Algebraic embeddings of complex and almost complex structures

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CIME School on Complex non-Kähler Geometry
Cetraro, Italy, July 9–13, 2018

A question raised by Fedor Bogomolov

Rough question

Can one produce an arbitrary compact complex manifold X, resp. an arbitrary compact Kähler manifold X by means of a "purely algebraic construction" ?

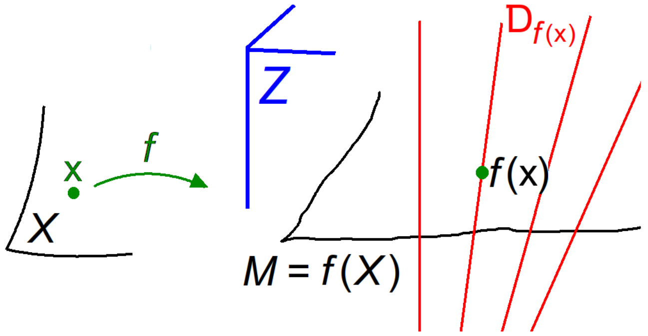
Let Z be a projective algebraic manifold, $\dim_{\mathbb{C}} Z = N$, equipped with a subbundle (or rather subsheaf) $\mathcal{D} \subset \mathcal{O}_Z(T_Z)$.

Assume that X^{2n} is a compact C^∞ real even dimensional manifold that is embedded in Z, as follows:

(i) $f : X \hookrightarrow Z$ is a smooth (say C^∞) embedding
(ii) $\forall x \in X, \quad f_* T_{X,x} \oplus \mathcal{D}_{f(x)} = T_{Z,f(x)}$.
(iii) $f(X) \cap \mathcal{D}_{\rm sing} = \emptyset$.

We say that $X \hookrightarrow (Z, \mathcal{D})$ is a transverse embedding.

Construction of an almost complex structure

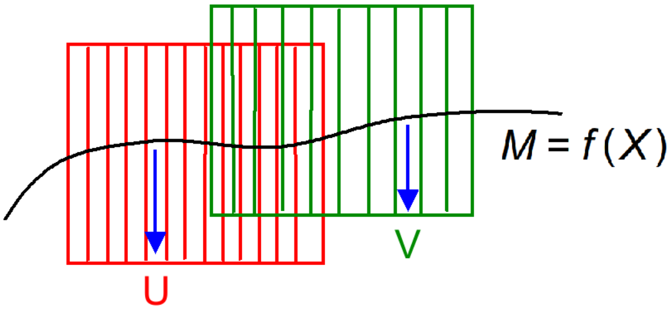


$f_* T_{X,x} = T_{M,f(x)} \simeq T_{Z,f(x)}/\mathcal{D}_{f(x)}$
is in a natural way
a complex vector space
$\Rightarrow$ almost complex structure J_f

Observation 1 (André Haefliger)

If $\mathcal{D} \subset T_Z$ is an algebraic foliation, i.e. $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$, then the almost complex structure J_f on X induced by $(Z, \mathcal{D})$ is integrable.

Proof: Any 2 charts yield a holomorphic transition map $U \to V$
$\Rightarrow$ holomorphic atlas

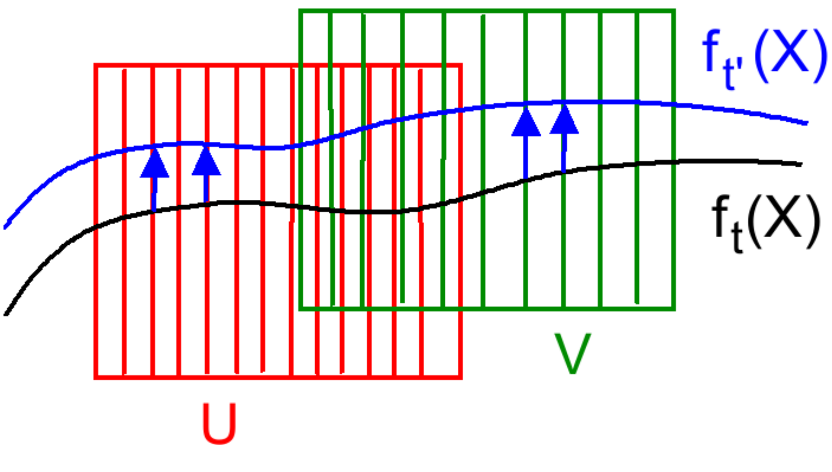


Invariance by transverse isotopies

Observation 2

If $\mathcal{D} \subset T_Z$ is an algebraic foliation and $f_t : X \hookrightarrow (Z, \mathcal{D})$ is an isotopy of transverse embeddings, $t \in [0,1]$, then all complex structures (X, J_{f_t}) are biholomorphic.

Proof:



A conjecture of Bogomolov

To each triple $(Z, \mathcal{D}, \alpha)$ where

- Z is a complex projective manifold
- $\mathcal{D} \subset T_Z$ is an algebraic foliation
- α is an isotopy class of transverse embeddings $f : X \hookrightarrow (Z, \mathcal{D})$

one can thus associate a biholomorphism class (X, J_f).

Conjecture (from RIMS preprint of Bogomolov, 1995)

One can construct in this way every compact complex manifold X.

Additional question 1

What if (X, ω) is Kähler ? Can one embed in such a way that ω is the pull-back of a transversal Kähler structure on $(Z, \mathcal{D})$?

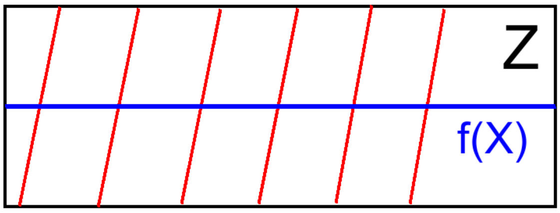
Additional question 2

Can one describe the non injectivity of the "Bogomolov functor" $(Z, D, \alpha) \mapsto (X, J_f)$, i.e. moduli spaces of such embeddings ?

There exist large classes of examples !

Example 1 : tori

If Z is an Abelian variety and $N \geq 2n$, every n-dimensional compact complex torus $X = \mathbb{C}^n/\Lambda$ can be embedded transversally to a linear codimension n foliation $\mathcal{D}$ on Z.



Example 2 : LVMB manifolds

One obtains a rich class, named after Lopez de Medrano, Verjovsky, Meersseman, Bosio, by considering foliations on $\mathbb{P}^N$ given by a commutative Lie subalgebra of the Lie algebra of $\mathrm{PGL}(N+1, \mathbb{C})$. The corresponding transverse varieties produced include e.g. Hopf surfaces and the Calabi-Eckmann manifolds $S^{2p+1} \times S^{2q+1}$.

What about the almost complex case ?

Easier question : drop the integrability assumption

Can one realize every compact almost complex manifold (X, J) by a transverse embedding into a projective algebraic pair $(Z, \mathcal{D})$, $\mathcal{D} \subset T_Z$, so that $J = J_f$?

Not surprisingly, there are constraints, and Z cannot be "too small". But how large exactly ?

Let $\Gamma^\infty(X, Z, \mathcal{D})$ the Fréchet manifold of transverse embeddings $f : X \hookrightarrow (Z, \mathcal{D})$ and $\mathcal{J}^\infty(X)$ the space of smooth almost complex structures on X.

Further question

When is $f \mapsto J_f$, $\quad \Gamma^\infty(X, Z, \mathcal{D}) \to \mathcal{J}^\infty(X)$ a submersion ?

Note: technically one has to consider rather Banach spaces of maps of $C^{r+\alpha}$ Hölder regularity.

Variation formula for J_f

First, the tangent space to the Fréchet manifold $\Gamma^\infty(X, Z, \mathcal{D})$ at a point f consists of

$$C^\infty(X, f^*T_Z) = C^\infty(X, f^*\mathcal{D}) \oplus C^\infty(X, T_X)$$

Theorem (D - Gaussier, arxiv:1412.2899, 2014, JEMS 2017)

Let $[\bullet, \bullet]$ be the Lie bracket of vector fields in T_Z,

$$\theta : \mathcal{D} \times \mathcal{D} \to T_Z/\mathcal{D}, \quad (\xi, \eta) \mapsto [\xi, \eta] \mod \mathcal{D}$$

be the torsion tensor of the holomorphic distribution $\mathcal{D}$, and $v \mapsto \overline{\partial}_{J_f} v$ the $\overline{\partial}$ operator of the almost complex structure (X, J_f).
Then the differential of the natural map $f \mapsto J_f$ along any infinitesimal variation $w = u + f_* v : X \to f^*T_Z = f^*\mathcal{D} \oplus f_*T_X$ of f is given by

$$dJ_f(w) = 2J_f\big(f_*^{-1}\theta(\overline{\partial}_{J_f} f, u) + \overline{\partial}_{J_f} v\big)$$

Sufficient condition for submersivity

Theorem (D - Gaussier, 2014)

Let $f : X \hookrightarrow (Z, \mathcal{D})$ be a smooth transverse embedding. Assume that f and the torsion tensor θ of $\mathcal{D}$ satisfy the following additional conditions :

(ii) f is a totally real embedding, i.e. $\overline{\partial} f(x) \in \mathrm{End}_{\overline{\mathbb{C}}}(T_{X,x}, T_{Z,f(x)})$ is injective at every point $x \in X$;

(ii) for every $x \in X$ and every $\eta \in \mathrm{End}_{\overline{\mathbb{C}}}(T_X)$, there exists a vector $\lambda \in \mathcal{D}_{f(x)}$ such that $\theta(\overline{\partial} f(x) \cdot \xi, \lambda) = \eta(\xi)$ for all $\xi \in T_X$.

Then there is a neighborhood $\mathcal{U}$ of f in $\Gamma^\infty(X, Z, \mathcal{D})$ and a neighborhood $\mathcal{V}$ of J_f in $\mathcal{J}^\infty(X)$ such that

$$\mathcal{U} \to \mathcal{V},\ f \mapsto J_f \text{ is a submersion.}$$

Remark. A necessary condition for (ii) to be possible is that $\mathrm{rank}\, \mathcal{D} = N - n \geq n^2 = \dim \mathrm{End}(T_X)$, i.e. $N \geq n + n^2$.

Existence of universal embedding spaces

Theorem (D - Gaussier, 2014)

For all integers $n \geq 1$ and $k \geq 4n$, there exists a complex affine algebraic manifold $Z_{n,k}$ of dimension $N = 2k + 2(k^2 + n(k - n))$ possessing a real structure (i.e. an anti-holomorphic algebraic involution) and an algebraic distribution $\mathcal{D}_{n,k} \subset T_{Z_{n,k}}$ of codimension n, with the following property:

for every compact n-dimensional almost complex manifold (X, J) admits an embedding $f : X \hookrightarrow Z^{\mathbb{R}}_{n,k}$ transverse to $\mathcal{D}_{n,k}$ and contained in the real part of $Z_{n,k}$, such that $J = J_f$.

The choice $k = 4n$ yields the explicit embedding dimension $N = 38n^2 + 8n$ (and a quadratic bound $N = O(n^2)$ is optimal by what we have seen previously).

Hint. $Z_{n,k}$ is produced by a fiber space construction mixing Grassmannians and twistor spaces.

Construction of the universal embedding

First observation. There exists a C^∞ embedding $\varphi : X \hookrightarrow \mathbb{R}^{2k}$, $k \geq 4n$, by the Whitney embedding theorem, and one can assume $N_{\varphi(X)} = (T_{\varphi(X)})^\perp$ to carry a complex structure for $k \geq 8n$; otherwise take $\Phi = \varphi \times \varphi : X \hookrightarrow \mathbb{R}^{2k} \times \mathbb{R}^{2k}$ and observe that

$$N_{\Phi(X)} \simeq N_X \oplus N_X \oplus T_X \simeq (\mathbb{C} \otimes_{\mathbb{R}} N_X) \oplus (T_X, J).$$

Second step. Assuming (N_X, J') almost complex, let $Z^{\mathbb{R}}_{n,k}$ be the set of triples (x, S, J) such that $S \in \mathrm{Gr}^{\mathbb{R}}(2k, 2n)$, $\operatorname{codim} S = 2n$, $J \in \mathrm{End}(\mathbb{R}^{2k})$, $J^2 = -\mathrm{Id}$, $J(S) \subset S$. Define

$$f : X \to Z^{\mathbb{R}}_{n,k}, \quad x \mapsto (\varphi(x), N_{\varphi(X),\varphi(x)}, \widetilde{J}(x))$$

where $\widetilde{J}$ is induced by $J(x) \oplus J'(x)$ on $\varphi_* T_X \oplus N_X$.

Third step. Complexify $Z^{\mathbb{R}}_{n,k}$ as a variety $Z_{n,k} = Z^{\mathbb{C}}_{n,k}$ and define an algebraic distribution $\mathcal{D}_{n,k} \subset T_{Z_{n,k}}$.

Definition of $Z_{n,k}$ and $\mathcal{D}_{n,k}$

We let $Z_{n,k} = Z^{\mathbb{C}}_{n,k}$ be the set of triples

$$(z, S, J) \in \mathbb{C}^{2k} \times \mathrm{Gr}^{\mathbb{C}}(2k, 2n) \times \mathrm{End}(\mathbb{C}^{2k})$$

with $J^2 = -\mathrm{Id}$, $J(S) = S$. Moreover we assume that we have "balanced" decompositions

$$S = S' \oplus S'', \qquad \dim S' = \dim S'' = n,$$
$$\mathbb{C}^{2k} = \Sigma' \oplus \Sigma'', \quad \dim \Sigma' = \dim \Sigma'' = k$$

for the i and $-i$ eigenspaces of $J_{|S}$ and J, $S' \subset \Sigma'$, $S'' \subset \Sigma''$.

Finally, if $\pi = \mathrm{pr}_1 : Z_{n,k} \to \mathbb{C}^{2k}$ is the first projection, we take $\mathcal{D}_{n,k}$ at point $w = (z, S, J)$ to be

$$\mathcal{D}_{n,k,w} := (d\pi)^{-1}(S' \oplus \Sigma'').$$

Since $\mathbb{C}^{2k} = \Sigma' \oplus \Sigma''$, we have

$$(T_{Z_{n,k}}/\mathcal{D}_{n,k})_w \simeq \Sigma'/S',$$

which on real points, is isomorphic to $(S^{\mathbb{R}})^\perp$.

Symplectic embeddings

Consider the case of a compact almost complex symplectic manifold (X, J, ω) where the symplectic form ω is assumed to be J-compatible, i.e. $J^*\omega = \omega$ and $\omega(\xi, J\xi) > 0$.

Definition

We say that a closed semipositive (1,1)-form β on Z is a transverse Kähler structure to $\mathcal{D} \subset T_Z$ if $\operatorname{Ker}\beta \subset \mathcal{D}$, i.e., if β induces a Kähler form on germs of complex submanifolds transverse to $\mathcal{D}$.

Theorem (D - Gaussier, 2014)

There also exist universal embedding spaces for compact almost complex symplectic manifolds, i.e. a certain triple $(Z, \mathcal{D}, \beta)$ as above, such that every (X, J, ω), $\dim_{\mathbb{C}} X = n$, $\{\omega\} \in H^2(X, \mathbb{Z})$, embeds transversally by $f : X \hookrightarrow (Z, \mathcal{D}, \beta)$, in such a way that

$$J = J_f \text{ and } \omega = f^*\beta.$$

Proof. Use the Tischler symplectic embedding $X \hookrightarrow (\mathbb{P}^{2n+1}, \omega_{\mathrm{FS}})$.

Integrability condition for an almost complex structure

Recall that the Nijenhuis tensor of an almost complex structure J is

$$N_J(\zeta, \eta) = 4\,\mathrm{Re}\,[\zeta^{0,1}, \eta^{0,1}]^{1,0} = [\zeta, \eta] - [J\zeta, J\eta] + J[\zeta, J\eta] + J[J\zeta, \eta].$$

The Newlander-Nirenberg theorem states that
(X, J) is complex analytic if and only if $N_J \equiv 0$.

In fact, we have the following relation between the torsion form θ of a distribution and the Nijenhuis tensor of the related transverse structure:

Nijenhuis tensor formula

If θ denotes the torsion of $(Z, \mathcal{D})$, the Nijenhuis tensor of the almost complex structure J_f induced by a transverse embedding $f : X \hookrightarrow (Z, \mathcal{D})$ is given by $\forall z \in X$, $\forall \zeta, \eta \in T_z X$

$$N_{J_f}(\zeta, \eta) = 4\,\theta(\overline{\partial}_{J_f} f(z) \cdot \zeta, \overline{\partial}_{J_f} f(z) \cdot \eta).$$

Solution of a weak Bogomolov conjecture

Theorem (D - Gaussier, 2014)

There exist universal embedding spaces $(W,\mathcal{E},\mathcal{S})=(W_{n,k},\mathcal{E}_{n,k},\mathcal{S}_{n,k})$ where $\dim W_{n,k} < \dim Z_{n,k} + n(\dim Z_{n,k} - 2n) = O(nk^2) = O(n^3)$, and $\mathcal{S} \subset \mathcal{E} \subset T_W$ are algebraic subsheaves satisfying $[\mathcal{S},\mathcal{S}] \subset \mathcal{E}$ (partial integrability), such that every compact $\mathbb{C}$-manifold (X,J) of given dimension n embeds transversally by $f: X \hookrightarrow (W_{n,k},\mathcal{E}_{n,k})$, i.e. $J = J_f$, with the additional constraint $\mathrm{Im}(\overline{\partial} f) \subset \mathcal{S}_{n,k}$.

Proof. By the Nijenhuis tensor formula, since $\overline{\partial}_{J_f} f$ is injective with values in $\mathcal{D}_{n,k}$, we see that $S = \overline{\partial}_{J_f} f(T_{X,x}) \subset \mathcal{D}_{n,k,f(x)}$ must be an n-dimensional complex subspace of $\mathcal{D}_{n,k,\,x} \subset T_{Z,f(x)}$ that is totally isotropic for θ, i.e. $\theta_{|S\times S} \equiv 0$.

We let $W_{n,k} \subset \mathrm{Gr}(\mathcal{D}_{n,k}, n)$ be the subvariety of the Grassmannian bundle consisting of the θ-isotropic n-subspaces, and lift $\mathcal{D}_{n,k} \subset T_{Z_{n,k}}$ to $\mathcal{E}_{n,k} \subset T_{W_{n,k}}$, $\mathcal{S}_{n,k}$ being the tautological isotropic subbundle.

Yau's challenge and S^6

In complex dimension 2, it is known that there exist compact almost complex manifolds that cannot be given a complex structure: by Van de Ven (1966), for X a complex surface,

$$p = c_1^2(X),\ q = c_2(X) \text{ is in the region } \{p \le 8q,\ p+q \equiv 0(12)\},$$

but the only restriction for X almost complex is $p + q \equiv 0(12)$.

Yau's challenge

For $n \ge 3$, find a compact almost complex n-fold that cannot be given a complex structure .

The sphere S^6 can be realized as the set of octonions $x \in \mathbb{O}$ such that $x^2 = -1$ ($\Leftrightarrow \mathrm{Re}\, x = 0$ and $|x| = 1$).

A natural non integrable almost complex structure is then given by

$$J_x h = xh,\ h \in T_{S^6,x} \Leftrightarrow \mathrm{Re}\, h = 0 \text{ and } xh + hx = 0.$$

S^6 is strongly suspected of not carrying a complex structure!

Application to complex structures on S^6

The octonion embedding $f : S^6 \hookrightarrow \mathbb{O} = \mathbb{R}^{2k}$, $k = 4$ (which has trivial rank 2 normal bundle), yields a universal embedding $\varphi : S^6 \to Z_{3,4}$ where $\dim Z_{3,4} = 46$, $\operatorname{rank} \mathcal{D}_{3,4} = 43$ (corank 3).

By passing to the Grassmannian bundle we get a map $\psi : S^6 \to W_{3,4}$ where $\dim W_{3,4} < 46 + 3 \times 40 = 166$, $W_{3,4}$ being equipped with bundles $\mathcal{E}_{3,4} \supset \mathcal{S}_{3,4}$ of respective coranks 3 and 43, and at the homotopy level the question is whether $\overline{\partial}\psi \subset \mathcal{E}_{3,4}$ can be retracted to a section with values in $\mathcal{S}_{3,4}$ over the whole S^6.

If the answer is negative, this would prove that there are no complex structures on S^6 (it is well known that S^6 admits only two almost complex structures up to homotopy, J_0 given by the octonions and its conjugate $-J_0$).

In general, this approach could yield topological obstructions for an almost complex structure to be homotopic to a complex structure.

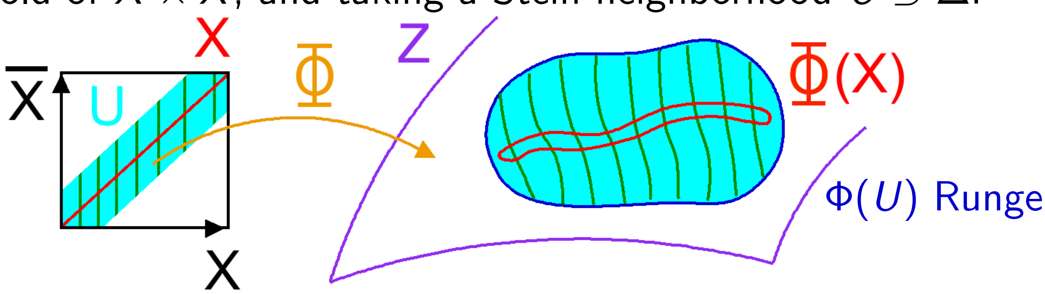
What about Bogomolov's original conjecture ?

Proposition (reduction of the conjecture to another one !)

Assume that holomorphic foliations can be approximated by Nash algebraic foliations uniformly on compact subsets of any polynomially convex open subset of $\mathbb{C}^N$.
Then every compact complex manifold can be approximated by compact complex manifolds that are embeddable in the sense of Bogomolov in foliated projective manifolds.

The proof uses the Grauert technique of embedding X as a totally real submanifold of $X \times \overline{X}$, and taking a Stein neighborhood $U \supset \Delta$.

Proof:



$\exists \Phi : U \to Z$ holomorphic embedding into Z affine algebraic (Stout).

Thank you for your attention