On the Henstock-Kurzweil integral (along with concerns about general math education in Europe)

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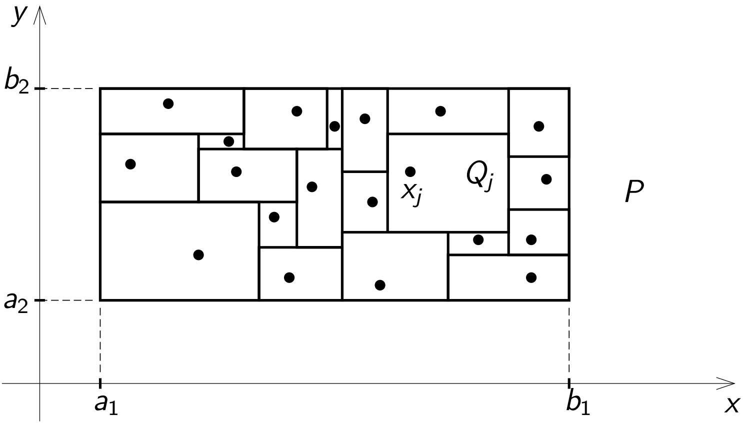
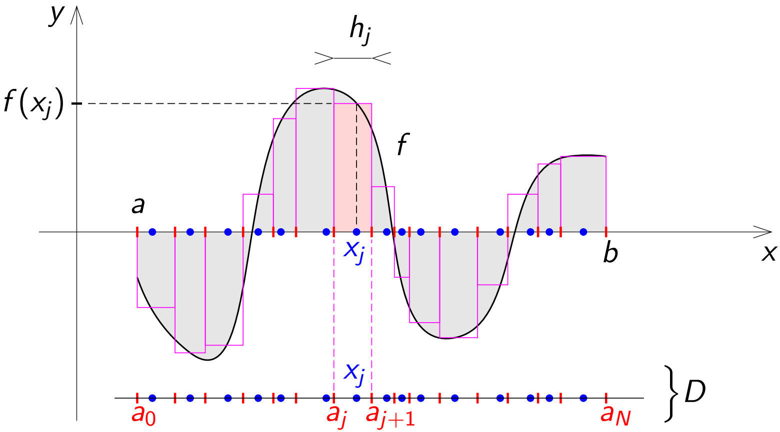
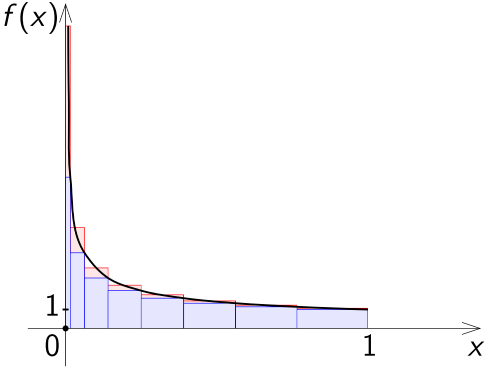
February 13, 2009 / University of Adelaide



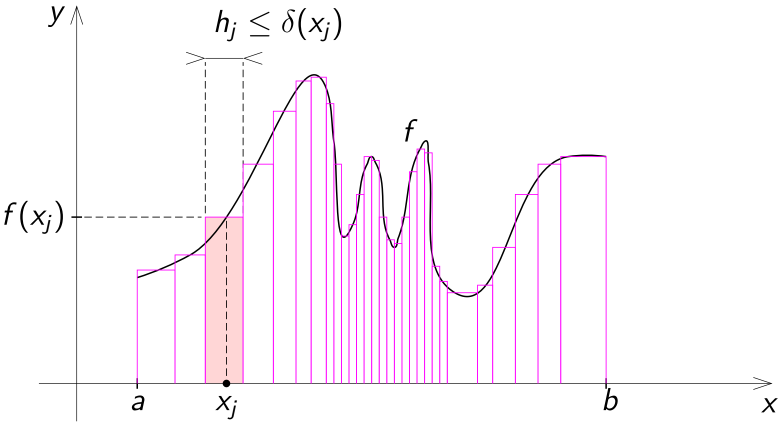
Tagged partition of a box P in $\mathbb{R}^n$.



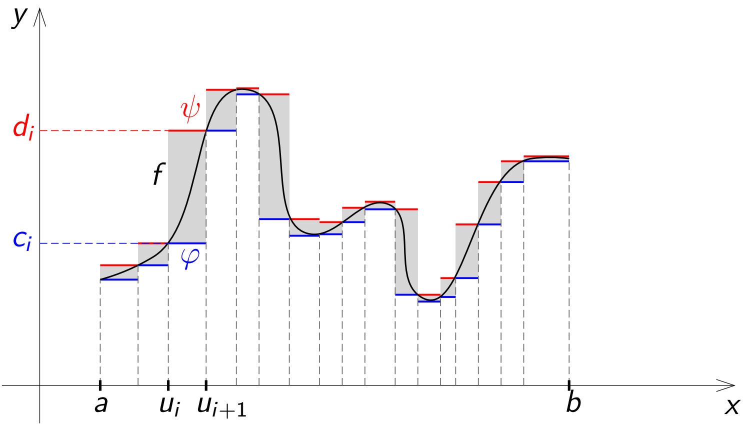
Riemann sum associated with f and tagged partition D.



Riemann sums associated with $f(x) = 1/\sqrt{x}$ **on** $[0, 1]$**.**



Riemann sum with variable steps



Every continuous function f is HK-integrable (no uniform continuity needed !)

1) Select k_0 such that $|A - \int_P f_k(x)\,dx| \le \varepsilon$ for $k \ge k_0$, and for each $x \in P$ an index $K(x) \ge k_0$ such that

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3) Then $\forall D = \{(Q_i, x_i)\}_{0\le i<N}$ δ-fine tagged partition of P

$$\big|S_D(f) - A\big| \le \Big|\sum_{0\le i<N} \big(f(x_i) - f_{K(x_i)}(x_i)\big)\,\mathrm{vol}(Q_i)\Big|_{\le \varepsilon\,\mathrm{vol}(P)}$$

$$+\Big|\sum_{0\le i<N}\Big(f_{K(x_i)}(x_i)\,\mathrm{vol}(Q_i) - \int_{Q_i} f_{K(x_i)}(x)\,dx\Big)\Big|_{\le \sum \varepsilon 2^{-k}}$$

$$+\Big|\sum_{0\le i<N}\int_{Q_i} f_{K(x_i)}(x)\,dx - A\Big|_{\le\varepsilon}$$