# Recent progress in the study of hyperbolic algebraic varieties

Jean-Pierre Demailly

Institut Fourier, Université de Grenoble I, France

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- $X = \overline{\mathbb{C}} \smallsetminus \{0, 1, \infty\} = \mathbb{C} \smallsetminus \{0, 1\}$ has no entire curves (Picard's theorem)
- A complex torus $X = \mathbb{C}^n/\Lambda$ ($\Lambda$ lattice) has a lot of entire curves. As $\mathbb{C}$ simply connected, every $f : \mathbb{C} \to X = \mathbb{C}^n/\Lambda$ lifts as $\tilde{f} : \mathbb{C} \to \mathbb{C}^n$,

$$\tilde{f}(t) = (\tilde{f}_1(t), \ldots, \tilde{f}_n(t))$$

and $\tilde{f}_j : \mathbb{C} \to \mathbb{C}$ can be arbitrary entire functions.

# Projective algebraic varieties

- Consider now the complex projective $n$-space

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- An entire curve $f : \mathbb{C} \to \mathbb{P}^n$ is given by a map

$$t \longmapsto [f_0(t) : f_1(t) : \ldots : f_n(t)]$$

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- More generally, look at a (complex) projective manifold, i.e.

$$X^n \subset \mathbb{P}^N, \qquad X = \{[z]\,;\ P_1(z) = ... = P_k(z) = 0\}$$

where $P_j(z) = P_j(z_0, z_1, \ldots, z_N)$ are homogeneous polynomials (of some degree $d_j$), such that $X$ is non singular.

# Kobayashi metric / hyperbolic manifolds

- For a complex manifold, $n = \dim_{\mathbb{C}} X$, one defines the Kobayashi pseudo-metric : $x \in X$, $\xi \in T_X$

  $$\kappa_x(\xi) = \inf\{\lambda > 0\,;\ \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f_*(0) = \xi\}$$

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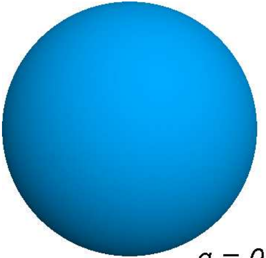
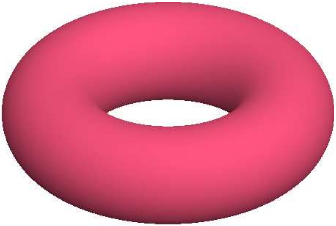
- **Theorem.** (Brody) If $X$ is compact then $X$ is Kobayashi hyperbolic if and only if there are no entire holomorphic curves $f : \mathbb{C} \to X$ (Brody hyperbolicity).
- Hyperbolic varieties are especially interesting for their expected diophantine properties :

  **Conjecture** (S. Lang) *If a projective variety $X$ defined over $\mathbb{Q}$ is hyperbolic, then $X(\mathbb{Q})$ is finite.*

# Complex curves ($n = 1$) : genus and curvature



$g = 0\,,\ K_X < 0$
(positive curvature)



$g = 1\,,\ K_X = 0$
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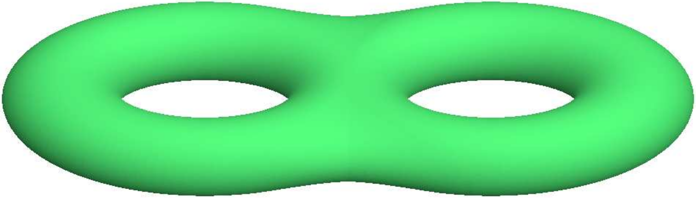

$g > 1\,,\ K_X > 0$
(negative curvature)

$K_X = \Lambda^n T_X^*, \quad \deg(K_X) = 2g - 2$

# Curves : hyperbolicity and curvature

- Case $n = 1$ (compact Riemann surfaces):

$$\begin{array}{ll} X = \mathbb{P}^1 & (g = 0, \quad T_X > 0) \\ X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) & (g = 1, \quad T_X = 0) \end{array}$$

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- If $g \geq 2$, $\quad X \simeq \mathbb{D}/\Gamma \quad (T_X < 0)$, then $X$ hyperbolic.
- **The *n*-dimensional case** (Kobayashi)
  *If $T_X$ is negatively curved ($T_X^* > 0$, i.e. ample), then $X$ is hyperbolic.*
  Recall that a holomorphic vector bundle $E$ is ample iff its symmetric powers $S^m E$ have global sections which generate 1-jets of (germs of) sections at any point $x \in X$.
- **Examples :** $X = \Omega/\Gamma$, $\Omega$ bounded symmetric domain.

# Varieties of general type

- **Definition** *A non singular projective variety $X$ is said to be of general type if the growth of pluricanonical sections*

$$\dim H^0(X, K_X^{\otimes m}) \sim cm^n, \qquad K_X = \Lambda^n T_X^*$$

*is maximal.*

(sections locally of the form $f(z)\,(dz_1 \wedge \ldots \wedge dz_n)^{\otimes m}$)

**Example**: A non singular hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree $d$ satisfies $K_X = \mathcal{O}(d-n-2)$,
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- **Conjecture GT.** *If a compact manifold $X$ is hyperbolic, then it should be of general type, and even better $K_X = \Lambda^n T_X^*$ should be of positive curvature (i.e. $K_X$ is ample, or equivalently $\exists$ Kähler metric $\omega$ such that Ricci$(\omega) < 0$).*

# Conjectural characterizations of hyperbolicity

- **Theorem.** *Let $X$ be projective algebraic. Consider the following properties :*
  $(\mathrm{P1})$ *$X$ is hyperbolic*
  $(\mathrm{P2})$ *Every subvariety $Y$ of $X$ is of general type.*
  $(\mathrm{P3})$ $\exists\varepsilon > 0$, $\forall C \subset X$ *algebraic curve*
  $$2g(\bar{C}) - 2 \geq \varepsilon \deg(C).$$
  ($X$ *"algebraically hyperbolic"*)
  $(\mathrm{P4})$ *$X$ possesses a jet-metric with negative curvature on its $k$-jet bundle $X_k$ [to be defined later], for $k \geq k_0 \gg 1$.*
  *Then* $(\mathrm{P4}) \Rightarrow (\mathrm{P1}),\ (\mathrm{P2}),\ (\mathrm{P3}),$
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  $(\mathrm{P1}) \Rightarrow (\mathrm{P3}),$
  *and if Conjecture GT holds,* $(\mathrm{P1}) \Rightarrow (\mathrm{P2})$.
- It is expected that all 4 properties (P1), (P2), (P3), (P4) are equivalent for projective varieties.

# Green-Griffiths-Lang conjecture

- **Conjecture** (Green-Griffiths-Lang = GGL) *Let $X$ be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.*

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- Combining the above conjectures, we get :
  **Expected consequence** (of GT + GGL)
  (P1) *$X$ is hyperbolic*
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  **Expected consequence** (of GT + GGL)
  (P1) *$X$ is hyperbolic*
  (P2) *Every subvariety $Y$ of $X$ is of general type are equivalent.*
- The main idea in order to attack GGL is to use differential equations. Let

$$\mathbb{C} \to X, \quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$$

be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on $X$.

# Definition of algebraic differential operators

- Consider algebraic differential operators which can be written locally in multi-index notation

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k} \end{aligned}$$

where $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are holomorphic coefficients on $X$ and $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its $k$-jet.

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\lambda \cdot f(t) = f(\lambda t), \quad (\lambda \cdot f)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)
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$\Rightarrow$ weighted degree $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$.

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- **Definition.** $E_{k,m}^{\mathrm{GG}}$ *is the sheaf (bundle) of algebraic differential operators of order $k$ and weighted degree $m$.*

# Vanishing theorem for differential operators

- **Fundamental vanishing theorem**
  ([Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]
  *Let* $P \in H^0(X, E^{\mathrm{GG}}_{k,m} \otimes \mathcal{O}(-A))$ *be a global algebraic differential operator whose coefficients vanish on some ample divisor* $A$*. Then for any* $f : \mathbb{C} \to X$, $P(f_{[k]}) \equiv 0$.

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- *Proof*. One can assume that $A$ is very ample and intersects $f(\mathbb{C})$. Also assume $f'$ bounded (this is not so restrictive by Brody !). Then all $f^{(k)}$ are bounded by Cauchy inequality. Hence

$$\mathbb{C} \ni t \mapsto P(f', f'', \ldots, f^{(k)})(t)$$

is a bounded holomorphic function on $\mathbb{C}$ which vanishes at some point. Apply Liouville's theorem ! $\square$

# Geometric interpretation of vanishing theorem

- Let $X_k^{\mathrm{GG}} = J_k(X)^*/\mathbb{C}^*$ be the projectivized $k$-jet bundle of $X$ = quotient of non constant $k$-jets by $\mathbb{C}^*$-action. Fibers are weighted projective spaces.
  **Observation.** *If $\pi_k : X_k^{\mathrm{GG}} \to X$ is canonical projection and $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is the tautological line bundle, then*

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  $$E_{k,m}^{\mathrm{GG}} = (\pi_k)_* \mathcal{O}_{X_k^{\mathrm{GG}}}(m)$$
- Saying that $f : \mathbb{C} \to X$ satisfies the differential equation $P(f_{[k]}) = 0$ means that
  $$f_{[k]}(\mathbb{C}) \subset Z_P$$
  where $Z_P$ is the zero divisor of the section
  $$\sigma_P \in H^0(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* \mathcal{O}(-A))$$
  associated with $P$.

- **Consequence of fundamental vanishing theorem.** *If* $P_j \in H^0(X, E^{\mathrm{GG}}_{k,m} \otimes \mathcal{O}(-A))$ *is a basis of sections then the image* $f(\mathbb{C})$ *lies in* $Y = \pi_k(\bigcap Z_{P_j})$*, hence property asserted by the GGL conjecture holds true if there are "enough independent differential equations" so that*

$$Y = \pi_k\Big(\bigcap_j Z_{P_j}\Big) \subsetneq X.$$

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- However, some differential equations are useless. On a surface with coordinates $(z_1, z_2)$, a Wronskian equation $f_1' f_2'' - f_2' f_1'' = 0$ tells us that $f(\mathbb{C})$ sits on a line, but $f_2''(t) = 0$ says that the second component is linear affine in time, an essentially meaningless information which is lost by a change of parameter $t \mapsto \varphi(t)$.

# Invariant differential operators

- The $k$-th order Wronskian operator

$$W_k(f) = f' \wedge f'' \wedge \ldots \wedge f^{(k)}$$

(locally defined in coordinates) has degree $m = \frac{k(k+1)}{2}$ and

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- **Definition.** *A differential operator $P$ of order $k$ and degree $m$ is said to be invariant by reparametrization if*

$$P(f \circ \varphi) = \varphi'^m P(f) \circ \varphi$$

*for any parameter change $t \mapsto \varphi(t)$. Consider their set*

$$E_{k,m} \subset E^{\mathrm{GG}}_{k,m} \quad \text{(\textit{a subbundle})}$$

(Any polynomial $Q(W_1, W_2, \ldots W_k)$ is invariant, but for $k \geq 3$ there are other invariant operators.)

# Category of directed manifolds

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  - Objects : pairs $(X, V)$, $X$ manifold$/\mathbb{C}$ and $V \subset \mathcal{O}(T_X)$
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  - "Absolute case" $(X, T_X)$
  - "Relative case" $(X, T_{X/S})$ where $X \to S$
  - "Integrable case" when $[V, V] \subset V$ (foliations)

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  - "Integrable case" when $[V, V] \subset V$ (foliations)
- **Foncteur "1-jet" :** $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :

$$\tilde{X} = P(V) = \text{bundle of projective spaces of lines in } V$$
$$\pi : \tilde{X} = P(V) \to X, \quad (x, [v]) \mapsto x, \quad v \in V_x$$
$$\tilde{V}_{(x,[v])} = \left\{\xi \in T_{\tilde{X},(x,[v])} \,;\, \pi_*\xi \in \mathbb{C}v \subset T_{X,x}\right\}$$

# Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to $V$

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

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- **Definition.** *Semple jet bundles :*
  – $(X_k, V_k) = k$-th iteration of fonctor $(X, V) \mapsto (\tilde{X}, \tilde{V})$
  – $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized $k$-jet of $f$.

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  – $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the projectivized $k$-jet of $f$.

- **Basic exact sequences**

$$0 \to T_{\tilde{X}/X} \to \tilde{V} \xrightarrow{\pi_\star} \mathcal{O}_{\tilde{X}}(-1) \to 0 \qquad \Rightarrow \text{rk}\, \tilde{V} = r = \text{rk}\, V$$

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# Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to $V$

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$
$$f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde{X}, \tilde{V}) \quad \text{(projectivized 1st-jet)}$$

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# Direct image formula

- For $n = \dim X$ and $r = \mathrm{rk}\, V$, get a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

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$$X_k^{\mathrm{GG,reg}}/G_k = J_k^{\mathrm{GG,reg}}/G_k$$
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- **Direct image formula.** $(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = E_{k,m} V^* =$ *invariant algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.*

# Results obtained so far

- Using this technology and deep results of McQuillan for curve foliations on surfaces, D. – El Goul proved in 1998
  **Theorem.** (solution of Kobayashi conjecture)
  *A very generic surface* $X \subset \mathbb{P}^3$ *of degree* $\geq 21$ *is hyperbolic.* (McQuillan got independently degree $\geq 35$).
- $\dim_{\mathbb{C}} X = n$. (S. Diverio, J. Merker, E. Rousseau [DMR09])
  If $X \subset \mathbb{P}^{n+1}$ is a generic $n$-fold of degree $d \geq d_n := 2^{n^5}$, then $\exists Y \subsetneq X$ s.t. every non constant $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$.
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  [also $d_3 = 593$, $d_4 = 3203$, $d_5 = 35355$, $d_6 = 172925$.]
- **Additional result.** (S. Diverio, S. Trapani, 2009)
  One can get $\mathrm{codim}_{\mathbb{C}}\, Y \geq 2$ and therefore a generic hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 593$ is hyperbolic.

# Algebraic structure of differential rings

- Although very interesting, results are currently limited by lack of knowledge on jet bundles and differential operators
- **Unknown !** *Is the ring of germs of invariant differential operators on* $(\mathbb{C}^n, T_{\mathbb{C}^n})$ *at the origin*
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- At least this is OK for $\forall n$, $k \le 2$ and $n = 2$, $k \le 4$:

$$\mathcal{A}_{1,n} = \mathcal{O}[f'_1, \ldots, f'_n]$$
$$\mathcal{A}_{2,n} = \mathcal{O}[f'_1, \ldots, f'_n, W^{[ij]}], \quad W^{[ij]} = f'_i f''_j - f'_j f''_i$$
$$\mathcal{A}_{3,2} = \mathcal{O}[f'_1, f'_2, W_1, W_2][W]^2, \quad W_i = f'_i DW - 3f''_i W$$
$$\mathcal{A}_{4,2} = \mathcal{O}[f'_1, f'_2, W_{11}, W_{22}, S][W]^6, \quad W_{ii} = f'_i DW_i - 5f''_i W_i$$

where $W = f'_1 f''_2 - f'_2 f''_1$ is 2-dim Wronskian and $S = (W_1 DW_2 - W_2 DW_1)/W$. Also known:

$\mathcal{A}_{3,3}$ (E. Rousseau [Rou06a]), $\mathcal{A}_{5,2}$ (J. Merker, [Mer08])

## Strategy : evaluate growth of differential operators

- The strategy of the proofs is that the algebraic structure of $\mathcal{A}_{k,n}$ allows to compute the Euler characteristic $\chi(X, E_{k,m} \otimes A^{-1})$, e.g. on surfaces

$$\chi(X, E_{k,m} \otimes A^{-1}) = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3).$$

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- Therefore many global differential operators exist for surfaces with $13c_1^2 - 9c_2 > 0$, e.g. surfaces of degree large enough in $\mathbb{P}^3$, $d \geq 15$ (end of proof uses stability)

# Trouble / more general perspectives

- Trouble is, in higher dimensions $n$, intermediate cohomology groups $H^q(X, E_{k,m}T^*_X)$, $0 < q < n$, don't vanish !!

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  *If $(X, V)$ is directed manifold of general type, i.e.* $\det V^*$ *big, then $\exists Y \subsetneq X$ such that every non-constant $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ is contained in $Y$.*

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- **Strategy.** OK by Ahlfors-Schwarz lemma if $r = \mathrm{rk}\, V = 1$. First try to get differential equations $f_{[k]}(\mathbb{C}) \subset Z \subsetneq X_k$. Take minimal such $k$. If $k = 0$, we are done! Otherwise $k \geq 1$ and $\pi_{k,k-1}(Z) = X_{k-1}$, thus $W = V_k \cap T_Z$ has rank $< \mathrm{rk}\, V_k = r$ and should have again $\det W^*$ big (unless some degeneration occurs ?). Use induction on $r$ !

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- **Needed induction step.** *If $(X, V)$ has $\det V^*$ big and $Z \subset X_k$ irreducible with $\pi_{k,k-1}(Z) = X_{k-1}$, then $(Z, W)$, $W = V_k \cap T_Z$ has $\mathcal{O}_{Z_\ell}(1)$ big on $(Z_\ell, W_\ell)$, $\ell \gg 0$.*

# Use holomorphic Morse inequalities !

- **Simple case of Morse inequalities**
  (Demailly, Siu, Catanese, Trapani)
  *If* $L = \mathcal{O}(A - B)$ *is a difference of big nef divisors* $A$, $B$, *then* $L$ *is big as soon as*
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- My PhD student S. Diverio has recently worked out this strategy for hypersurfaces $X \subset \mathbb{P}^{n+1}$, with

$$L = \bigotimes_{1 \le j < k} \pi_{k,j}^* \mathcal{O}_{X_j}(2 \cdot 3^{k-j-1}) \otimes \mathcal{O}_{X_k}(1),$$

$$B = \pi_{k,0}^* \mathcal{O}_X(2 \cdot 3^{k-1}), \quad A = L + B \Rightarrow L = A - B.$$

  In this way, one obtains equations of order $k = n$, when $d \ge d_n$ and $n \le 6$ (although the method might work also for $n > 6$). One can check that

$$d_2 = 15, \quad d_3 = 82, \quad d_4 = 329, \quad d_5 = 1222, \quad d_6 \text{ exists.}$$

# A differentiation technique by Yum-Tong Siu

One uses an important idea due to Yum-Tong Siu, itself based on ideas of Claire Voisin and Herb Clemens, and then refined by M. Păun [Pau08], E. Rousseau [Rou06b] and J. Merker [Mer09].

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The idea consists of studying vector fields on the relative jet space of the universal family of hypersurfaces of $\mathbb{P}^{n+1}$. Let $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ be the universal hypersurface, i.e.

$$\mathcal{X} = \{(z, a)\,;\ a = (a_\alpha) \text{ s.t. } P_a(z) = \sum a_\alpha z^\alpha = 0\},$$

$\Omega \subset \mathbb{P}^{N_d}$ the open subset of $a$'s for which $X_a = \{P_a(z) = 0\}$ is smooth, and let

$$p : \mathcal{X} \to \mathbb{P}^{n+1}, \quad \pi : \mathcal{X} \to \Omega \subset \mathbb{P}^{N_d}$$

be the natural projections.

Let

$$p_k : \mathcal{X}_k \to \mathcal{X} \to \mathbb{P}^{n+1}, \quad \pi_k : \mathcal{X}_k \to \Omega \subset \mathbb{P}^{N_d}$$

be the relative Green-Griffiths $k$-jet space of $\mathcal{X} \to \Omega$. Then J. Merker [Mer09] has shown that global sections $\eta_j$ of

$$\mathcal{O}(T_{\mathcal{X}_k}) \otimes p_k^* \mathcal{O}_{\mathbb{P}^{n+1}}(k^2 + 2k) \otimes \pi_k^* \mathcal{O}_{\mathbb{P}^{N_d}}(1)$$

generate the bundle at all points of $\mathcal{X}_k^{\mathrm{reg}}$ for $k = n = \dim X_a$. From this, it follows that if $P$ is a non zero global section over $\Omega$ of $E_{k,m}^{\mathrm{GG}} T_{\mathcal{X}}^* \otimes p_k^* \mathcal{O}_{\mathbb{P}^{n+1}}(-s)$ for some $s$, then for a suitable collection of $\eta = (\eta_1, \ldots, \eta_m)$, the $m$-th derivatives

$$D_{\eta_1} \ldots D_{\eta_m} P$$

yield sections of $H^0(\mathcal{X}, E_{k,m}^{\mathrm{GG}} T_{\mathcal{X}}^* \otimes p_k^* \mathcal{O}_{\mathbb{P}^{n+1}}(m(k^2 + 2k) - s))$ whose joint base locus is contained in $\mathcal{X}_k^{\mathrm{sing}}$, whence the result.

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