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# Kobayashi conjecture on the generic hyperbolicity of algebraic hypersurfaces in projective space

Jean-Pierre Demailly

Institut Fourier, Université Grenoble Alpes & Académie des Sciences de Paris

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A complex space $X$ is said to be Kobayashi hyperbolic if the Kobayashi pseudodistance $d_{\mathrm{Kob}} : X \times X \to \mathbb{R}_+$ is a distance (i.e. everywhere non degenerate).

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**Theorem** (Brody, 1978)

For a compact complex manifold $X$, $dim_{\mathbb{C}} X = n$, TFAE:

(i) $X$ is Kobayashi hyperbolic

(ii) $X$ is Brody hyperbolic, i.e. $\nexists$ entire curves $f : \mathbb{C} \to X$

(iii) The Kobayashi infinitesimal pseudometric $\mathbf{k}_x$ is everywhere non degenerate

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Our interest is the study of hyperbolicity for projective varieties.
In dim $n = 1$, $X$ is hyperbolic iff genus $g \geq 2$.

# Main conjectures

## Conjecture of General Type (CGT)

- A compact complex variety $X$ is volume hyperbolic $\iff$ $X$ is of general type, i.e. $K_X$ is big [implication $\Longleftarrow$ is well known].

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## Green-Griffiths-Lang Conjecture (GGL)

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## Arithmetic counterpart (Lang 1987) – very optimistic ?

If $X$ is projective and defined over a number field $\mathbb{K}_0$, the smallest locus $Y = \mathrm{GGL}(X)$ in GGL's conjecture is also the smallest $Y$ such that $X(\mathbb{K}) \smallsetminus Y$ is finite $\forall \mathbb{K}$ number field $\supset \mathbb{K}_0$.

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### Consequence of CGT + GGL

A compact complex manifold $X$ should be Kobayashi hyperbolic iff it is projective and every subvariety $Y$ of $X$ is of general type.

# Kobayashi conjecture on generic hyperbolicity

**Kobayashi conjecture (1970)**

- Let $X^n \subset \mathbb{P}^{n+1}$ be a (very) generic hypersurface of degree $d \geq d_n$ large enough. Then $X$ is Kobayashi hyperbolic.

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Using "jet technology" and deep results of McQuillan for curve foliations on surfaces, the following has been proved:

**Theorem (D., El Goul, 1998)**

A very generic surface $X^2 \subset \mathbb{P}^3$ of degree $d \geq 21$ is hyperbolic. Independently McQuillan got $d \geq 35$.

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In 2012, Yum-Tong Siu announced a proof of the case of arbitrary dimension $n$, with a non explicit $d_n$ (and a rather involved proof).

# Results on the generic Green-Griffiths conjecture

By a combination of an algebraic existence theorem for jet differentials and of Y.T. Siu's technique of "slanted vector fields" (itself derived from ideas of H. Clemens, L. Ein and C. Voisin), the following was proved:

**Theorem (S. Diverio, J. Merker, E. Rousseau, 2009)**

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$$d_n = \left\lfloor \frac{n^4}{3}\big(n\log(n\log(24n))\big)^n \right\rfloor \qquad \text{(D-, 2012)},$$

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**Theorem (S. Diverio, S. Trapani, 2009)**

Additionally, a generic hypersurface $X^3 \subset \mathbb{P}^4$ of degree $d \geq 593$ is hyperbolic.

# Recent proof of the Kobayashi conjecture

In 2016, Brotbek gave a shorter and more geometric proof of Y.T. Siu's result on the Kobayashi conjecture, using again jet techniques.

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Let $Z$ be a projective $n+1$-dimensional projective manifold and $A \to Z$ a very ample line bundle. Let $\sigma \in H^0(Z, dA)$ be a generic section. Then, for $d \gg 1$ large, the hypersurface $X_\sigma = \sigma^{-1}(0)$ is hyperbolic.

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The initial proof of Brotbek did not provide effective bounds. Through various improvements, Deng Ya showed in his PhD thesis:

**Theorem (Y. Deng, May 2016)**

In the above setting, a generic hypersurface $X_\sigma = \sigma^{-1}(0)$ is hyperbolic as soon as

$$d \geq d_n = (n+1)^{n+2}(n+2)^{2n+7} = O(n^{3n+9}).$$

# Solution of a conjecture of Debarre (2005)

In the same vein, the following results have also been proved.

**Solution of Debarre's conjecture (Brotbek-Darondeau & Xie, 2015)**

Let $Z$ be a projective $n+c$-dimensional projective manifold and $A \to Z$ a very ample line bundle. Let $\sigma_j \in H^0(Z, d_jA)$ be generic sections, $1 \le j \le c$. Then, for $c \geq n$ and $d_j \gg 1$ large, the $n$-dimensional complete intersection $X_\sigma = \bigcap \sigma_j^{-1}(0) \subset Z$ has an ample cotangent bundle $T^*_{X_\sigma}$.

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$$d_{n,c} = 4\nu(2N-1)^{2\nu+1} + 6N - 3 = O((2N)^{N+3}), \qquad \nu = \lfloor \tfrac{N+1}{2} \rfloor.$$

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The proof is obtained by selecting carefully certain special sections $\sigma_j$ associated with "lacunary" polynomials of high degree.

# Category of directed manifolds

**Goal.** More generally, we are interested in curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$ where $V$ is a subbundle of $T_X$, or possibly a singular linear subspace, i.e. a closed irreducible analytic subspace such that $\forall x \in X$, $V_x := V \cap T_{X,x}$ is linear.

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**Definition (Category of directed manifolds)**

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## Canonical sheaf of a directed manifold $(X,V)$

When $V$ is nonsingular, i.e. a subbundle, one simply sets

$$K_V=\det(V^*)\quad\text{(as a line bundle).}$$

# Canonical sheaf of a singular pair (X,V)

When $V$ is singular, we first introduce the rank 1 sheaf ${}^b\mathcal{K}_V$ of sections of $\det V^*$ that are locally bounded with respect to a smooth ambient metric on $T_X$. One can show that ${}^b\mathcal{K}_V$ is equal to the integral closure of the image of the natural morphism

$$\mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*) \to \mathcal{L}_V := \text{invert. sheaf } \mathcal{O}(\Lambda^r V^*)^{**}$$

that is, if the image is $\mathcal{L}_V \otimes \mathcal{J}_V$, $\mathcal{J}_V \subset \mathcal{O}_X$,

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### Consequence

If $\mu : \widetilde{X} \to X$ is a modification and $\widetilde{X}$ is equipped with the pull-back directed structure $\widetilde{V} = \overline{\widetilde{\mu}^{-1}(V)}$, then

$$ {}^b\mathcal{K}_V \subset \mu_*({}^b\mathcal{K}_{\widetilde{V}}) \subset \mathcal{L}_V$$

and $\mu_*({}^b\mathcal{K}_{\widetilde{V}})$ increases with $\mu$.

# Canonical sheaf of a singular pair (X,V) [cont.]

By Noetherianity, one can define a sequence of rank 1 sheaves

$$\mathcal{K}_V^{[m]} = \lim_{\mu} \uparrow \mu_*({}^b\mathcal{K}_{\widetilde{V}})^{\otimes m}, \quad \mu_*({}^b\mathcal{K}_V)^{\otimes m} \subset \mathcal{K}_V^{[m]} \subset \mathcal{L}_V^{\otimes m}$$

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**Remark**

The blow-up $\mu$ for which the limit is attained may depend on $m$. We do not know if there is a $\mu$ that works for all $m$.

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## Remark

The blow-up $\mu$ for which the limit is attained may depend on $m$. We do not know if there is a $\mu$ that works for all $m$.

This generalizes the concept of reduced singularities of foliations, which is known to work in that form only for surfaces.

## Definition

We say that $(X, V)$ is of general type if the pluricanonical sheaf sequence $\mathcal{K}_V^{[\bullet]}$ is big, i.e. $H^0(X, \mathcal{K}_V^{[m]})$ provides a generic embedding of $X$ for a suitable $m \gg 1$.

# Definition of algebraic differential operators

Let $(\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, $\quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$ be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on $X$. It has a local Taylor expansion

$$f(t) = x + t\xi_1 + \ldots + t^k\xi_k + O(t^{k+1}), \quad \xi_s = \frac{1}{s!}\nabla^s f(0)$$

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One considers the Green-Griffiths bundle $E^{\mathrm{GG}}_{k,m}V^*$ of polynomials of weighted degree $m$ written locally in coordinate charts as

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(x)\xi_1^{\alpha_1}\ldots\xi_k^{\alpha_k}, \quad \xi_s \in V,$$

also viewed as algebraic differential operators

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2}\ldots f^{(k)}(t)^{\alpha_k}. \end{aligned}$$

# Definition of algebraic differential operators [cont.]

Here $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its $k$-jet, and $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are supposed to holomorphic functions on $X$.

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The reparametrization action : $f \mapsto f \circ \varphi_\lambda$, $\varphi_\lambda(t) = \lambda t$, $\lambda \in \mathbb{C}^*$ yields $(f \circ \varphi_\lambda)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$, whence a $\mathbb{C}^*$-action

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$E_{k,m}^{\mathrm{GG}}$ is precisely the set of polynomials of weighted degree $m$, corresponding to coefficients $a_{\alpha_1\ldots\alpha_k}$ with $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$.

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### Direct image formula

If $J_k^{\mathrm{nc}}V$ is the set of non constant $k$-jets, one defines the Green-Griffiths bundle to be $X_k^{\mathrm{GG}} = J_k^{\mathrm{nc}}V/\mathbb{C}^*$ and $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ to be the associated tautological rank 1 sheaf. Then we have

$$\pi_k : X_k^{\mathrm{GG}} \to X, \qquad E_{k,m}^{\mathrm{GG}}V^* = (\pi_k)_*\mathcal{O}_{X_k^{\mathrm{GG}}}(m)$$

# Generalized GGL conjecture

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If $(X, V)$ is directed manifold of general type, i.e. $\mathcal{K}_V^{[\bullet]}$ is big, then $\exists Y \subsetneq X$ such that $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one has $f(\mathbb{C}) \subset Y$.

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**Remark.** Elementary by Ahlfors-Schwarz if $r = \mathrm{rk}\, V = 1$.
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$\forall P \in H^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-A))$ : global diff. operator on $X$
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## Theorem on existence of jet differentials (D-, 2010)

Let $(X, V)$ be of general type, such that ${}^b\mathcal{K}_V^{\otimes p}$ is a big rank 1 sheaf.
Then $\exists$ many global sections $P$, $m \gg k \gg 1 \Rightarrow \exists$ alg. hypersurface $Z \subsetneq X_k^{\mathrm{GG}}$ s.t. all entire $f : (\mathbb{C}, T_{\mathbb{C}}) \mapsto (X, V)$ satisfy $f_{[k]}(\mathbb{C}) \subset Z$.

# $1^{\mathrm{st}}$ step: take a Finsler metric on $k$-jet bundles

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$$\Psi_{h_k}(f):=\Big(\sum_{1\le s\le k}\varepsilon_s\|\nabla^s f(0)\|_{h(x)}^{2p/s}\Big)^{1/p},\quad 1=\varepsilon_1\gg\varepsilon_2\gg\cdots\gg\varepsilon_k.$$

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Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric $h_k$ on $L_k := \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$, with curvature form $(x, \xi_1, \ldots, \xi_k) \mapsto$

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\, dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor $\Theta_{V^*,h^*}$ and $\omega_{\mathrm{FS},k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\mathrm{GG}} \to X$.

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The expression gets simpler by using polar coordinates

$$x_s = |\xi_s|_h^{2p/s}, \qquad u_s = \xi_s/|\xi_s|_h = \nabla^s f(0)/|\nabla^s f(0)|.$$

# $2^{\mathrm{nd}}$ step: probabilistic interpretation of the curvature

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha} \overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in $\xi$. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors $u_s$ in the unit sphere bundle $SV \subset V$.

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The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2p/s} = 1$, so we can take here $x_s \geq 0$, $\sum x_s = 1$. This is essentially a sum of the form $\sum \frac{1}{s}\gamma(u_s)$ where $u_s$ are random points of the sphere, and so as $k \to +\infty$ this can be estimated by a "Monte-Carlo" integral

$$\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big) \int_{u\in SV} \gamma(u)\, du.$$

As $\gamma$ is quadratic here, $\int_{u\in SV} \gamma(u)\, du = \frac{1}{r}\,\mathrm{Tr}(\gamma)$.

# $3^{\rm rd}$ step: getting the main cohomology estimates

$\Rightarrow$ the leading term only involves the trace of $\Theta_{V^*,h^*}$, i.e. the curvature of $(\det V^*, \det h^*)$, that can be taken $> 0$ if $\det V^*$ is big.

## Corollary of holomorphic Morse inequalities (D-, 2010)

Let $(X, V)$ be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, $(V, h)$ and $(F, h_F)$ hermitian. Define

$$L_k = \mathcal{O}_{X_k^{\rm GG}}(1) \otimes \pi_k^* \mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big) F\Big),$$

$$\eta = \Theta_{\det V^*, \det h^*} + \Theta_{F, h_F}.$$

Then for all $q \geq 0$ and all $m \gg k \gg 1$ such that $m$ is sufficiently divisible, we have upper and lower bounds [$q = 0$ most useful!]

$$h^q(X_k^{\rm GG}, \mathcal{O}(L_k^{\otimes m})) \leq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r} \Big(\int_{X(\eta,q)} (-1)^q \eta^n + \frac{C}{\log k}\Big)$$

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# And now ... the Semple jet bundles

- **Fonctor "1-jet" :** $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :

  $\tilde{X} = P(V) =$ bundle of projective spaces of lines in $V$

  $\pi : \tilde{X} = P(V) \to X, \quad (x, [v]) \mapsto x, \quad v \in V_x$

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- **Basic exact sequences**

  $$0\to T_{X_k/X_{k-1}}\to V_k\xrightarrow{(\pi_k)_\star}\mathcal{O}_{X_k}(-1)\to 0\qquad\Rightarrow \operatorname{rk} V_k = r$$

  $$0\to\mathcal{O}_{X_k}\to\pi_k^\star V_{k-1}\otimes\mathcal{O}_{X_k}(1)\to T_{X_k/X_{k-1}}\to 0\ \ \text{(Euler)}$$

# Direct image formula for Semple bundles

For $n = \dim X$ and $r = \operatorname{rk} V$, one gets a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\operatorname{rk} V_k = r$,
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## Theorem

$X_k$ is a smooth compactification of $X_k^{\mathrm{GG,reg}}/\mathbb{G}_k = J_k^{\mathrm{GG,reg}}/\mathbb{G}_k$, where $\mathbb{G}_k$ is the group of $k$-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization: $(f, \varphi) \mapsto f \circ \varphi$, and $J_k^{\mathrm{reg}}$ is the space of $k$-jets of regular curves.

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## Direct image formula for invariant differential operators

$E_{k,m}V^* := (\pi_{k,0})_*\mathcal{O}_{X_k}(m) =$ sheaf of algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ such that $P((f \circ \varphi)_{[k]}) = \varphi'^m P(f_{[k]}) \circ \varphi$.

# Induced directed structure on a subvariety

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Alternatively, one could also take $W$ to be the closure of $T_{Z'} \cap V_k$ in the $k$-th stage $(\mathcal{X}_k, \mathcal{A}_k)$ of the "absolute Semple tower" associated with $(\mathcal{X}_0, \mathcal{A}_0) = (X, T_X)$ (so as to deal only with nonsingular ambient Semple bundles).

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This produces an induced directed subvariety

$$(Z, W) \subset (X_k, V_k).$$

It is easy to show that $\pi_{k,k-1}(Z) = X_{k-1} \Rightarrow \operatorname{rk} W < \operatorname{rk} V_k = \operatorname{rk} V$.

# Sufficient criterion for the GGL conjecture

**Definition**

Let $(X, V)$ be a directed pair where $X$ is projective algebraic. We say that $(X, V)$ is “strongly of general type” if it is of general type and for every irreducible alg. subvariety $Z \subsetneq X_k$ that projects onto $X$, $X_k \not\subset D_k := P(T_{X_{k-1}/X_{k-2}})$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, i.e. ${}^b\mathcal{K}_W \otimes \mathcal{O}_{X_k}(m)_{|Z}$ is big for some $m \in \mathbb{Q}_+$, after a suitable blow-up.

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**Theorem (D-, 2014)**

If $(X, V)$ is strongly of general type, the Green-Griffiths-Lang conjecture holds true for $(X, V)$, namely there $\exists Y \subsetneq X$ such that every non constant holomorphic curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfies $f(\mathbb{C}) \subset Y$.

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**Proof:** Induction on rank $V$, using existence of jet differentials.

# Related stability property

## Definition

Fix an ample divisor $A$ on $X$. For every irreducible subvariety $Z \subset X_k$ that projects onto $X_{k-1}$ for $k \geq 1$, $Z \not\subset D_k$, and $Z = X = X_0$ for $k = 0$, we define the slope of the corresponding directed variety $(Z, W)$ to be $\mu_A(Z, W) =$

$$\frac{\inf\left\{\lambda \in \mathbb{Q}\,;\ \exists m \in \mathbb{Q}_+,\ {}^b\mathcal{K}_W \otimes \big(\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(\lambda A)\big)_{|Z} \text{ big on } Z\right\}}{\mathrm{rank}\, W}.$$

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Notice that $(X, V)$ is of general type iff $\mu_A(X, V) < 0$.

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We say that $(X, V)$ is $A$-jet-stable (resp. $A$-jet-semi-stable) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leq \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above.

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Notice that $(X, V)$ is of general type iff $\mu_A(X, V) < 0$.

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**Observation.** If $(X, V)$ is of general type and $A$-jet-semi-stable, then $(X, V)$ is strongly of general type.

# Criterion for the generalized Kobayashi conjecture

**Definition**

Let $(X, V)$ be a directed pair where $X$ is projective algebraic. We say that $(X, V)$ is "algebraically jet-hyperbolic" if for every irreducible alg. subvariety $Z \subsetneq X_k$ s.t. $X_k \not\subset D_k$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ either has $W = 0$ or is of general type modulo $X_k \to X$.

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## Theorem (D-, 2014)

If $(X, V)$ is algebraically jet-hyperbolic, then $(X, V)$ is Kobayashi (or Brody) hyperbolic, i.e. there are no entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.

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Now, the hope is that a (very) generic complete intersection $X = H_1 \cap \ldots \cap H_c \subset \mathbb{P}^{n+c}$ of codimension $c$ and degrees $(d_1, \ldots, d_c)$ s.t. $\sum d_j \geq 2n + c$ yields $(X, T_X)$ algebraically jet-hyperbolic.

# Invariance of “directed plurigenera” ?

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One way to check the above property, at least with non optimal bounds, would be to show some sort of Zariski openness of the properties "strongly of general type" or "algebraically jet-hyperbolic". One would need e.g. to know the answer to

**Question**

Let $(\mathcal{X}, \mathcal{V}) \to S$ be a proper family of directed varieties over a base $S$, such that $\pi : \mathcal{X} \to S$ is a nonsingular deformation and the directed structure on $X_t = \pi^{-1}(t)$ is $V_t \subset T_{X_t}$, possibly singular. Under which conditions is

$$t \mapsto h^0(X_t, \mathcal{K}_{V_t}^{[m]})$$

locally constant over $S$ ?

This would be very useful since one can easily produce jet sections for hypersurfaces $X \subset \mathbb{P}^{n+1}$ admitting meromorphic connections with low pole order (Siu, Nadel).

# Proof of the non optimal Kobayashi conjecture (Brotbek)

Let $A \to Z$ be a very ample line bundle, and $X_\sigma$ the hypersurface associated with $\sigma \in H^0(Z, dA)$, $d \gg 1$. One looks at special sections

$$\sigma = \sum_{|I|=\delta} a_I \tau^{(p+k)I}, \quad a_I \in H^0(Z, \eta A),\ \tau_j \in H^0(Z, A),\ 1 \leq j \leq N$$

where the $\tau_j$ are generic and $d = \ell + (p+k)\delta$, $p \gg 1$. Let $\mathcal{X} \to S$ be the corresponding family of hypersurfaces, $\sigma \in S$, and let $\mathcal{X}_k \to S$ be the Semple construction relative to $\mathcal{X} \to S$.

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A construction similar to Nadel's meromorphic connections with low pole orders then produces certain Wronskian operators, and one shows that for generic $\sigma$, a certain fonctorial blow-up $\mu_k : \widehat{\mathcal{X}}_k \to \mathcal{X}_k$ of the $k$-th stage carries an ample invertible sheaf

$$L_k = \mu_k^*\big(\mathcal{O}_{\mathcal{X}_k}(a_1, a_2, \ldots, a_k) \otimes \mathcal{J}_k \otimes \pi_{k,0} A^{-1}\big) \ \text{ over } X_\sigma,$$

where $\mathcal{J}_k$ is the ideal sheaf associated with a suitable family of Wronkian operators. This is enough to prove the conjecture. ☐

# The end