsilon \to \mathcal{F}^{q+1}_\varepsilon$ over X, with $\mathcal{F}^q_\varepsilon(V) =$ smooth (n,q)-forms

$$f(z,w) = \sum_{|J|=q} f_J(z,w)\, dw_1 \wedge \ldots \wedge dw_n \wedge d\overline{z}_J, \quad (z,w) \in U_\varepsilon \cap (V \times \overline{X}),$$

such that $f_J(z,w)$ is holomorphic in w, and for all $K \Subset V$ one has

$$f(z,w) \in L^2(p^{-1}(K)) \ \text{ and } \ \overline\partial_z f(z,w) \in L^2(p^{-1}(K)).$$

An immediate consequence of this definition is:

Proposition

$\overline\partial = \overline\partial_z$ yields a complex of sheaves $(\mathcal{F}^\bullet_\varepsilon, \overline\partial)$, and the kernel $\mathrm{Ker}\,\overline\partial : \mathcal{F}^0_\varepsilon \to \mathcal{F}^1_\varepsilon$ coincides with $\mathcal{B}_\varepsilon$.

If we define $\mathcal{O}_{L^2}(B_\varepsilon)$ to be the sheaf of $L^2_{\rm loc}$ sections f of B_ε such that $\overline\partial f = 0$ in the sense of distributions, then we exactly have $\mathcal{O}_{L^2}(B_\varepsilon) = \mathcal{B}_\varepsilon$ as a sheaf.

Bergman sheaves are "very ample"

Theorem

Assume that $\varepsilon > 0$ is taken so small that $\psi(z,w) := |\log h_z(w)|^2$ is strictly plurisubharmonic up to the boundary on the compact set $\overline{U}_\varepsilon \subset X \times \overline{X}$. Then the complex of sheaves $(\mathcal{F}^\bullet_\varepsilon, \overline{\partial})$ is a resolution of $\mathcal{B}_\varepsilon$ by soft sheaves over X (actually, by $\mathcal{C}^\infty_X$-modules), and for every holomorphic vector bundle $E \to X$ we have

$$H^q(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) = H^q\big(\Gamma(X, \mathcal{F}^\bullet_\varepsilon \otimes \mathcal{O}(E)), \overline{\partial}\big) = 0, \quad \forall q \geq 1.$$

Moreover the fibers $B_{\varepsilon,z} \otimes E_z$ are always generated by global sections of $H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$.

In that sense, B_ε is a "very ample holomorphic vector bundle" (as a Hilbert bundle of infinite dimension).
The proof is a direct consequence of Hörmander's L^2 estimates.

Caution !!

B_ε is NOT a locally trivial *holomorphic* bundle.

Embedding into a Hilbert Grassmannian

Corollary of the very ampleness of Bergman sheaves

Let X be an arbitrary compact complex manifold, $E \to X$ a holomorphic vector bundle (e.g. the trivial bundle). Consider the Hilbert space $\mathbb{H} = H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$. Then one gets a "holomorphic embedding" into a Hilbert Grassmannian,

$$\Psi : X \to \mathrm{Gr}(\mathbb{H}), \quad z \mapsto S_z,$$

mapping every point $z \in X$ to the infinite codimensional closed subspace S_z consisting of sections $f \in \mathbb{H}$ such that $f(z) = 0$ in $B_{\varepsilon,z}$, i.e. $f_{|p^{-1}(z)} = 0$.

The main problem with this "holomorphic embedding" is that the holomorphicity is to be understood in a weak sense, for instance the map Ψ is not even continuous with respect to the strong metric topology of $\mathrm{Gr}(\mathbb{H})$, given by
$d(S, S') =$ Hausdorff distance of the unit balls of S, S'.

Chern connection of Bergman bundles

Since we have a natural $\nabla^{0,1} = \overline{\partial}$ connection on B_ε, and a natural hermitian metric as well, it follows from the usual formalism that B_ε can be equipped with a unique Chern connection.

Model case: $X = \mathbb{C}^n$, $\gamma =$ standard hermitian metric.
Then one sees that a orthonormal frame of B_ε is given by

$$e_\alpha(z,w) = \pi^{-n/2}\varepsilon^{-|\alpha|-n}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\,(w-\overline{z})^\alpha, \quad \alpha \in \mathbb{N}^n.$$

This frame is non holomorphic! The $(0,1)$-connection $\nabla^{0,1} = \overline{\partial}$ is given by

$$\nabla^{0,1} e_\alpha = \overline{\partial}_z e_\alpha(z,w) = \varepsilon^{-1}\sum_{1\le j\le n}\sqrt{\alpha_j(|\alpha|+n)}\,d\overline{z}_j \otimes e_{\alpha-c_j}$$

where $c_j = (0,\ldots,1,\ldots,0) \in \mathbb{N}^n$.

Curvature of Bergman bundles

Let $\Theta_{B_\varepsilon,h} = \nabla^2$ be the curvature tensor of B_ε with its natural Hilbertian metric h. Remember that

$$\Theta_{B_\varepsilon,h} = \nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0} \in C^\infty(X, \Lambda^{1,1}T_X^\star \otimes \mathrm{Hom}(B_\varepsilon, B_\varepsilon)),$$

and that one gets an associated quadratic Hermitian form on $T_X \otimes B_\varepsilon$ such that

$$\widetilde{\Theta}_\varepsilon(v\otimes\xi) = \langle \Theta_{B_\varepsilon,h}\sigma(v,Jv)\xi,\xi\rangle_h$$

for $v \in T_X$ and $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_\varepsilon$.

Definition

One says that the curvature tensor is Griffiths positive if

$$\widetilde{\Theta}_\varepsilon(v\otimes\xi) > 0, \quad \forall 0 \ne v \in T_X, \quad \forall 0 \ne \xi \in B_\varepsilon,$$

and Nakano positive if

$$\widetilde{\Theta}_\varepsilon(\tau) > 0, \quad \forall 0 \ne \tau \in T_X \otimes B_\varepsilon.$$

Calculation of the curvature tensor for $X = \mathbb{C}^n$

A simple calculation of ∇^2 in the orthonormal frame (e_α) leads to:

Formula

In the model case $X = \mathbb{C}^n$, the curvature tensor of the Bergman bundle (B_ε, h) is given by

$$\widetilde{\Theta}_\varepsilon(v \otimes \xi) = \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \Bigg(\Big| \sum_j \sqrt{\alpha_j}\, \xi_{\alpha - c_j} v_j \Big|^2 + \sum_j (|\alpha| + n)\, |\xi_\alpha|^2 |v_j|^2 \Bigg).$$

Consequence

In $\mathbb{C}^n$, the curvature tensor $\Theta_\varepsilon(v \otimes \xi)$ is Nakano positive.

On should observe that $\widetilde{\Theta}_\varepsilon(v \otimes \xi)$ is an unbounded quadratic form on B_ε with respect to the standard metric $\|\xi\|^2 = \sum_\alpha |\xi_\alpha|^2$.

However there is convergence for all $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_{\varepsilon'}$, $\varepsilon' > \varepsilon$, since then $\sum_\alpha (\varepsilon'/\varepsilon)^{2|\alpha|} |\xi_\alpha|^2 < +\infty$.

Curvature of Bergman bundles (general case)

Bergman curvature formula on a general hermitian manifold

Let X be a compact complex manifold equipped with a C^ω hermitian metric γ, and $B_\varepsilon = B_{\gamma,\varepsilon}$ the associated Bergman bundle.
Then its curvature is given by an asymptotic expansion

$$\widetilde{\Theta}_\varepsilon(z, v \otimes \xi) = \sum_{p=0}^{+\infty} \varepsilon^{-2+p} Q_p(z, v \otimes \xi), \quad v \in T_X, \quad \xi \in B_\varepsilon$$

where $Q_0(z, v \otimes \xi) = Q_0(v \otimes \xi)$ is given by the model case $\mathbb{C}^n$:

$$Q_0(v \otimes \xi) = \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \Bigg(\Big| \sum_j \sqrt{\alpha_j}\, \xi_{\alpha - c_j} v_j \Big|^2 + \sum_j (|\alpha| + n)\, |\xi_\alpha|^2 |v_j|^2 \Bigg).$$

The other terms $Q_p(z, v \otimes \xi)$ are real analytic; Q_1 and Q_2 depend respectively on the torsion and curvature tensor of γ.
In particular $Q_1 = 0$ is γ is Kähler.

A consequence of the above formula is that B_ε is strongly Nakano positive for $\varepsilon > 0$ small enough.

Idea of proof of the asymptotic expansion

The formula is in principle a special case of a more general result proved by Wang Xu, expressing the curvature of weighted Bergman bundles $\mathcal{H}_t$ attached to a smooth family $\{D_t\}$ of strongly pseudoconvex domains. Wang's formula is however in integral form and not completely explicit.

Here, one simply uses the real analytic Taylor expansion of $\mathrm{logh} : X \times \overline{X} \to T_X$ (inverse diffeomorphism of exph)

$$\mathrm{logh}_z(w) = w - \overline{z} + \sum z_j a_j(w-\overline{z}) + \sum \overline{z}_j a'_j(w-\overline{z})$$

$$+ \sum z_j z_k b_{jk}(w-\overline{z}) + \sum \overline{z}_j \overline{z}_k b'_{jk}(w-\overline{z})$$

$$+ \sum z_j \overline{z}_k c_{jk}(w-\overline{z}) + O(|z|^3),$$

which is used to compute the difference with the model case $\mathbb{C}^n$, for which $\mathrm{logh}_z(w) = w - \overline{z}$.

Back to holomorphic Morse inequalities

Idea for the general case. Let $\gamma \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ and $u \in \gamma$ a smooth form. As we have seen, one can find a sequence of Hermitian line bundles (L_k, h_k, ∇_k) such that

$$\theta_k = \frac{i}{2\pi}\nabla_k^2 = ku + \beta_k, \quad \beta_k = O(k^{-1/b_2}).$$

Then $d\theta_k = 0 \Rightarrow \overline{\partial}\beta_k^{0,2} = 0$, and as U_ε is Stein, $\mathrm{pr}_1^*\,\beta_k^{0,2} = \overline{\partial}\eta_k$ with a C^∞ $(0,1)$-form $\eta_k = O(k^{-1/b_2})$. This shows that $\tilde{L}_k := pr_1^* L_k$ becomes a holomorphic line bundle when equipped with the connection $\tilde{\nabla}_k = \mathrm{pr}_1^*\nabla_k - \eta_k$, which has a curvature form $\Theta_{\tilde{L}_k,\tilde{\nabla}_k} = k\,\mathrm{pr}_1^*\,u + O(k^{-1/b_2})$. Two possibilities emerge:

- correct the small eigenvalue eigenfunctions $\mathrm{pr}_1^*\,\sigma_{k,\ell}$ given by Laeng's method to actually get holomorphic sections of $\tilde{L}_k$ on U_ε.
- directly deal with the Hilbert Dolbeault complex of $(\mathrm{pr}_1)^{L^2}_*(\mathcal{O}_{U_\varepsilon}(\tilde{L}_k))$, and use Bergman estimates instead of dimension counts in Morse inequalities.

Other potential target: invariance of plurigenera for polarized families of compact Kähler manifolds?

Conjecture

Let $\pi : \mathcal{X} \to S$ be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base S. Assume that the family admits a polarization, i.e. a closed smooth $(1,1)$-form ω such that $\omega_{|X_t}$ is positive definite on each fiber $X_t := \pi^{-1}(t)$. Then the plurigenera

$$p_m(X_t) = h^0(X_t, mK_{X_t}) \text{ are independent of } t \text{ for all } m \geq 0.$$

The conjecture is known to be true for a projective family $\mathcal{X} \to S$:

- Siu and Kawamata (1998) in the case of varieties of general type
- Siu (2000) and Păun (2004) in the arbitrary projective case

The proof is based on an iterated application of the Ohsawa-Takegoshi L^2 extension theorem w.r.t. an ample line bundle $\mathcal{A}$ on $\mathcal{X}$: replace $\mathcal{A}$ by a Bergman bundle in the Kähler case ?

The end

Thank you for your attention