Holomorphic Morse inequalities and volume of (1,1) cohomology classes

Jean-Pierre Demailly

Institut Fourier, Université de Grenoble I, France

CMI, Chennai, December 18, 2008

RMS-SMF-IMSc-CMI Conference
held in Chennai, December 15–19, 2008

Positive cones

Definition. Let X be a compact Kähler manifold.

- The Kähler cone is the set $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an open convex cone.

Positive cones

Definition. Let X be a compact Kähler manifold.

- The Kähler cone is the set $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an open convex cone.
- The pseudo-effective cone is the set $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{T\}$ of closed positive $(1,1)$ currents. This is a closed convex cone.
  (by weak compactness of bounded sets of currents).

Positive cones

Definition. Let X be a compact Kähler manifold.

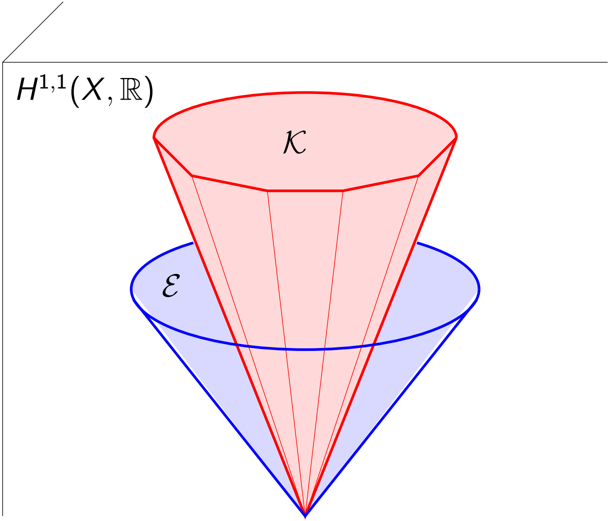
- The Kähler cone is the set $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an open convex cone.
- The pseudo-effective cone is the set $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{T\}$ of closed positive $(1,1)$ currents. This is a closed convex cone.
(by weak compactness of bounded sets of currents).
- Always true: $\overline{\mathcal{K}} \subset \mathcal{E}$.

Positive cones

Definition. Let X be a compact Kähler manifold.

- The Kähler cone is the set $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an open convex cone.
- The pseudo-effective cone is the set $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{T\}$ of closed positive $(1,1)$ currents. This is a closed convex cone. (by weak compactness of bounded sets of currents).
- Always true: $\overline{\mathcal{K}} \subset \mathcal{E}$.
- One can have: $\overline{\mathcal{K}} \subsetneq \mathcal{E}$: if X is the surface obtained by blowing-up $\mathbb{P}^2$ in one point, then the exceptional divisor $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha^2 = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

Kähler (red) cone and pseudoeffective (blue) cone



Neron Severi parts of the cones

In case X is projective, it is interesting to consider the "algebraic part" of our "transcendental cones" $\mathcal{K}$ and $\mathcal{E}$. which consist of suitable integral divisor classes.

Cohomology classes of algebraic divisors live in $H^2(X,\mathbb{Z})$.

Neron Severi parts of the cones

In case X is projective, it is interesting to consider the "algebraic part" of our "transcendental cones" $\mathcal{K}$ and $\mathcal{E}$. which consist of suitable integral divisor classes.

Cohomology classes of algebraic divisors live in $H^2(X,\mathbb{Z})$.

- **Neron-Severi lattice and Neron-Severi space**

$$\begin{aligned}\mathrm{NS}(X) &:= H^{1,1}(X,\mathbb{R}) \cap \big(H^2(X,\mathbb{Z})/\{\text{torsion}\}\big),\\ \mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}.\end{aligned}$$

Neron Severi parts of the cones

In case X is projective, it is interesting to consider the "algebraic part" of our "transcendental cones" $\mathcal{K}$ and $\mathcal{E}$. which consist of suitable integral divisor classes.

Cohomology classes of algebraic divisors live in $H^2(X,\mathbb{Z})$.

- **Neron-Severi lattice and Neron-Severi space**

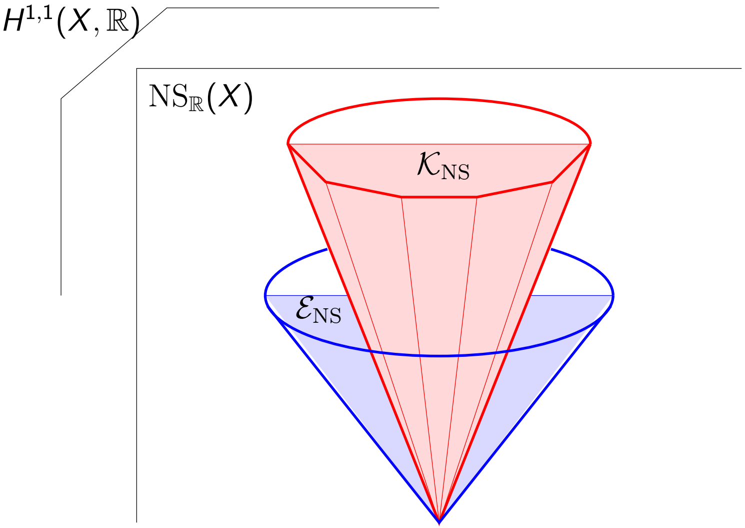
$$\begin{aligned} \mathrm{NS}(X) &:= H^{1,1}(X,\mathbb{R}) \cap \big(H^2(X,\mathbb{Z})/\{\mathrm{torsion}\}\big), \\ \mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}. \end{aligned}$$

- **Algebraic parts of $\mathcal{K}$ and $\mathcal{E}$**

$$\begin{aligned} \mathcal{K}_{\mathrm{NS}} &:= \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X), \\ \mathcal{E}_{\mathrm{NS}} &:= \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X). \end{aligned}$$

The rest we refer to as the "transcendental part"

Neron Severi parts of the cones



ample / nef / effective / big divisors

Theorem (Kodaira+successors, D90). *Assume X projective.*

- $\mathcal{K}_{\mathrm{NS}}$ *is the open cone generated by ample (or very ample) divisors A (Recall that a divisor A is said to be very ample if the linear system $H^0(X,\mathcal{O}(A))$ provides an embedding of X in projective space).*

ample / nef / effective / big divisors

Theorem (Kodaira+successors, D90). *Assume X projective.*

- $\mathcal{K}_{\mathrm{NS}}$ *is the open cone generated by ample (or very ample) divisors A (Recall that a divisor A is said to be very ample if the linear system $H^0(X, \mathcal{O}(A))$ provides an embedding of X in projective space).*
- *The closed cone $\overline{\mathcal{K}}_{\mathrm{NS}}$ consists of the closure of the cone of nef divisors D (or nef line bundles L), namely effective integral divisors D such that $D \cdot C \geq 0$ for every curve C.*

ample / nef / effective / big divisors

Theorem (Kodaira+successors, D90). *Assume X projective.*

- $\mathcal{K}_{\mathrm{NS}}$ *is the open cone generated by ample (or very ample) divisors A (Recall that a divisor A is said to be very ample if the linear system $H^0(X, \mathcal{O}(A))$ provides an embedding of X in projective space).*
- *The closed cone $\overline{\mathcal{K}}_{\mathrm{NS}}$ consists of the closure of the cone of nef divisors D (or nef line bundles L), namely effective integral divisors D such that $D \cdot C \geq 0$ for every curve C.*
- $\mathcal{E}_{\mathrm{NS}}$ *is the closure of the cone of effective divisors, i.e. divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}_+$.*

ample / nef / effective / big divisors

Theorem (Kodaira+successors, D90). *Assume X projective.*

- $\mathcal{K}_{\mathrm{NS}}$ *is the open cone generated by ample (or very ample) divisors A (Recall that a divisor A is said to be very ample if the linear system $H^0(X, \mathcal{O}(A))$ provides an embedding of X in projective space).*
- *The closed cone $\overline{\mathcal{K}}_{\mathrm{NS}}$ consists of the closure of the cone of nef divisors D (or nef line bundles L), namely effective integral divisors D such that $D \cdot C \geq 0$ for every curve C.*
- $\mathcal{E}_{\mathrm{NS}}$ *is the closure of the cone of effective divisors, i.e. divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}_+$.*
- *The interior $\mathcal{E}^\circ_{\mathrm{NS}}$ is the cone of big divisors, namely divisors D such that $h^0(X, \mathcal{O}(kD)) \geq c\, k^{\dim X}$ for k large.*

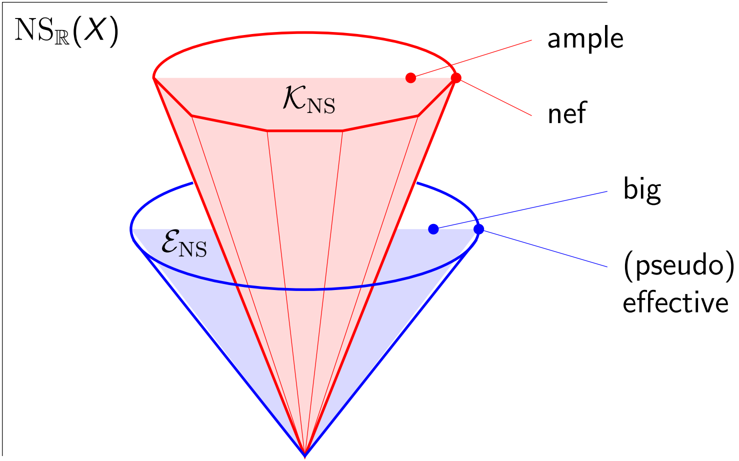
ample / nef / effective / big divisors

Theorem (Kodaira+successors, D90). *Assume X projective.*

- $\mathcal{K}_{\mathrm{NS}}$ *is the open cone generated by ample (or very ample) divisors A (Recall that a divisor A is said to be very ample if the linear system $H^0(X,\mathcal{O}(A))$ provides an embedding of X in projective space).*
- *The closed cone $\overline{\mathcal{K}}_{\mathrm{NS}}$ consists of the closure of the cone of nef divisors D (or nef line bundles L), namely effective integral divisors D such that $D\cdot C\ge 0$ for every curve C.*
- $\mathcal{E}_{\mathrm{NS}}$ *is the closure of the cone of effective divisors, i.e. divisors $D=\sum c_jD_j$, $c_j\in\mathbb{R}_+$.*
- *The interior $\mathcal{E}^{\circ}_{\mathrm{NS}}$ is the cone of big divisors, namely divisors D such that $h^0(X,\mathcal{O}(kD))\ge c\,k^{\dim X}$ for k large.*

Proof: L^2 estimates for $\overline{\partial}$ / Bochner-Kodaira technique

ample / nef / effective / big divisors



Characterization of the Kähler cone

Theorem (Demailly-Paun 2004).
Consider the "numerically positive cone"

$$\mathcal{P} = \{\alpha \in H^{1,1}(X, \mathbb{R})\,;\ \int_Y \alpha^p > 0\}$$

where $Y \subset X$ irreducible analytic subset, $\dim Y = p$.
The Kähler cone $\mathcal{K}$ is one of the connected components of $\mathcal{P}$.

Characterization of the Kähler cone

Theorem (Demailly-Paun 2004).
Consider the "numerically positive cone"

$$\mathcal{P} = \left\{\alpha \in H^{1,1}(X,\mathbb{R})\,;\ \int_Y \alpha^p > 0\right\}$$

where $Y \subset X$ *irreducible analytic subset,* $\dim Y = p$.
The Kähler cone $\mathcal{K}$ *is one of the connected components of* $\mathcal{P}$.

Corollary (DP2004). *Let* X *be a compact Kähler manifold.*
$\alpha \in H^{1,1}(X,\mathbb{R})$ *is nef* $(\alpha \in \overline{\mathcal{K}}) \Leftrightarrow$
$\int_Y \alpha \wedge \omega^{p-1} \geq 0$, $\forall \omega$ *Kähler,* $\forall Y \subset X$ *irreducible,* $\dim Y = p$.

Characterization of the Kähler cone

Theorem (Demailly-Paun 2004).
Consider the "numerically positive cone"

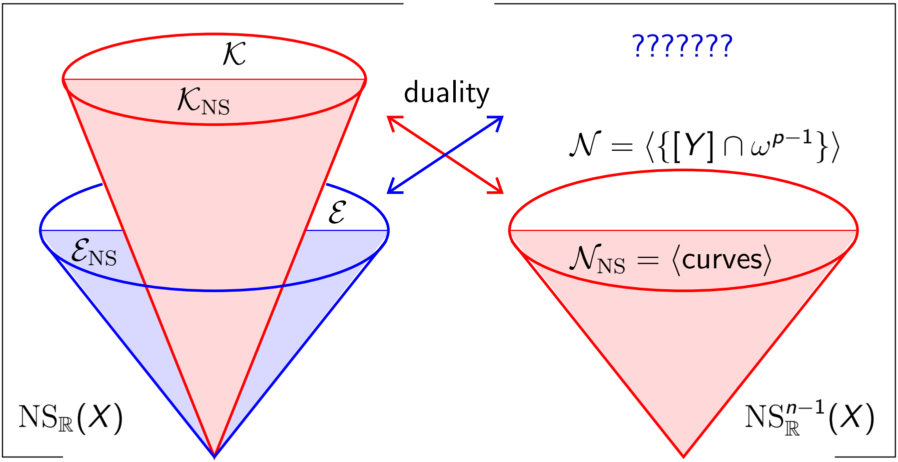
$$\mathcal{P} = \Big\{\alpha \in H^{1,1}(X,\mathbb{R})\,;\ \int_Y \alpha^p > 0\Big\}$$

where $Y \subset X$ irreducible analytic subset, $\dim Y = p$.
The Kähler cone $\mathcal{K}$ is one of the connected components of $\mathcal{P}$.

Corollary (DP2004). *Let X be a compact Kähler manifold.*
$\alpha \in H^{1,1}(X,\mathbb{R})$ *is nef* $(\alpha \in \overline{\mathcal{K}}) \Leftrightarrow$
$\int_Y \alpha \wedge \omega^{p-1} \geq 0$, $\forall \omega$ *Kähler,* $\forall Y \subset X$ *irreducible,* $\dim Y = p$.

Re-interpretation. *the dual of the nef cone $\overline{\mathcal{K}}$ is the closed convex cone in $H_{\mathbb{R}}^{n-1,n-1}(X)$ generated by cohomology classes of currents of the form $[Y] \wedge \omega^{p-1}$ in $H^{n-1,n-1}(X,\mathbb{R})$.*

Duality theorem for $\mathcal{K}$



(divisors) $\alpha \cdot \Gamma$ (curves)

$H^{1,1}(X, \mathbb{R}) \longleftarrow$ Serre duality $\longrightarrow H^{n-1,n-1}(X, \mathbb{R})$

Variation of complex structure

Suppose $\pi : \mathcal{X} \to S$ is a deformation of compact Kähler manifolds. Put $X_t = \pi^{-1}(t)$, $t \in S$ and let

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & 0 \\ * & \nabla^{1,1} & * \\ 0 & * & \nabla^{0,2} \end{pmatrix}$$

be the Gauss-Manin connection on the Hodge bundle $t \mapsto H^2(X_t, \mathbb{C})$, relative to the decomposition $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

Variation of complex structure

Suppose $\pi : \mathcal{X} \to S$ is a deformation of compact Kähler manifolds. Put $X_t = \pi^{-1}(t)$, $t \in S$ and let

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & 0 \\ * & \nabla^{1,1} & * \\ 0 & * & \nabla^{0,2} \end{pmatrix}$$

be the Gauss-Manin connection on the Hodge bundle $t \mapsto H^2(X_t, \mathbb{C})$, relative to the decomposition $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

Theorem (Demailly-Păun 2004). *Let $\pi : \mathcal{X} \to S$ be a deformation of compact Kähler manifolds over an irreducible base S. Then there exists a countable union $S' = \bigcup S_\nu$ of analytic subsets $S_\nu \subsetneq S$, such that the Kähler cones $\mathcal{K}_t \subset H^{1,1}(X_t, \mathbb{C})$ of the fibers $X_t = \pi^{-1}(t)$ are $\nabla^{1,1}$-invariant over $S \smallsetminus S'$ under parallel transport with respect to $\nabla^{1,1}$.*

Approximation of currents, Zariski decomposition

- **Definition.** *On X compact Kähler, a Kähler current T is a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.*

Approximation of currents, Zariski decomposition

- **Definition.** *On X compact Kähler, a Kähler current T is a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.*
- **Theorem.** $\alpha \in \mathcal{E}^\circ \Leftrightarrow \alpha \ni T$*, a Kähler current.*

  We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.

Approximation of currents, Zariski decomposition

- **Definition.** *On X compact Kähler, a Kähler current T is a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.*
- **Theorem.** $\alpha \in \mathcal{E}^\circ \Leftrightarrow \alpha \ni T$, *a Kähler current.*

  We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.
- **Theorem** (D92). *Any Kähler current T can be written*

$$T = \lim T_m$$

where $T_m \in \alpha = \{T\}$ has logarithmic poles, i.e. $\exists$ a modification $\mu_m : \widetilde{X}_m \to X$ such that

$$\mu_m^\star T_m = [E_m] + \gamma_m$$

where E_m is an effective $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coefficients in $\frac{1}{m}\mathbb{Z}$ and γ_m is a Kähler form on $\widetilde{X}_m$.

Idea of proof of analytic Zariski decomposition (1)

Locally one can write $T = i\partial\overline{\partial}\varphi$ for some strictly plurisubharmonic potential φ on X. The approximating potentials φ_m of φ are defined as

$$\varphi_m(z) = \frac{1}{2m} \log \sum_{\ell} |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \{f \in \mathcal{O}(\Omega)\,;\ \int_{\Omega} |f|^2 e^{-2m\varphi} dV < +\infty\}.$$

The Ohsawa-Takegoshi L^2 extension theorem (applied to extension from a single isolated point) implies that there are enough such holomorphic functions, and thus $\varphi_m \geq \varphi - C/m$. On the other hand $\varphi = \lim_{m\to+\infty} \varphi_m$ by a Bergman kernel trick and by the mean value inequality.

Idea of proof of analytic Zariski decomposition (2)

The Hilbert basis $(g_{\ell,m})$ is a family of local generators of the multiplier ideal sheaf $\mathcal{I}(mT) = \mathcal{I}(m\varphi)$. The modification $\mu_m : \widetilde{X}_m \to X$ is obtained by blowing-up this ideal sheaf, with

$$\mu_m^\star \mathcal{I}(mT) = \mathcal{O}(-mE_m).$$

for some effective $\mathbb{Q}$-divisor E_m with normal crossings on $\widetilde{X}_m$. Now, we set $T_m = i\partial\overline{\partial}\varphi_m$ and $\gamma_m = \mu_m^* T_m - [E_m]$. Then $\gamma_m = i\partial\overline{\partial}\psi_m$ where

$$\psi_m = \frac{1}{2m} \log \sum_\ell |g_{\ell,m} \circ \mu_m / h|^2 \quad \text{locally on } \widetilde{X}_m$$

and h is a generator of $\mathcal{O}(-mE_m)$, and we see that γ_m is a smooth semi-positive form on $\widetilde{X}_m$. The construction can be made global by using a gluing technique, e.g. via partitions of unity, and γ_m can be made Kähler by a perturbation argument.

Algebraic analogue

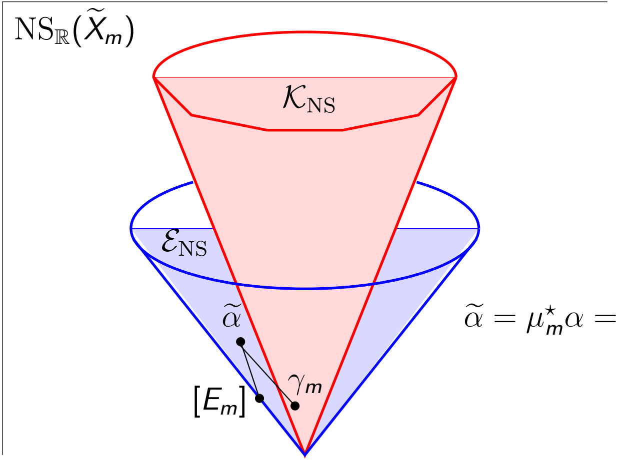
The more familiar algebraic analogue would be to take $\alpha = c_1(L)$ with a big line bundle L and to blow-up the base locus of $|mL|$, $m \gg 1$, to get a $\mathbb{Q}$-divisor decomposition

$$\mu_m^\star L \sim E_m + D_m, \qquad E_m \ \text{effective,} \quad D_m \ \text{free.}$$

Such a blow-up is usually referred to as a "log resolution" of the linear system $|mL|$, and we say that $E_m + D_m$ is an approximate Zariski decomposition of L.

We will also use the terminology of "approximate Zariski decomposition" for the above decomposition of Kähler currents with logarithmic poles.

Analytic Zariski decomposition



$$\widetilde{\alpha} = \mu_m^\star \alpha = [E_m] + \gamma_m$$

Fujita approximation / concept of volume

Theorem. (Fujita 1994) *If L is a big line bundle and $\mu_m^*(mL) = [E_m] + [D_m]$ (E_m=fixed part, D_m=moving part)*

$$\lim_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL) = \lim_{m\to+\infty} D_m^n.$$

This quantity will be called $\mathrm{Vol}(c_1(L))$. More generally :

Fujita approximation / concept of volume

Theorem. (Fujita 1994) *If L is a big line bundle and* $\mu_m^*(mL) = [E_m] + [D_m]$ *(E_m=fixed part, D_m=moving part)*

$$\lim_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL) = \lim_{m\to+\infty} D_m^n.$$

This quantity will be called $\mathrm{Vol}(c_1(L))$. More generally :

Definition (Boucksom 2002). *Let $\alpha \in \mathcal{E}^\circ$ be a big class*
The volume (movable self-intersection) of α is

$$\mathrm{Vol}(\alpha) = \sup_{T\in\alpha} \int_{\widetilde{X}} \gamma^n > 0$$

with Kähler currents $T \in \alpha$ with log poles,
and $\mu^\star T = [E] + \gamma$ where $\mu : \widetilde{X} \to X$ modification.

Fujita approximation / concept of volume

Theorem. (Fujita 1994) *If L is a big line bundle and $\mu_m^*(mL) = [E_m] + [D_m]$ (E_m=fixed part, D_m=moving part)*

$$\lim_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL) = \lim_{m\to+\infty} D_m^n.$$

This quantity will be called $\mathrm{Vol}(c_1(L))$. More generally :

Definition (Boucksom 2002). *Let $\alpha \in \mathcal{E}^\circ$ be a big class The volume (movable self-intersection) of α is*

$$\mathrm{Vol}(\alpha) = \sup_{T\in\alpha} \int_{\widetilde{X}} \gamma^n > 0$$

with Kähler currents $T \in \alpha$ with log poles, and $\mu^\star T = [E] + \gamma$ where $\mu : \widetilde{X} \to X$ modification.

If $\alpha \in \mathcal{K}$, then $\mathrm{Vol}(\alpha) = \alpha^n = \int_X \alpha^n$.

Fujita approximation / concept of volume

Theorem. (Fujita 1994) *If L is a big line bundle and $\mu_m^*(mL) = [E_m] + [D_m]$ (E_m=fixed part, D_m=moving part)*

$$\lim_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL) = \lim_{m\to+\infty} D_m^n.$$

This quantity will be called $\mathrm{Vol}(c_1(L))$. More generally :

Definition (Boucksom 2002). *Let $\alpha \in \mathcal{E}^\circ$ be a big class The volume (movable self-intersection) of α is*

$$\mathrm{Vol}(\alpha) = \sup_{T\in\alpha} \int_{\widetilde X} \gamma^n > 0$$

with Kähler currents $T \in \alpha$ with log poles, and $\mu^\star T = [E] + \gamma$ where $\mu : \widetilde X \to X$ modification.

If $\alpha \in \mathcal{K}$, then $\mathrm{Vol}(\alpha) = \alpha^n = \int_X \alpha^n$.

Theorem (Boucksom 2002). α contains $T_{\min}$ and $\mathrm{Vol}(\alpha) = \lim_{m\to+\infty} \int_X \gamma_m^n$ for the approximation of $T_{\min}$.

Movable intersection theory

Theorem (Boucksom 2002) *Let X be a compact Kähler manifold and*

$$H^{k,k}_{\geq 0}(X) = \big\{\{T\} \in H^{k,k}(X,\mathbb{R})\,;\ T \textit{ closed} \geq 0\big\}.$$

Movable intersection theory

Theorem (Boucksom 2002) *Let X be a compact Kähler manifold and*

$$H^{k,k}_{\geq 0}(X) = \{\{T\} \in H^{k,k}(X,\mathbb{R})\,;\ T \text{ closed} \geq 0\}.$$

- $\forall k = 1, 2, \ldots, n$,
  $\exists$ *canonical "movable intersection product"*
  $$\mathcal{E}\times\cdots\times\mathcal{E} \to H^{k,k}_{\geq 0}(X), \quad (\alpha_1,\ldots,\alpha_k) \mapsto \langle\alpha_1\cdot\alpha_2\cdots\alpha_{k-1}\cdot\alpha_k\rangle$$
  such that $\mathrm{Vol}(\alpha) = \langle\alpha^n\rangle$ *whenever* α *is a big class.*

Movable intersection theory

Theorem (Boucksom 2002) *Let X be a compact Kähler manifold and*

$$H^{k,k}_{\geq 0}(X) = \big\{\{T\} \in H^{k,k}(X,\mathbb{R})\,;\ T \text{ closed} \geq 0\big\}.$$

- $\forall k = 1, 2, \ldots, n$,
  $\exists$ *canonical "movable intersection product"*

  $$\mathcal{E}\times\cdots\times\mathcal{E} \to H^{k,k}_{\geq 0}(X), \quad (\alpha_1, \ldots, \alpha_k) \mapsto \langle\alpha_1\cdot\alpha_2\cdots\alpha_{k-1}\cdot\alpha_k\rangle$$

  such that $\mathrm{Vol}(\alpha) = \langle\alpha^n\rangle$ *whenever* α *is a big class.*
- *The product is increasing, homogeneous of degree 1 and superadditive in each argument, i.e.*

  $$\langle\alpha_1\cdots(\alpha_j'+\alpha_j'')\cdots\alpha_k\rangle \geq \langle\alpha_1\cdots\alpha_j'\cdots\alpha_k\rangle+\langle\alpha_1\cdots\alpha_j''\cdots\alpha_k\rangle.$$

  It coincides with the ordinary intersection product when the $\alpha_j \in \overline{\mathcal{K}}$ *are nef classes.*

Construction of the movable intersection product

First assume that all classes α_j are big, i.e. $\alpha_j \in \mathcal{E}^\circ$. Fix a smooth closed $(n-k, n-k)$ *semi-positive* form u on X. We select Kähler currents $T_j \in \alpha_j$ with logarithmic poles, and simultaneous more and more accurate log-resolutions $\mu_m : \widetilde{X}_m \to X$ such that

$$\mu_m^\star T_j = [E_{j,m}] + \gamma_{j,m}.$$

We define

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{m \to +\infty} \uparrow \{(\mu_m)_\star(\gamma_{1,m} \wedge \gamma_{2,m} \wedge \ldots \wedge \gamma_{k,m})\}$$

as a weakly convergent subsequence. The main point is to show that there is actually convergence and that the limit is unique in cohomology ; this is based on "monotonicity properties" of the Zariski decomposition.

Transcendental Holomorphic Morse inequalities

Conjecture. *For any class $\alpha \in H^{1,1}(X,\mathbb{R})$ and $\theta \in \alpha$ smooth*

$$\mathrm{Vol}(\{\alpha\}) \geq \int_{X(\theta,\leq 1)} \theta^n$$

where $\mathrm{Vol}(\alpha) := 0$ *if* $\alpha \notin \mathcal{E}^\circ$ *and*

$$X(\theta,q) = \{x \in X\,;\ \theta(x) \text{ has signature } (n-q,q)\}$$

$$X(\theta,\leq q) = \textstyle\bigcup_{0\leq j\leq q} X(\theta,j).$$

Conjecture. *For any class* $\alpha \in H^{1,1}(X, \mathbb{R})$ *and* $\theta \in \alpha$ *smooth*

$$\mathrm{Vol}(\{\alpha\}) \geq \int_{X(\theta, \leq 1)} \theta^n$$

where $\mathrm{Vol}(\alpha) := 0$ *if* $\alpha \notin \mathcal{E}^\circ$ *and*

$$X(\theta, q) = \{x \in X\,;\ \theta(x) \text{ has signature } (n-q, q)\}$$
$$X(\theta, \leq q) = \textstyle\bigcup_{0 \leq j \leq q} X(\theta, j).$$

Theorem (D 1985) (Holomorphic Morse inequalities)
The above is true when $\alpha = c_1(L)$ is integral. Then, with $\theta = \frac{i}{2\pi}\Theta_{L,h} \in \alpha$

$$H^0(X, L^{\otimes k}) \geq \frac{k^n}{n!} \int_{X(\theta, \leq 1)} \theta^n - o(k^n)$$

(and more generally, bounds for all $H^q(X, L^{\otimes k})$ hold true).

Three equivalent properties

Lemma. *A, B nef divisors on X projective. Then*

$$\mathrm{Vol}(A - B) \geq A^n - nA^{n-1} \cdot B.$$

Elementary / easy corollary of Morse inequalities.

Three equivalent properties

Lemma. *A, B nef divisors on X projective. Then*

$$\mathrm{Vol}(A-B) \ge A^n - nA^{n-1}\cdot B.$$

Elementary / easy corollary of Morse inequalities.

Theorem. *Let X be compact Kähler. We have* $\iff$

(1) $\forall\alpha,\beta\in\overline{\mathcal{K}}$, $\mathrm{Vol}(\alpha-\beta)\ge\alpha^n - n\alpha^{n-1}\cdot\beta$. (Weak Morse)

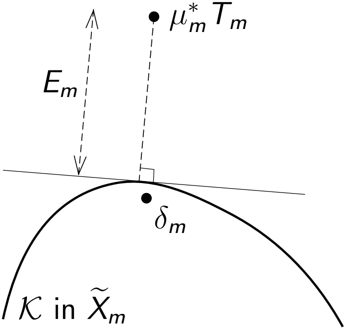
(2) $\forall\alpha,\beta\in\mathcal{E}$, $\mathrm{Vol}(\alpha-\beta)\ge\mathrm{Vol}(\alpha) - n\int_0^1\langle\alpha-t\beta\rangle^{n-1}\cdot\beta\,dt$.

(3) *Orthogonality property : Let $\alpha=\{T\}\in\mathcal{E}^\circ$ big, and $\mu_m^\star T_m = [E_m]+\gamma_m$ approximate Zariski decomposition. Then $\gamma_m^{n-1}\cdot E_m\to 0$ as $\mathrm{Vol}(\gamma_m)\to\mathrm{Vol}(\alpha)$.*

Proof. (2) $\Rightarrow$ (1) obvious.
What remains to show is : (1) $\Rightarrow$ (3), (3) $\Rightarrow$ (2).

Morse implies orthogonality

(1) $\Rightarrow$ (3). The proof is similar to the case of projecting a point onto a convex set, where the segment to closest point is orthogonal to tangent plane.



Orthogonality implies differential estimate

(3) $\Rightarrow$ (2): Take a parametrized) approximate Zariski Decomposition :

$$\mu^*(\alpha - t\beta) \ni [E_t] + \gamma_t$$

where $E_t = \sum c_j(t)E_j$. Take d/dt :

$$-\mu^*\beta \ni \sum \dot{c}_j(t)E_j + \dot{\gamma}_t$$

while

$$\mathrm{Vol}(\alpha - t\beta) \simeq \int_{\widetilde{X}} \gamma_t^n, \quad \frac{d}{dt}\mathrm{Vol}(\alpha - t\beta) \simeq n\int_{\widetilde{X}} \gamma_t^{n-1}\dot{\gamma}_t.$$

Since $\int_{\widetilde{X}} \gamma_t^{n-1} \cdot E_j$ small (by orthogonality), we get

$$\frac{d}{dt}\mathrm{Vol}(\alpha - t\beta) \simeq n\int_{\widetilde{X}} \gamma_t^{n-1} \cdot (-\mu^*\beta) = -n\int_{\widetilde{X}} \mu_*(\gamma_t^{n-1}) \cdot \beta \Rightarrow$$

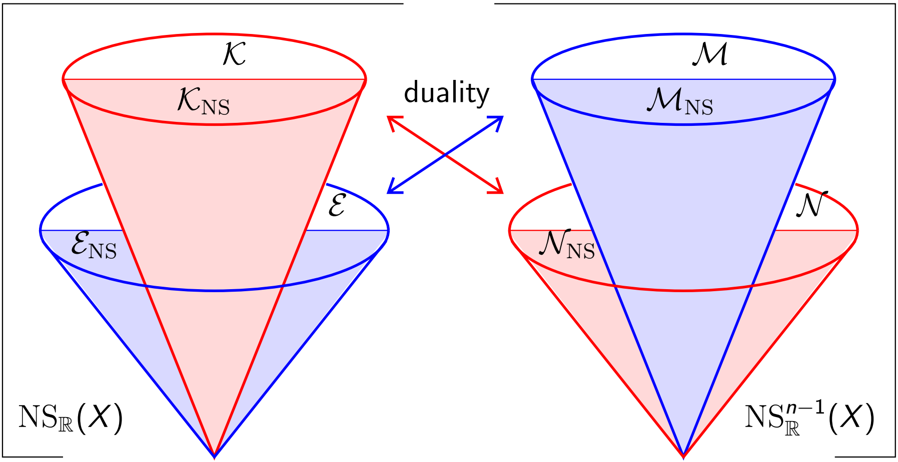
$$\frac{d}{dt}\mathrm{Vol}(\alpha - t\beta) \simeq -n\int_{\widetilde{X}} \langle(\alpha - t\beta)^{n-1}\rangle \cdot \beta.$$

Positive cones in $H^{n-1,n-1}(X)$ and Serre duality

Definition. *Let X be a compact Kähler manifold.*

- *Cone of $(n-1, n-1)$ positive currents*
  $\mathcal{N} = \overline{\mathit{cone}}\{\{T\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ T \text{ closed} \geq 0\}.$
- *Cone of effective curves*
  $$\mathcal{N}_{\mathrm{NS}} = \mathcal{N} \cap \mathrm{NS}_{\mathbb{R}}^{n-1,n-1}(X),$$
  $$= \overline{\mathit{cone}}\{\{C\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ C \text{ effective curve}\}.$$
- *Cone of movable curves : with $\mu : \widetilde{X} \to X$, let*
  $\mathcal{M}_{\mathrm{NS}} = \overline{\mathit{cone}}\{\{C\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ [C] = \mu_\star(H_1 \cdots H_{n-1})\}$
  where H_j = ample hyperplane section of $\widetilde{X}$.
- *Cone of movable currents : with $\mu : \widetilde{X} \to X$, let*
  $\mathcal{M} = \overline{\mathit{cone}}\{\{T\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ T = \mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})\}$
  where $\widetilde{\omega}_j$ = Kähler metric on $\widetilde{X}$.

Main duality theorem



$H^{1,1}(X,\mathbb{R})$ $\longleftarrow$ Serre duality $\longrightarrow$ $H^{n-1,n-1}(X,\mathbb{R})$

Proof of duality between $\mathcal{E}_{\mathrm{NS}}$ and $\mathcal{M}_{\mathrm{NS}}$

Theorem (Boucksom-Demailly-Păun-Peternell 2004).
For X projective, a class α is in $\mathcal{E}_{\mathrm{NS}}$ (pseudo-effective) if and only if it is dual to the cone $\mathcal{M}_{\mathrm{NS}}$ of moving curves.

Proof of the theorem. We want to show that $\mathcal{E}_{\mathrm{NS}} = \mathcal{M}_{\mathrm{NS}}^{\vee}$. By obvious positivity of the integral pairing, one has in any case

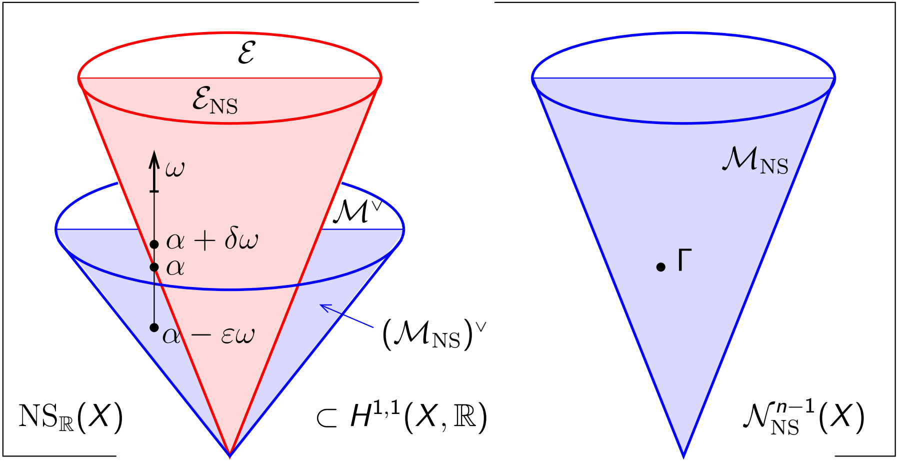
$$\mathcal{E}_{\mathrm{NS}} \subset (\mathcal{M}_{\mathrm{NS}})^{\vee}.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$ on the boundary of $\mathcal{E}_{\mathrm{NS}}$ which is in the interior of $\mathcal{N}_{\mathrm{NS}}^{\vee}$. Hence

$$(*) \qquad \alpha \cdot \Gamma \geq \varepsilon\omega \cdot \Gamma$$

for every moving curve Γ, while $\langle \alpha^n \rangle = \mathrm{Vol}(\alpha) = 0$.

Schematic picture of the proof



Then use approximate Zariski decomposition of $\{\alpha + \delta\omega\}$ and orthogonality relation to contradict $(*)$ with $\Gamma = \langle \alpha^{n-1} \rangle$.

Characterization of uniruled varieties

Recall that a projective variety is called uniruled if it can be covered by a family of rational curves $C_t \simeq \mathbb{P}^1_{\mathbb{C}}$.

Theorem (Boucksom-Demailly-Paun-Peternell 2004)
A projective manifold X has K_X pseudo-effective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$, if and only if X is not uniruled.

Characterization of uniruled varieties

Recall that a projective variety is called uniruled if it can be covered by a family of rational curves $C_t \simeq \mathbb{P}^1_{\mathbb{C}}$.

Theorem (Boucksom-Demailly-Paun-Peternell 2004)
A projective manifold X has K_X pseudo-effective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$, if and only if X is not uniruled.

Proof (of the non trivial implication). If $K_X \notin \mathcal{E}_{\mathrm{NS}}$, the duality pairing shows that there is a moving curve C_t such that $K_X \cdot C_t < 0$. The standard "bend-and-break" lemma of Mori then implies that there is family Γ_t of rational curves with $K_X \cdot \Gamma_t < 0$, so X is uniruled.

Conjecture. (BDPP 2004) The same is expected to be true for X compact Kähler.

Weak Kähler Morse inequalities (new approach)

Theorem (D 2008) Let X be compact Kähler, γ a Kähler class on X and $E = \sum c_j E_j \geq 0$ a divisor with normal crossings. Then, if $\mathrm{Vol}_{X|Y}$ denotes the "restricted" volume on Y ("sections" on Y which extend to X)

$$\mathrm{Vol}\left(\gamma + \sum c_j E_j\right) \geq \mathrm{Vol}(\gamma) + n \sum_j \int_0^{c_j} \mathrm{Vol}_{X|E_j}(\gamma + tE_j)\, dt$$

$$+ n(n-1) \sum_{j<k} \int_0^{c_j} \int_0^{c_k} \mathrm{Vol}_{X|E_j \cap E_k}(\gamma + t_j E_j + t_k E_k)\, dt_j dt_k$$

$$+ n(n-1)(n-2) \sum_{j<k<\ell} \int_0^{c_j} \int_0^{c_k} \int_0^{c_\ell} \mathrm{Vol}_{X|E_j \cap E_k \cap E_\ell} \ \ldots$$

Weak Kähler Morse inequalities (new approach)

Theorem (D 2008) Let X be compact Kähler, γ a Kähler class on X and $E = \sum c_j E_j \geq 0$ a divisor with normal crossings. Then, if $\mathrm{Vol}_{X|Y}$ denotes the "restricted" volume on Y ("sections" on Y which extend to X)

$$\mathrm{Vol}\left(\gamma + \sum c_j E_j\right) \geq \mathrm{Vol}(\gamma) + n \sum_j \int_0^{c_j} \mathrm{Vol}_{X|E_j}(\gamma + tE_j)\, dt$$

$$+\, n(n-1) \sum_{j<k} \int_0^{c_j} \int_0^{c_k} \mathrm{Vol}_{X|E_j \cap E_k}(\gamma + t_j E_j + t_k E_k)\, dt_j dt_k$$

$$+\, n(n-1)(n-2) \sum_{j<k<\ell} \int_0^{c_j} \int_0^{c_k} \int_0^{c_\ell} \mathrm{Vol}_{X|E_j \cap E_k \cap E_\ell} \ \ldots$$

The proof relies on pluripotential theory (glueing psh functions).

This should imply the orthogonality estimate in the Kähler case, and therefore also the duality theorem (work in progress).