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On the existence of global orbifold jet differentials

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Aim of the lecture

- Our goal is to study (nonconstant) entire curves $f : \mathbb{C} \to X$ drawn in a projective variety/$\mathbb{C}$. The variety X is said to be Brody hyperbolic if there are no such curves.
- More generally, if $\Delta = \sum \Delta_j$ is a reduced normal crossing divisor in X, we want to study entire curves $f : \mathbb{C} \to X \smallsetminus \Delta$ drawn in the complement of Δ. If there are no such curves, we say that the log pair (X, Δ) is Brody hyperbolic.
- Even more generally, if $\Delta = \sum (1 - \frac{1}{\rho_j})\Delta_j \subset X$ is a normal crossing divisor, we want to study entire curves $f : \mathbb{C} \to X$ meeting each component Δ_j of Δ with multiplicity $\geq \rho_j$.
  The pair (X, Δ) is called an orbifold (in the sense of Campana).
  Here $\rho_j \in \,]1, \infty]$, where $\rho_j = \infty$ corresponds to the logarithmic case. Usually $\rho_j \in \{2, 3, ..., \infty\}$, but $\rho_j \in \mathbb{R}_{>1}$ will be allowed.
- The strategy is to show that under suitable conditions, orbifold entire curves must satisfy algebraic differential equations.

k-jets of curves and k-jet bundles

Let X be a nonsingular n-dimensional projective variety over $\mathbb{C}$.

Definition of k-jets

For $k \in \mathbb{N}^*$, a k-jet of curve $f_{[k]} : (\mathbb{C}, 0)_k \to X$ is an equivalence class of germs of holomorphic curves $f : (\mathbb{C}, 0) \to X$, written $f = (f_1, \ldots, f_n)$ in local coordinates $(z_1, \ldots, z_n)$ on an open subset $U \subset X$, where two germs are declared to be equivalent if they have the same Taylor expansion of order k at 0 :

$$f(t) = x + t\xi_1 + t^2\xi_2 + \cdots + t^k\xi_k + O(t^{k+1}), \quad t \in D(0, \varepsilon) \subset \mathbb{C},$$

and $x = f(0) \in U$, $\xi_s \in \mathbb{C}^n$, $1 \le s \le k$.

Notation

Let J^kX be the bundle of k-jets of curves, and $\pi_k : J^kX \to X$ the natural projection, where the fiber $(J^kX)_x = \pi_k^{-1}(x)$ consists of k-jets of curves $f_{[k]}$ such that $f(0) = x$.

Algebraic differential operators

Let $t \mapsto z = f(t)$ be a germ of curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet at any point $t = 0$. Look at the $\mathbb{C}^*$-action induced by *dilations* $\lambda \cdot f(t) := f(\lambda t)$, $\lambda \in \mathbb{C}^*$, for $f_{[k]} \in J^kX$.

Taking a (local) connection ∇ on T_X and putting $\xi_s = f^{(s)}(0) = \nabla^s f(0)$, we get a trivialization $J^kX \simeq (T_X)^{\oplus k}$ and the $\mathbb{C}^*$ action is given by

$$(*) \qquad \lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k).$$

We consider the Green-Griffiths sheaf $E_{k,m}(X)$ of homogeneous polynomials of weighted degree m on J^kX defined by

$$P(x\,;\, \xi_1, \ldots, \xi_k) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(x)\, \xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k}, \quad \sum_{s=1}^k s|\alpha_s| = m.$$

Here, we assume the coefficients $a_{\alpha_1\alpha_2\ldots\alpha_k}(x)$ to be holomorphic in x, and view P as a differential operator $P(f) = P(f\,;\, f', f'', \ldots, f^{(k)})$,

$$P(f)(t) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\ f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k}.$$

Graded algebra of algebraic differential operators

In this way, we get a graded algebra $\bigoplus_m E_{k,m}(X)$ of differential operators. As sheaf of rings, in each coordinate chart $U \subset X$, it is a pure polynomial algebra isomorphic to

$$\mathcal{O}_X[f_j^{(s)}]_{1\le j\le n,\,1\le s\le k} \quad \text{where} \quad \deg f_j^{(s)} = s.$$

If a change of coordinates $z \mapsto w = \psi(z)$ is performed on U, the curve $t \mapsto f(t)$ becomes $t \mapsto \psi \circ f(t)$ and we have inductively

$$(\psi \circ f)^{(s)} = (\psi' \circ f) \cdot f^{(s)} + Q_{\psi,s}(f', \ldots, f^{(s-1)})$$

where $Q_{\psi,s}$ is a polynomial of weighted degree s.

By filtering by the partial degree of $P(x; \xi_1, \ldots, \xi_k)$ successively in ξ_k, $\xi_{k-1}, \ldots, \xi_1$, one gets a multi-filtration on $E_{k,m}(X)$ such that the graded pieces are

$$G^\bullet E_{k,m}(X) = \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}T_X^* \otimes \cdots \otimes S^{\ell_k}T_X^*.$$

Logarithmic jet differentials

Take a logarithmic pair (X, Δ), $\Delta = \sum \Delta_j$ normal crossing divisor.

Fix a point $x \in X$ which belongs exactly to p components, say $\Delta_1, \ldots, \Delta_p$, and take coordinates $(z_1, \ldots, z_n)$ so that $\Delta_j = \{z_j = 0\}$.

$\Longrightarrow$ log differential operators : polynomials in the derivatives

$$(\log f_j)^{(s)}, \quad 1 \le j \le p \quad \text{and} \quad f_j^{(s)}, \quad p+1 \le j \le n.$$

Alternatively, one gets an algebra of logarithmic jet differentials, denoted $\bigoplus_m E_{k,m}(X, \Delta)$, that can be expressed locally as

$$\mathcal{O}_X\big[(f_1)^{-1}f_1^{(s)}, \ldots, (f_p)^{-1}f_p^{(s)}, f_{p+1}^{(s)}, \ldots, f_n^{(s)}\big]_{1\le s\le k}.$$

One gets a multi-filtration on $E_{k,m}(X, \Delta)$ with graded pieces

$$G^\bullet E_{k,m}(X, \Delta) = \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}T_X^*\langle\Delta\rangle \otimes \cdots \otimes S^{\ell_k}T_X^*\langle\Delta\rangle$$

where $T_X^*\langle\Delta\rangle$ is the logarithmic tangent bundle, locally free sheaf generated by $\frac{dz_1}{z_1}, \ldots, \frac{dz_p}{z_p}, dz_{p+1}, \ldots, dz_n$.

Orbifold jet differentials

Consider an orbifold (X,Δ), $\Delta=\sum(1-\frac{1}{\rho_j})\Delta_j$ a SNC divisor.

Assuming $\Delta_1=\{z_1=0\}$ and f having multiplicity $q\geq\rho_1>1$ along Δ_1, then $f_1^{(s)}$ still vanishes at order $\geq(q-s)_+$, thus $(f_1)^{-\beta}f_1^{(s)}$ is bounded as soon as $\beta q\leq(q-s)_+$, i.e. $\beta\leq(1-\frac{s}{q})_+$. Thus, it is sufficient to ask that $\beta\leq(1-\frac{s}{\rho_1})_+$. At a point $x\in|\Delta_1|\cap\ldots\cap|\Delta_p|$, the condition for a monomial of the form

$$(*)\qquad f_1^{-\beta_1}\ldots f_p^{-\beta_p}\prod_{s=1}^k(f^{(s)})^{\alpha_s},\quad (f^{(s)})^{\alpha_s}=(f_1^{(s)})^{\alpha_{s,1}}\ldots(f_n^{(s)})^{\alpha_{s,n}},$$

$\alpha_s\in\mathbb{N}^n$, $\beta_1,\ldots,\beta_p\in\mathbb{N}$, to be bounded, is to require that

$$(**)\qquad \beta_j\leq\sum_{s=1}^k\alpha_{s,j}\Big(1-\frac{s}{\rho_j}\Big)_+,\quad 1\leq j\leq p.$$

Definition

$E_{k,m}(X,\Delta)$ is taken to be the algebra generated by monomials $(*)$ of degree $\sum s|\alpha_s|=m$, satisfying partial degree inequalities $(**)$.

Orbifold jet differentials [continued]

It is important to notice that if we consider the log pair $(X,\lceil\Delta\rceil)$ with $\lceil\Delta\rceil=\sum\Delta_j$, then

$$E_{k,m}(X,\Delta)\ \text{ is a graded subalgebra of }\ E_{k,m}(X,\lceil\Delta\rceil).$$

The subalgebra $E_{k,m}(X,\Delta)$ still has a multi-filtration induced by the one on $E_{k,m}(X,\lceil\Delta\rceil)$, and, at least for $\rho_j\in\mathbb{Q}$, we formally have

$$G^\bullet E_{k,m}(X,\Delta)\subset\bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m}S^{\ell_1}T_X^*\langle\Delta^{(1)}\rangle\otimes\cdots\otimes S^{\ell_k}T_X^*\langle\Delta^{(k)}\rangle$$

where $T_X^*\langle\Delta^{(s)}\rangle$ is the "s-th orbifold cotangent sheaf" generated by

$$z_j^{-(1-s/\rho_j)_+}d^{(s)}z_j,\quad 1\leq j\leq p,\qquad d^{(s)}z_j,\quad p+1\leq j\leq n$$

(which makes sense only after taking some Galois cover of X ramifying at sufficiently large order along Δ_j).

Projectivized jets and direct image formula

Green Griffiths bundles

Consider $X_k := J^k X/\mathbb{C}^* = \mathrm{Proj} \bigoplus_m E_{k,m}(X)$. This defines a bundle $\pi_k : X_k \to X$ of weighted projective spaces whose fibers are the quotients of $(\mathbb{C}^n)^k \smallsetminus \{0\}$ by the $\mathbb{C}^*$ action

$$\lambda \cdot (\xi_1, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k).$$

Correspondingly, there is a tautological rank 1 sheaf $\mathcal{O}_{X_k}(m)$ [only invertible when $\mathrm{lcm}(1, ..., k) \mid m$], and a direct image formula

$$E_{k,m}(X) = (\pi_k)_* \mathcal{O}_{X_k}(m)$$

In the logarithmic case, we define similarly

$$X_k\langle\Delta\rangle := \mathrm{Proj} \bigoplus_m E_{k,m}(X, \Delta)$$

and let $\mathcal{O}_{X_k\langle\Delta\rangle}(1)$ be the corresponding tautological sheaf, so that

$$E_{k,m}(X, \Delta) = (\pi_k)_* \mathcal{O}_{X_k\langle\Delta\rangle}(m)$$

Generalized Green-Griffiths-Lang conjecture

Generalized GGL conjecture (very optimistic ?)

If (X, Δ) is an orbifold of general type, in the sense that $K_X + \Delta$ is a big $\mathbb{R}$-divisor, then there is a proper algebraic subvariety $Y \subsetneq X$ containing all orbifold entire curves $f : \mathbb{C} \to (X, \Delta)$ (not contained in Δ and having multiplicity $\geq \rho_j$ along Δ_j).

One possible strategy is to show that such orbifold entire curves f must satisfy a lot of algebraic differential equations of the form $P(f; f', ..., f^{(k)}) = 0$ for $k \gg 1$. This is based on:

Fundamental vanishing theorem

[Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996], ...
Let A be an ample divisor on X. Then, for all global jet differential operators on (X, Δ) with coefficients vanishing on A, i.e. $P \in H^0(X, E_{k,m}(X, \Delta) \otimes \mathcal{O}(-A))$, and for all orbifold entire curves $f : \mathbb{C} \to (X, \Delta)$, one has $P(f_{[k]}) \equiv 0$.

Proof of the fundamental vanishing theorem

Simple case. First consider the absolute case ($\Delta = 0$), and assume that f is a Brody curve, i.e. $\|f'\|_\omega$ bounded for some hermitian metric ω on X. By raising P to a power, we can assume A very ample, and view P as a $\mathbb{C}$ valued differential operator whose coefficients vanish on a very ample divisor A.

The Cauchy inequalities imply that all derivatives $f^{(s)}$ are bounded in any relatively compact coordinate chart. Hence $u_A(t) = P(f_{[k]})(t)$ is bounded, and must thus be
constant by Liouville's theorem.

Since A is very ample, we can move $A \in |A|$ such that A hits $f(\mathbb{C}) \subset X$. Bu then u_A vanishes somewhere and so $u_A \equiv 0$.

Logarithmic and orbifold cases. In the general case, the proof is more tricky. One possible way is to use Nevanlinna theory, and especially the logarithmic derivative lemma.

Holomorphic Morse inequalities

Theorem (D, 1985, L. Bonavero 1996)

Let $L \to X$ be a holomorphic line bundle on a compact complex manifold. Assume L equipped with a *singular hermitian metric* $h = e^{-\varphi}$ with analytic singularities in $\Sigma \subset X$, and $\theta = \frac{i}{2\pi}\Theta_{L,h}$. Let

$$X(\theta, q) := \{x \in X \smallsetminus \Sigma\,;\ \theta(x) \text{ has signature } (n-q, q)\}$$

be the q-index set of the $(1,1)$-form θ, and

$$X(\theta, \le q) = \bigcup_{j \le q} X(\theta, j).$$

Then

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes \mathcal{I}(m\varphi)) \le \frac{m^n}{n!} \int_{X(\theta,\le q)} (-1)^q \theta^n + o(m^n),$$

where $\mathcal{I}(m\varphi) \subset \mathcal{O}_X$ denotes the multiplier ideal sheaf

$$\mathcal{I}(m\varphi)_x = \{f \in \mathcal{O}_{X,x}\,;\ \exists U \ni x \text{ s.t. } \textstyle\int_U |f|^2 e^{-m\varphi} dV < +\infty\}.$$

Holomorphic Morse inequalities [continued]

Consequence of the holomorphic Morse inequalities

For $q = 1$, with the same notation as above, we get a lower bound

$$h^0(X, L^{\otimes m}) \geq h^0(x, l^{\otimes m} \otimes \mathcal{I}(m\varphi))$$
$$\geq h^0(x, l^{\otimes m} \otimes \mathcal{I}(m\varphi)) - h^1(x, l^{\otimes m} \otimes \mathcal{I}(m\varphi))$$
$$\geq \frac{m^n}{n!} \int_{x(\theta, \leq 1)} \theta^n - o(m^n).$$

here θ is a real $(1, 1)$ form of arbitrary signature on x.

when $\theta = \alpha - \beta$ for some explicit (1,1)-forms $\alpha, \beta \geq 0$ (not necessarily closed), an easy lemma yields

$$\mathbf{1}_{x(\alpha-\beta, \leq 1)}\ (\alpha - \beta)^n \geq \alpha^n - n\alpha^{n-1} \wedge \beta$$

hence

$$h^0(X, L^{\otimes m}) \geq \frac{m^n}{n!} \int_X (\alpha^n - n\alpha^{n-1} \wedge \beta) - o(m^n).$$

Finsler metric on the k-jet bundles

Assume that T_X is equipped with a C^∞ connection ∇ and a (possibly singular) hermitian metric h. One then defines a "weighted Finsler metric" on $J^k X$ by taking $p = k!$ and

$$\Psi_{h_k}(f_{[k]}) := \Big(\sum_{1 \leq s \leq k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \Big)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

Letting $\xi_s = \nabla^s f(0)$, this can be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k}(1)$, and the curvature form of L_k is obtained by computing $\frac{i}{2\pi}\partial\overline{\partial} \log \Psi_{h_k}(f_{[k]})$ as a function of $(x, \xi_1, \ldots, \xi_k)$.

Modulo negligible error terms of the form $O(\varepsilon_{s+1}/\varepsilon_s)$, this gives

$$\Theta_{L_k, h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi} \sum_{1 \leq s \leq k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor $\Theta_{T_X^*, h^*}$ and $\omega_{\mathrm{FS},k}$ is the weighted Fubini-Study metric on the fibers of $X_k \to X$.

Evaluation of Morse integrals

The above expression gets simpler by using polar coordinates

$$x_s = |\xi_s|_h^{2p/s}, \qquad u_s = \xi_s/|\xi_s|_h = \nabla^s f(0)/|\nabla^s f(0)|.$$

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ.

By holomorphic Morse inequalities, we need to evaluate an integral

$$\int_{X_k(\Theta_{L_h,h_k},\le 1)} \Theta_{L_k,h_k}^{N_k}, \qquad N_k = \dim X_k = n + (kn-1),$$

and we have to integrate over the parameters $z \in X$, $x_s \in \mathbb{R}_+$ and u_s in the unit sphere bundle $\mathbb{S}(T_X,1) \subset T_X$.

Since the weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2p/s} = 1$, we can take here $\sum x_s = 1$, i.e. (x_s) in the $(k-1)$-dimensional simplex Δ^{k-1}.

Probabilistic interpretation of the curvature

Now, the signature of Θ_{L_k,h_k} depends only on the vertical terms, i.e.

$$\sum_{1\le s\le k} \frac{1}{s} x_s q(u_s), \quad q(u_s) := \frac{i}{2\pi} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j.$$

After averaging over $(x_s) \in \Delta^{k-1}$ and computing the rational number $\int \omega_{\mathrm{FS},k}(\xi)^{nk-1} = \frac{1}{(k!)^n}$, what is left is to evaluate Morse integrals with respect to (u_s) of "horizontal" $(1,1)$-forms given by sums $\sum \frac{1}{s} q(u_s)$, where u_s are "random points" on the unit sphere.

As $k \to +\infty$, this sum yields asymptotically a "Monte-Carlo" integral

$$\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big) \int_{u\in\mathbb{S}(T_X,1)} q(u)\, du.$$

Since q is quadratic in u, we have $\displaystyle\int_{u\in\mathbb{S}(T_X,1)} q(u)\, du = \frac{1}{n}\mathrm{Tr}(q)$ and

$$\mathrm{Tr}(q) = \mathrm{Tr}(\Theta_{T_X^*,h^*}) = \Theta_{\det T_X^*,\det h^*} = \Theta_{K_X,\det h^*}.$$

Probabilistic cohomology estimate

Theorem 1 (D-, Pure and Applied Math. Quarterly 2011)

Let $A \to X$ be a $\mathbb{Q}$-line bundle on X, (T_X, h) and (A, h_A) hermitian structures on T_X and A. Let $\eta = \Theta_{K_X, \det h^*} - \Theta_{A, h_A}$ and

$$L_k = \mathcal{O}_{X_k}(1) \otimes \pi_k^* \mathcal{O}_X\Big(-\frac{1}{kn}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)A\Big).$$

Then for m sufficiently divisible, we have a lower bound

$$h^0(X_k, L_k^{\otimes m}) = h^0\Big(X, E_{k,m}(X) \otimes \mathcal{O}_X\Big(-\frac{m}{kn}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)A\Big)\Big)$$
$$\geq \frac{m^{n+kn-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^n} \Big(\int_{X(\eta, \leq 1)} \eta^n - \frac{C}{\log k}\Big).$$

There are in fact similar upper and lower bounds for $h^q(X_k, L_k^{\otimes m})$.

Corollary

If K_X is big and A is $\mathbb{Q}$-ample and small, then η can be taken > 0, so $h^0(X_k, L_k^{\otimes m}) \geq c_{n,k} m^{n+kn-1}$ with $c_{n,k} > 0$, for $m \gg k \gg 1$.

Non probabilistic cohomology estimate

Instead of using a Monte-Carlo integral, one can rather assume that $\Theta_{T_X^*, h^*}$ admits an explicit lower bound of the form

$$\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \xi_i \overline{\xi}_j u_\alpha \overline{u}_\beta \geq -\sum \gamma_{ij} \xi_i \overline{\xi}_j \, |u|^2,$$

where $\gamma = i \sum \gamma_{ij} dz_1 \wedge d\overline{z}_j \geq 0$ is a $(1,1)$-form on X. In case X is embedded in $\mathbb{P}^N$, one can always take $\gamma = \Theta_{\mathcal{O}(2)} = 2\omega_{\rm FS}$. By Morse inequalities in the difference form $\mathbf{1}_{X(\alpha-\beta, \leq 1)}\,(\alpha - \beta)^n$, one gets

Theorem 2 (D-, Acta Math. Vietnamica 2012)

Assume $k \geq n$ and $m \gg 1$. With the same notation as in Theorem 1, the dimensions $h^0(X_k, L_k^{\otimes m})$ are bounded below by

$$m^{n+kn-1} \int_X c_{n,k} \big(\Theta_{K_X} + n\gamma\big)^n - c'_{n,k} \big(\Theta_{K_X} + n\gamma\big)^{n-1} \wedge \big(\Theta_{A,h_A} + n\gamma\big)$$

where $c_{n,k}, c'_{n,k} \in \mathbb{Q}_{>0}$ are explicit universal constants.

Logarithmic situation

In the case of a log pair (X,Δ), one reproduce essentially the same calculations, by replacing the cotangent bundle T^*_X by the logarithmic cotangent bundle $T^*_X\langle\Delta\rangle$. This gives

Theorem 3 (probabilistic estimate)

Put $\eta=\Theta_{K_X+\Delta,\det h^*}-\Theta_{A,h_A}$. For $m\gg k\gg 1$, the dimensions

$h^0\big(X,E_{k,m}(X,\Delta)\otimes\mathcal{O}_X\big(-\frac{1}{kn}(1+\frac12+\ldots+\frac1k)A\big)\big)$

are bounded below by

$$\frac{m^{n+kn-1}}{(n+kr-1)!}\,\frac{(\log k)^n}{n!\,(k!)^n}\bigg(\int_{X(\eta,\le 1)}\eta^n-\frac{C}{\log k}\bigg).$$

Theorem 4 (non probabilistic estimate)

Assume $\Theta_{T^*_X\langle\Delta\rangle}\ge-\gamma\otimes\mathrm{Id}$. For $k\ge n$, $m\gg 1$, there are bounds

$$m^{n+kn-1}\int_X c_{n,k}\big(\Theta_{K_X+\Delta}+n\gamma\big)^n-c'_{n,k}\big(\Theta_{K_X+\Delta}+n\gamma\big)^{n-1}\wedge\big(\Theta_{A,h_A}+n\gamma\big).$$

Orbifold situation

Consider now the orbifold case (X,Δ), $\Delta=\sum(1-\frac{1}{\rho_j})\Delta_j$.

In this case, the solution is to work on the logarithmic projectivized jet bundle $X_k\langle\lceil\Delta\rceil\rangle$, with Finsler metrics $\Psi_{h_k}(f_{[k]})$ of the form

$$\Bigg(\sum_{1\le s\le k}\varepsilon_s\bigg(\sum_{j=1}^{p}|f_j|^{-2(1-\frac{s}{\rho_j})_+}|f_j^{(s)}(0)|^2+\sum_{j=p+1}^{n}|f_j^{(s)}(0)|^2\bigg)_{h_s(x)}^{p/s}\Bigg)^{1/p},$$

where $h_s(x)$ is a hermitian metric on the s-th orbifold bundle $T^*_X\langle\Delta^{(s)}\rangle$ and $x=f(0)$. This gives:

Theorem 5 (non probabilistic estimate)

Assume that $\Theta_{T^*_X\langle\Delta^{(s)}\rangle}\ge-\gamma\otimes\mathrm{Id}$ and that $\Theta_{K_X+\Delta^{(s)}}\le\delta$, for all $s=1,\ldots,k$, with $(1,1)$-forms $\gamma,\delta\ge 0$. $\exists c_{n,k}>0$, $c''_{n,k}\le(3n\log k)^n$ such that $h^0\big(X,E_{k,m}(X,\Delta)\otimes\mathcal{O}_X\big(-\frac{1}{kn}(1+\frac12+\ldots+\frac1k)A\big)\big)$ is bounded below for $k\ge n$, $m\gg 1$, by $c_{n,k}\,m^{n+kn-1}\times$

$$\bigg[\int_X\bigwedge_{\ell=1}^{n}\big(\Theta_{K_X+\Delta^{(\ell)}}+n\gamma\big)-c''_{n,k}\big(\delta+n\gamma\big)^{n-1}\wedge\big(\Theta_{A,h_A}+n\gamma\big)-o(1)\bigg].$$