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Variational approach for complex Monge-Ampère equations and geometric applications

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Abstract and goals

• Recent work by Berman, Berndtsson, Boucksom, Eyssidieux, Guedj, Jonsson, Zeriahi (among others) leads to a new variational approach for the solution of Monge-Ampère equations on compact Kähler manifolds.

• The method can be made independent of the previous PDE technicalities of Yau's approach.

• It is based on the study of certain functionals (Ding-Tian, Mabuchi) on the space of Kähler metrics, and their geodesic convexity due to X.X. Chen and Berman-Berndtsson in its full generality.

• Applications include the existence and uniqueness of Kähler-Einstein metrics on $\mathbb{Q}$-Fano varieties with log terminal singularities, and a new proof by Berman-Boucksom-Jonsson of a uniform version of the Yau-Tian-Donaldson conjecture solved around 2013 by Chen-Donaldson-Sun.

Kähler-Einstein metrics

To a Kähler metric on a compact complex n fold X

$$\omega = i \sum_{1\le j,k\le n} \omega_{jk}(z)\, dz_j \wedge d\overline{z}_k, \quad d\omega = 0$$

one associates its Ricci curvature form

$$\mathrm{Ricci}(\omega) = \Theta_{\Lambda^n T_X, \Lambda^n \omega} = -dd^c \log\det(\omega_{jk})$$

where $d^c = \frac{1}{4i\pi}(\partial - \overline{\partial})$, $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. The Kähler metric ω is said to be Kähler-Einstein if

$$(*) \qquad \mathrm{Ricci}(\omega) = \lambda\omega \quad \text{for some } \lambda \in \mathbb{R}.$$

This requires $\lambda\omega \in c_1(X)$, hence $(*)$ can be solved only when $c_1(X)$ is positive definite, negative definite or zero, and after rescaling ω by a constant, one can always assume that $\lambda \in \{0, 1, -1\}$.

Kähler-Einstein $\Longleftrightarrow$ Monge-Ampère equation (1)

Fix a reference Kähler metric ω_0 and put $\omega = \omega_0 + dd^c\varphi$. The KE condition $(*)$ is equivalent to

$$(**) \qquad (\omega_0 + dd^c\varphi)^n = e^{-\lambda\varphi + f}\omega_0^n.$$

• When $\lambda = -1$ and $c_1(X) < 0$, i.e. $c_1(K_X) > 0$, Aubin has shown in 1978 that there is always a unique solution, hence a unique Kähler metric $\omega \in c_1(K_X)$ such that

$$\mathrm{Ricci}(\omega) = -\omega.$$

This is a very natural generalization of the existence of constant curvature metrics on complex algebraic curves, implied by Poincaré's uniformization theorem in dimension 1.

Kähler-Einstein $\Longleftrightarrow$ Monge-Ampère equation (2)

• For $\lambda = 0$ and $c_1(X) = 0$, a celebrated result of Yau (solution of the Calabi conjecture, 1978) states that there exists a unique metric $\omega = \omega_0 + dd^c\varphi$ in the given cohomology class $\{\omega_0\}$ such that $\mathrm{Ricci}(\omega) = 0$.Moreover, the Monge-Ampère equation

$$(\omega_0 + dd^c\varphi)^n = e^f\omega_0^n$$

has a unique solution whenever $\int_X e^f\omega_0^n = \int_X \omega_0^n$.
Equivalently, the Ricci curvature form can be prescribed to be equal any given smooth closed $(1,1)$-form

$$\mathrm{Ricci}(\omega) = \rho,$$

provided that $\rho \in c_1(X)$.

The case of Fano manifolds

For $\lambda = +1$, the equation to solve is

$$(**) \qquad (\omega_0 + dd^c\varphi)^n = e^{-\varphi+f}\omega_0^n.$$

This is possible only if $-K_X$ $(= \Lambda^n T_X)$ is ample. One then says that X is a Fano manifold.
When solutions exist, it is known by Bando Mabuchi (1987) that they are unique up to the action of the identity component $\mathrm{Aut}^0(X)$ in the complex Lie group of biholomorphisms of X.

Berman-Boucksom-Jonsson 2015

Let X be a Fano manifold with finite automorphism group. Then X admits a Kähler-Einstein metric if and only if it is uniformly K-stable.

Recently, Chen, Donaldson and Sun got this result under the more general assumption that X is K-stable (Bourbaki/Ph. Eyssidieux, january 2015).

The case of log Fano varieties

Definition

A log Fano pair is a klt pair (X, Δ) such that X is projective and the $\mathbb{Q}$-divisor $A = -(K_X + \Delta)$ is ample.

Here X is a normal compact complex variety X and Δ an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. By Hironaka, there exists a log resolution $\pi : \tilde{X} \to X$ of (X, Δ), i.e. a modification of X over the complement of the singular loci of X and Δ, such that the pull-back of Δ and of $X_{\rm sing}$ consists of simple normal crossing (snc) divisors in $\tilde{X}$. One writes

$$\pi^*(K_X + \Delta) = K_{\tilde{X}} + E, \qquad E = \textstyle\sum_j a_j E_j$$

for some $\mathbb{Q}$-divisor E whose push-forward to X is Δ (since $X_{\rm sing}$ has codimension 2, the components E_j that lie over $X_{\rm sing}$ yield $\pi_* E_j = 0$). The coefficient $-a_j \in \mathbb{Q}$ is known as the discrepancy of (X, Δ) along E_j.

The klt condition ("Kawamata Log Terminal")

Definition

(X, Δ) is klt if $a_j < 1$ for all j.

Let r be a positive integer such that $r(K_X + \Delta)$ is Cartier, and σ a local generator of $\mathcal{O}(r(K_X + \Delta))$ on some open set $U \subset X$. Then the (n, n) form

$$|\sigma|^{2/r} := i^{n^2}\,\sigma^{1/r} \wedge \overline{\sigma^{1/r}}$$

is a volume form with poles along $S = \mathrm{Supp}\,\Delta \cup X_{\rm sing}$.
By the change of variable formula, the local integrability can be checked by pulling back σ to $\tilde{X}$, in which case it is easily seen that the integrability occurs if and only if $a_j < 1$ for all j, i.e. when (X, Δ) is klt.
In local coordinates

$$|\sigma|^{2/r} \sim \frac{\text{volume form}}{\prod |z_j|^{2a_j}}.$$

Singular Monge-Ampère equation

By definition (X, Δ) log Fano $\Longrightarrow c_1(X, \Delta) \ni \omega_0$ Kähler.
Every form $\omega = \omega_0 + dd^c\psi \in \{\omega_0\}$ can be seen as the curvature form of a smooth hermitian metric h on $\mathcal{O}(-(K_X + \Delta))$, whose weight is $\phi = u_0 + \psi$ where u_0 is a local potential of ω_0, hence

$$\omega = \omega_0 + dd^c\psi = dd^c\phi$$

where ϕ is understood as the weight of a global metric formally denoted $h = e^{-\phi}$ on the $\mathbb{Q}$-line bundle $\mathcal{O}(-(K_X + \Delta))$.
The inverse e^{ϕ} is a hermitian metric on $\mathcal{O}(K_X + \Delta)$. If σ is a local generator of $\mathcal{O}(r(K_X + \Delta))$, the product $|\sigma|^{2/r}e^{\phi} = e^{\psi+u_0}$ is (locally) a smooth positive function whenever φ is smooth, hence

$$e^{-\phi} = |\sigma|^{2/r}e^{-(\psi+u_0)}$$

is an integrable volume form on X with poles along $S := \mathrm{Supp}\,\Delta \cup \{\text{singularities}\}$. The KE condition can be rewritten

$$(dd^c\phi)^n = c\,e^{-\phi} \quad \text{on } X \smallsetminus S \Leftrightarrow \mathrm{Ricci}(\omega) = \omega + [\Delta].$$

The space of Kähler metrics

Let $A \in H^{1,1}_{\partial\overline{\partial}}(X, \mathbb{R})$ be a Kähler $\partial\overline{\partial}$-cohomology class, and let

$$\omega_0 = \alpha + dd^c\psi_0 = dd^c\phi_0 \in A$$

be a Kähler metric.
Here we are mostly interested in the Fano case $A = -K_X$ and the log Fano case $A = -(K_X + \Delta)$. Let $V = \int_X \omega_0^n = A^n$ be the volume of ω_0.

Definition

The space $\mathcal{K}_A$ of Kähler metrics (resp. $\mathcal{P}_A$ of Kähler potentials) is the set of Kähler metrics ω (resp. functions ψ) such that

$$\omega = \omega_0 + dd^c\psi > 0.$$

Here $\phi = u_0 + \psi$ is thought intrinsically as a hermitian metric $h = e^{-\phi}$ on A with strictly plurisubharmonic (psh) weight ϕ.

Clearly $\mathcal{K}_A \simeq \mathcal{P}_A/\mathbb{R}$.

The Riemannian structure on $\mathcal{P}_A$

The basic operator of interest on $\mathcal{P}_A$ is the Monge-Ampère operator

$$\mathcal{P}_A \to \mathcal{M}_+, \qquad \mathrm{MA}(\phi) = (dd^c\phi)^n = (\omega_0 + dd^c\psi)^n$$

According to Mabuchi the space $\mathcal{P}_A$ can be seen as some sort of infinite dimensional Riemannian manifold: a "tangent vector" to $\mathcal{P}_A$ is an infinitesimal variation $\delta\phi \in C^\infty(X,\mathbb{R})$ of ϕ (or ψ), and the infinitesimal Riemannian metric at a point $h = e^{-\phi}$ is given by

$$\|\delta\phi\|_2^2 = \frac{1}{V}\int_X (\delta\phi)^2\,\mathrm{MA}(\phi).$$

X.X. Chen and his collaborators have studied the metric and geometric properties of the space $\mathcal{P}_A$, showing in particular that it is a path metric space (a non trivial assertion in this infinite dimensional setting) of nonpositive curvature in the sense of Alexandrov. A key step has been to produce almost $C^{1,1}$-geodesics which minimize the geodesic distance.

Basic functionals (1)

Given $\phi_0, \phi \in \mathcal{P}_A$, one defines:

- The Monge-Ampère functional

$$E_{\phi_0}(\phi) = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X (\phi-\phi_0)(dd^c\phi)^j \wedge (dd^c\phi_0)^{n-j}$$

$$(***) \qquad = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X \psi(\omega_0+dd^c\psi)^j \wedge \omega_0^{n-j}, \quad \psi = \phi-\phi_0.$$

It is a primitive of the Monge-Ampère operator in the sense that $dE_{\phi_0}(\phi) = \frac{1}{V}\,\mathrm{MA}(\phi)$, i.e. for any path $[T,T'] \ni t \mapsto \phi_t$, one has

$$\frac{d}{dt}E_{\phi_0}(\phi_t) = \frac{1}{V}\int_X \dot\phi_t\,\mathrm{MA}(\phi_t) \quad \text{where } \dot\phi_t = \frac{d}{dt}\phi_t.$$

This is easily checked by a differentiation under the integral sign.

Basic functionals (2)

As a consequence E satisfies the cocycle relation

$$E_{\phi_0}(\phi_1) + E_{\phi_1}(\phi_2) = E_{\phi_0}(\phi_2),$$

so its dependence on ϕ_0 is only up to a constant.
Finally, if ϕ_t depends linearly on t, one has $\ddot{\phi}_t = \frac{d^2}{dt^2}\phi_t = 0$ and a further differentiation of $(***)$ yields

$$\frac{d^2}{dt^2} E_{\phi_0}(\phi_t) = \frac{n}{V}\int_X \dot{\phi}_t \, dd^c\dot{\phi}_t \wedge (dd^c\phi_t)^{n-1}$$
$$= -\frac{n}{V}\int_X d\dot{\phi}_t \wedge d^c\dot{\phi}_t \wedge (dd^c\phi_t)^{n-1} \leq 0.$$

It follows that E_{ϕ_0} is concave on $\mathcal{P}_A$.

The J and J^* functionals

- The concavity of E implies the nonnegativity of $J_{\phi_0}(\phi) := dE_{\phi_0}(\phi_0)\cdot(\phi-\phi_0) - E_{\phi_0}(\phi)$, This quantity is called the Aubin J-energy functional

$$J_{\phi_0}(\phi) = V^{-1}\int_X (\phi-\phi_0)(dd^c\phi_0)^n - E_{\phi_0}(\phi) \geq 0.$$

- By exchanging the roles of ϕ, ϕ_0 and putting $J^*_{\phi_0}(\phi) = J_\phi(\phi_0) \geq 0$, the cocycle relation for E yields $E_\phi(-\phi_0) = -E_{\phi_0}(\phi)$. The transposed J-energy functional is

$$J^*_{\phi_0}(\phi) := E_{\phi_0}(\phi) - V^{-1}\int_X (\phi-\phi_0)(dd^c\phi)^n$$
$$= E_{\phi_0}(\phi) - V^{-1}\int_X \psi(\omega_0 + dd^c\psi)^n \geq 0, \quad \psi = \phi - \phi_0.$$

The symmetric I functional

• The I-functional is the symmetric functional defined by

$$I_{\phi_0}(\phi) = I_\phi(\phi_0) := -\frac{1}{V}\int_X (\phi-\phi_0)\big(\mathrm{MA}(\phi)-\mathrm{MA}(\phi_0)\big)$$
$$= \sum_{j=0}^{n-1} V^{-1}\int_X d(\phi-\phi_0)\wedge d^c(\phi-\phi_0)\wedge(dd^c\phi)^j\wedge(dd^c\phi_0)^{n-1-j} \geq 0.$$

In fact $I_{\phi_0}(\phi) = J_{\phi_0}(\phi) + J^*_{\phi_0}(\phi)$, and one can also write

$$I_{\phi_0}(\phi) = V^{-1}\left(\int_X \psi\,\omega_0^n - \int_X \psi(\omega_0+dd^c\psi)^n\right).$$

It satisfies the quasi-triangle inequality: $\exists c_n > 0$ s.t.

$$I_{\phi_0}(\phi) \leq c_n\big(I_{\phi_0}(\phi_1) + I_{\phi_1}(\phi)\big). \quad \forall\ \phi_0,\ \phi_1,\ \phi \in \mathcal{P}_A.$$

The Ding and Mabuchi functionals (1)

• In the Fano or log Fano setting, the Ding functional is defined by

$$D_{\phi_0} = L - L(\phi_0) - E_{\phi_0}, \quad \text{where} \quad L(\phi) = -\log\int_X e^{-\phi}.$$

Recall: $e^{-\phi}$ is integrable by the klt condition.

• Given probability measures μ,ν on a space X, the relative entropy $\mathrm{Entr}_\mu(\nu) \in [0,+\infty]$ of ν with respect to μ is defined as the integral

$$\mathrm{Entr}_\mu(\nu) := \int_X \log\left(\frac{d\nu}{d\mu}\right) d\nu,$$

if ν is absolutely continuous w.r.t. μ; $\mathrm{Entr}_\mu(\nu) = +\infty$ otherwise.
Pinsker inequality: for all proba measures μ,ν one has

$$\mathrm{Entr}_\mu(\nu) \geq \frac{1}{2}\|\mu-\nu\|^2 \geq 0.$$

In particular, $\mu = \nu \iff \mathrm{Entr}_\mu(\nu) = 0$.

The Ding and Mabuchi functionals (2)

In the Fano or log Fano situation, the entropy functional $H_{\phi_0}(\phi)$ is defined to be the entropy of the probability measure $\frac{1}{V}(dd^c\phi)^n$ with respect to $e^{L(\phi_0)}e^{-\phi_0}$, namely

$$H_{\phi_0}(\phi) = \int_X \log\left(\frac{(dd^c\phi)^n/V}{e^{L(\phi_0)}e^{-\phi_0}}\right)\frac{(dd^c\phi)^n}{V} \geq 0.$$

- The Mabuchi functional is then defined by

$$M_{\phi_0} = H_{\phi_0} - J^*_{\phi_0}.$$

One gets the more explicit expression

$$M_{\phi_0}(\phi) = \int_X \log\left(\frac{e^{\phi}(dd^c\phi)^n}{V}\right)\frac{(dd^c\phi)^n}{V} - E_{\phi_0}(\phi) - L(\phi_0).$$

Comparison properties

Observation

If c is a constant, then

$$E_{\phi_0}(\phi + c) = E_{\phi_0}(\phi) + c \quad \text{and} \quad L(\phi + c) = L(\phi) + c.$$

On the other hand, the functionals $I_{\phi_0}, J_{\phi_0}, J^*_{\phi_0}, D_{\phi_0}, H_{\phi_0}, M_{\phi_0}$ are invariant by $\phi \mapsto \phi + c$ and therefore descend to the quotient space $\mathcal{K}_A = \mathcal{P}_A/\mathbb{R}$ of Kähler metrics $\omega = dd^c\phi \in A$.

Comparison between I, J, J^*

The functionals I, J, J^* are essentially growth equivalent:

$$\frac{1}{n}J_{\phi}(\phi_0) \leq J_{\phi_0}(\phi) \leq \frac{n+1}{n}J_{\phi_0}(\phi) \leq I_{\phi_0}(\phi) \leq (n+1)J_{\phi_0}(\phi).$$

Comparison between Ding and Mabuchi functionals

Proposition

Let (X, Δ) be a log Fano manifold. Then $M_{\phi_0}(\phi) \geq D_{\phi_0}(\phi)$ and, in case of equality, ϕ must be Kähler-Einstein.

Proof. From the definitions one gets

$$M - D = (H - J^*)(L - L(\phi_0) - E),$$

$$E_{\phi_0}(\phi) - J^*_{\phi_0}(\phi) = V^{-1} \int_X (\phi - \phi_0)(dd^c\phi)^n,$$

$$M_{\phi_0}(\phi) - D_{\phi_0}(\phi)$$

$$= \int_X \left(\log \left(\frac{(dd^c\phi)^n/V}{e^{L(\phi_0)}e^{-\phi_0}} \right) + (\phi - \phi_0) \right) \frac{(dd^c\phi)^n}{V} + L(\phi_0) - L(\phi)$$

$$= \int_X \log \left(\frac{(dd^c\phi)^n/V}{e^{L(\phi)}e^{-\phi}} \right) \frac{(dd^c\phi)^n}{V} \geq 0.$$

In case of equality, Pinsker implies KE condition: $\frac{(dd^c\phi)^n}{V} = e^{L(\phi)}e^{-\phi}$

Non pluripolar products

- Bedford-Taylor Monge-Ampère products : for $u_j \in L^\infty_{\mathrm{loc}}$, one sets inductively

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k := dd^c(u_1\, dd^c u_2 \wedge \ldots \wedge dd^c u_k)$$

- Non pluripolar products (Guedj-Zeriahi)

Let $\mathcal{P}(X, \omega_0)$ be the set of ω_0-psh potentials, i.e. $\phi = \phi_0 + \psi$ such that $dd^c\phi = \omega_0 + dd^c\psi \geq 0$.

The functions $\psi_\nu := \max\{\psi, -\nu\}$ are again ω_0-psh and bounded for all $\nu \in \mathbb{N}$. The Monge-Ampère measures $(\omega_0 + dd^c\psi_\nu)^n$ are therefore well-defined in the sense of Bedford-Taylor, and one defines for any bidegree (p, p) a positive current

$$T = \langle(\omega_1 + dd^c\psi_1) \wedge \ldots \wedge (\omega_p + dd^c\psi_p)\rangle = \lim_{\nu\to+\infty}$$

$$\mathbf{1}_{\bigcap\{\psi_j > -\nu\}}(\omega_1 + dd^c \max\{\psi_1, -\nu\}) \wedge \ldots \wedge (\omega_p + dd^c \max\{\psi_p, -\nu\})$$

Basic fact: T is still closed [Proof uses ideas of Skoda & Sibony].

Space of potentials of finite energy

One introduces for any $p \in [1, +\infty[$ the space

$$\mathcal{E}^p(X, \omega_0) := \left\{\phi = \phi_0 + \psi\,;\ \int_X |\psi|^p\,\mathrm{MA}(\omega_0 + dd^c\psi) < +\infty\right\},$$

and $\int_X \mathrm{MA}(\omega_0 + dd^c\psi) = \int_X \omega_0^n$ ("full non pluripolar mass"). One says that functions $\psi \in \mathcal{E}^p(X, \omega_0)$ have *finite $\mathcal{E}^p$-energy*. One also denotes by

$$\mathcal{T}^p(X, \omega_0) \subset \mathcal{T}^p_{\mathrm{full}}(X, \omega_0)$$

the corresponding set of *currents with finite $\mathcal{E}^p$-energy*, which can be identified with the quotient space

$$\mathcal{T}^p(X, \omega_0) = \mathcal{E}^p(X, \omega_0)/\mathbb{R} \quad \text{via} \quad \phi \mapsto dd^c\phi = \omega_0 + dd^c\psi.$$

It is important to note that $\mathcal{T}^p(X, \omega_0)$ is not a closed subset of $\mathcal{T}(X, \omega_0)$ for the weak topology.

Finite energy extension of the functionals

Finite energy extension of the functionals

All functionals E, L, I, J, J^*, D, H, M have a natural extension to arguments $\phi, \phi_0 \in \mathcal{E}^1(X, \omega_0)$, and I, J, J^*, D, H, M descend to $\mathcal{T}^1(X, \omega_0) = \mathcal{E}^1(X, \omega_0)/\mathbb{R}$.

Theorem (BBGZ)

The map $T = \omega_0 + dd^c\psi \mapsto V^{-1}\langle T^n\rangle$ is a bijection between $\mathcal{T}^1(X, \omega_0)$ and the space of probability measures $\mathcal{M}^1(X, \omega_0)$ of finite energy.

Here one uses the Legendre-Fenchel transform

$$E_0^*(\mu) := \sup_{\phi = \phi_0 + \psi \in \mathcal{E}^1(X, \omega_0)} \left(E_0(\psi) - \int_X \psi\,\mu\right) \in [0, +\infty]$$

where $E_0(\psi) = E_{\phi_0}(\phi_0 + \psi)$, and μ has finite energy if $E_0^*(\mu) < +\infty$.

Sufficient conditions for existence of KE metrics

Theorem (BBEGZ)

For a current $\omega = dd^c\phi \in \mathcal{T}^1(X, A)$, the following conditions are equivalent.

(i) ω is a Kähler-Einstein metric for (X, Δ).

(ii) The Ding functional reaches its infimum at ϕ : $D_{\phi_0}(\phi) = \inf_{\mathcal{E}^1(X,A)/\mathbb{R}} D_{\phi_0}$.

(iii) The Mabuchi functional reaches its infimum at ϕ : $M_{\phi_0}(\phi) = \inf_{\mathcal{E}^1(X,A)/\mathbb{R}} M_{\phi_0}$.

Corollary (BBEGZ)

Let X be a $\mathbb{Q}$-Fano variety with log terminal singularities.

(i) The identity component $\mathrm{Aut}^0(X)$ of the automorphism group of X acts transitively on the set of KE metrics on X,

(ii) If the Mabuchi functional of X is proper, then $\mathrm{Aut}^0(X) = \{1\}$ and X admits a unique Kähler-Einstein metric.

Test configurations

Definition

A test configuration $(\mathcal{X}, \mathcal{A})$ for a ($\mathbb{Q}$-)polarized projective variety (X, A) consists of the following data :

(i) a flat and proper morphism $\pi : \mathcal{X} \to \mathbb{C}$ of algebraic varieties; one denotes by $X_t = \pi^{-1}(t)$ the fiber over $t \in \mathbb{C}$.

(ii) a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{C}$;

(iii) an isomorphism $X_1 \simeq X$.

(iv) a $\mathbb{C}^*$-linearized ample line bundle $\mathcal{A}$ on $\mathcal{X}$; one puts $A_t = \mathcal{A}_{|X_t}$.

(v) an isomorphism $(X_1, A_1) \simeq (X, A)$ extending the one in (iii).

K stability (and uniform K-stability) is defined in termes of certain numerical invariants attached to arbitrary test configurations.

Donaldson-Futaki invariants

Donaldson-Futaki invariant

Let $N_m = h^0(X, mA)$ and $w_m \in \mathbb{Z}$ be the weight of the $\mathbb{C}^*$-action on the determinant $\det H^0(\mathcal{X}_0, m\mathcal{A}_0)$. Then there is an asymptotic expansion

$$\frac{w_m}{mN_m} = F_0 + m^{-1}F_1 + m^{-2}F_2 + \ldots\ .$$

and one defines $\mathrm{DF}(\mathcal{X}, \mathcal{A}) := -2F_1$.

Definition

The polarized variety (X, A) is said K-stable if $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq 0$ for all normal test configurations, with equality iff $(\mathcal{X}, \mathcal{A})$ is trivial.

Generalized Yau-Tian-Donaldson conjecture

Let (X, A) be a polarized variety. Then X admits a cscK metric (short hand for Kähler metric with constant scalar curvature) $\omega \in c_1(A)$ if and only if (X, A) is K-stable.

Uniform K-stability

The Duistermaat-Heckman measure $\mathrm{DH}_{(X,A)}$ is the proba distribution measure of the $\mathbb{C}^*$-action weights:

$$\mathrm{DH}_{(X,A)} = \lim_{m\to\infty} \sum_{\lambda\in\mathbb{Z}} \frac{\dim H^0(X, mA)_\lambda}{\dim H^0(X, mA)}\, \delta_{\lambda/m}, \quad \delta_p := \text{Dirac at } p,$$

where $H^0(X, mA) = \bigoplus_{\lambda\in\mathbb{Z}} H^0(X, mA)_\lambda$ is the weight space decomposition. For each $p \in [1, \infty]$, the L^p-norm $\|(\mathcal{X}, \mathcal{A})\|_p$ of an ample test configuration $(\mathcal{X}, \mathcal{A})$ is defined as the L^p norm

$$\|(\mathcal{X}, \mathcal{A})\|_p = \left(\int_{\mathbb{R}} |\lambda - b(\mu)|^p\, d\mu(\lambda)\right)^{1/p}, \quad b(\mu) = \int_{\mathbb{R}} \lambda\, d\mu(\lambda).$$

Definition (Székelyhidi)

The polarized variety (X, A) is said to be L^p-uniformly K-stable if there exists $\delta > 0$ such that $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq \delta\, \|(\mathcal{X}, \mathcal{A})\|_p$ for all normal test configurations. [Note: only possible if $p < \frac{n}{n-1)}$.]

Sufficiency of uniform K-stability

Berman-Boucksom-Jonsson 2015

Let X be a Fano manifold with finite automorphism group. Then X admits a Kähler-Einstein metric if and only if it is uniformly K-stable (in a related and simpler "non archimedean" sense).

Let $A = -K_X$. A ray $(\phi_t)_{t\geq 0}$ in $\mathcal{P}_A$ corresponds to an S^1-invariant metric Φ on the pull-back of $-K_X$ to the product of X with the punctured unit disc $\mathbb{D}^*$. The ray is called subgeodesic when Φ is plurisubharmonic (psh for short). Denoting by F any of the functionals M, D or J, there is a limit

$$\lim_{t\to+\infty} \frac{F(\phi_t)}{t} = F^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})$$

Here F^{NA} can be seen as the corresponding "non-Archimedean" functional.