Potential Theory in Several Complex Variables

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This work is the second part of our survey article on Monge-Ampère operators. We are concerned here with the theory of complex n-dimensional capacities generalizing the usual logarithmic capacity in $\mathbb{C}$, and with the related notions of pluripolar and negligible sets. Decisive progress in the theory have been made by Bedford-Taylor [B-T1], [B-T2]. The present exposition, which is an expansion of lectures given in Nice at the Centre International de Mathématiques Pures et Appliquées (ICPAM) in 1989, borrows much to these papers. The last section on comparison of capacities is based on the work of Alexander and Taylor [A-T]. We are indebted to Z. Błocki and D. Coman for pointing out a few mistakes in the original version. Z. Błocki also suggested to derive the improved logarithmic growth estimate (due independently to H. El Mir and J. Siciak) from our proof of Josefson's theorem on the equivalence between locally pluripolar and globally pluripolar sets.

10. Capacities, Regularity and Capacitability

The goal of this section is to discuss a few fundamental notions and results of capacity theory. The reader will find a much more complete study in U. Cegrell's memoir [Ceg]. All topological spaces occurring here are assumed to be *Hausdorff.*

(10.1) Definition. *Let Ω be a topological space. A capacity is a set function $\overline{c} : E \mapsto \overline{c}(E)$ defined on all subsets $E \subset \Omega$ with values in $[0, +\infty]$, satisfying the axioms* (a,b,c) *below:*

(a) *If $E_1 \subset E_2 \subset \Omega$, then $\overline{c}(E_1) \le \overline{c}(E_2)$.*

(b) *If $E_1 \subset E_2 \subset \ldots \subset \Omega$, then $\overline{c}(\bigcup E_j) = \lim_{j\to+\infty} \overline{c}(E_j)$.*

(c) *If $K_1 \supset K_2 \supset \ldots$ are compact subsets, then $\overline{c}(\bigcap K_j) = \lim_{j\to+\infty} \overline{c}(K_j)$.*

The capacity $\overline{c}$ is said to be subadditive if moreover $\overline{c}(\emptyset) = 0$ and $\overline{c}$ satisfies the further axiom:

(d) *If $E_1, E_2, \ldots$ are subsets of Ω, then $\overline{c}(\bigcup E_j) \leq \sum_j \overline{c}(E_j)$.*

In our applications, we will have to consider set functions which are only defined on the collection of Borel subsets. We thus introduce:

(10.2) Definition. *A precapacity is a set function $c : E \mapsto c(E)$ defined on all Borel subsets $E \subset \Omega$ with values in $[0, +\infty]$, satisfying axioms* 10.1 (a), (b). *The precapacity c is said to be inner regular if all Borel subsets satisfy*

(i) $$c(E) = \sup_{K \text{ compact} \subset E} c(K).$$

Similarly, c is said to be outer regular if all Borel subsets E satisfy

(o) $$c(E) = \inf_{G \text{ open} \supset E} c(G)$$

When c is a precapacity and $E \subset \Omega$ is an arbitrary subset, the inner capacity $c_\star(E)$ and the outer capacity $c^\star(E)$ are defined by

(10.3 i) $$c_\star(E) = \sup_{K \text{ compact} \subset E} c(K),$$

(10.3 o) $$c^\star(E) = \inf_{G \text{ open} \supset E} c(G).$$

(10.4) Proposition. *Let c be a precapacity. If c is outer regular, then $c^\star$ is a capacity. Moreover $c^\star$ is subadditive as soon as c is subadditive.*

Proof. It is clear that $c^\star$ satisfies 10.1 (a). Moreover, for any set $E \subset \Omega$, there is a countable intersection $\widetilde{E} = \bigcap G_\ell$ of open sets containing E such that $c(\widetilde{E}) = c^\star(E)$ (take $G_\ell \supset E$ with $c(G_\ell) < c^\star(E) + 1/\ell$). When $E_1 \subset E_2 \ldots$, we can arrange that $\widetilde{E}_1 \subset \widetilde{E}_2 \ldots$, after replacing $\widetilde{E}_j$ by $\bigcap_{k \geq j} \widetilde{E}_k$ if necessary. Since c satisfies 10.1 (b) for Borel subsets, we conclude that $\widetilde{E} = \bigcup \widetilde{E}_j$ has precapacity $c(\widetilde{E}) = \lim c(\widetilde{E}_j) = \lim c^\star(E_j)$, hence by the outer regularity

$$c^\star(E) \leq c^\star(\widetilde{E}) = c(\widetilde{E}) = \lim c^\star(E_j).$$

The opposite inequality is clear, thus $c^\star$ satisfies 10.1 (b). Finally, if K_j is a decreasing sequence of compact sets, any open set G containing their intersection contains one of the sets K_j, so $c(G) \geq \lim c^\star(K_j)$ and $c^\star(\bigcap K_j) \geq \lim c^\star(K_j)$. The opposite inequality is again clear, thus $c^\star$ satisfies 10.1 (c). The final assertion concerning subadditivity is easy. □

(10.5) Example. Let Ω be a separable locally compact space and let (μ_α) be a family of positive Radon measures on Ω. Then $c(E) = \sup \mu_\alpha(E)$ is a subadditive precapacity; this follows from the standard properties of measures (countable additivity, monotone convergence theorem). The precapacity c is called the *upper envelope* of the family of measures (μ_α). In general, c does not satisfy the additivity property

$$E_1, E_2 \text{ disjoint } \Rightarrow c(E_1 \cup E_2) = c(E_1) + c(E_2)\,;$$

for a specific example, consider the measures $\mu_1 = \delta_0$, $\mu_2 = d\lambda$ on $\mathbb{R}$ and the sets $E_1 = \{0\}$, $E_2 =]0,1]$; then

$$c(\{0\}) = 1, \quad c(]0,1]) = 1, \quad c([0,1]) = 1.$$

Moreover, the precapacity $c = \sup \mu_\alpha$ is inner regular because all Radon measures on a separable locally compact space are inner regular. However, c need not be outer regular: for instance, take $d\mu_\alpha(x) = \alpha^{-1}\rho(x/\alpha)dx$ on $\mathbb{R}$, $\alpha > 0$, where $\rho \geq 0$ is a function with support in $[-1,1]$ and $\int_{\mathbb{R}} \rho(x)dx = 1$; then $c(\{0\}) = 0$ but every neighborhood of 0 has capacity 1. □

(10.6) Definition. *Let c be a precapacity on Ω. A set $E \subset \Omega$ is said to be c-capacitable if $c_\star(E) = c^\star(E)$.*

By definition, the precapacity c is (inner and outer) regular if and only if all Borel subsets are c-capacitable.

We are now going to prove a general capacitability theorem due to G. Choquet. Before doing so, we need a few results about K-analytic spaces.

(10.7) Definition. *Let X be a topological space. Then*

(a) *a F_σ subset of X is a countable union of closed subsets of X ;*

(b) *a $F_{\sigma\delta}$ subset of X is a countable intersection of F_σ subsets of X.*

(c) *the space X is said to be a K_σ (resp. $K_{\sigma\delta}$) space if it is homeomorphic to some F_σ (resp. $F_{\sigma\delta}$) subset of a compact space W.*

(10.8) Properties.

(a) *Every closed subset F of a $K_{\sigma\delta}$ space X is a $K_{\sigma\delta}$ space.*

(b) *Every countable disjoint sum $\coprod X_j$ of $K_{\sigma\delta}$ spaces is a $K_{\sigma\delta}$ space.*

(c) *Every countable product $\prod X_j$ of $K_{\sigma\delta}$ spaces is a $K_{\sigma\delta}$ space.*

Proof. (a) Write $X = \bigcap_{\ell\geq 1} G_\ell$ and $G_\ell = \bigcup_{m\geq 1} K_{\ell m}$ where $K_{\ell m}$ are closed subsets of a compact space W. If $\overline{F}$ is the closure of F in W, we have

$$F = X \cap \overline{F} = \bigcap_{\ell\geq 1} G_\ell \cap \overline{F}, \qquad G_\ell \cap \overline{F} = \bigcup_{m\geq 1} K_{\ell m} \cap \overline{F}.$$

(b) Let $(X_j)_{j\geq 1}$ be $K_{\sigma\delta}$ spaces and write for each j

$$X_j = \bigcap_{\ell\geq 1} G_\ell^j, \qquad G_\ell^j = \bigcup_{m\geq 1} K_{\ell m}^j$$

where $K_{\ell m}^j$ is a closed subspace of a compact space W_j, and let $\{\star\}$ be a one-point topological space. Then $\coprod W_j$ can be embedded in the compact space $W = \prod(W_j \amalg \{\star\})$ via the obvious map which sends $w \in W_j$ to $(\star,\ldots,\star,w,\star,\ldots)$ with w in the j-th position. Now $X = \coprod X_j$ can be written

$$X = \bigcap_{\ell\geq 1} G_\ell, \qquad G_\ell = \bigcup_{m\geq 1} \coprod_{j\geq 1} K_{\ell m}^j.$$

As $K_{\ell m}^j$ is sent onto a closed set by the embedding $\coprod W_j \to W$, we conclude that X is a $K_{\sigma\delta}$ space.

(c) With the notations of (b), write $X = \prod X_j$ as

$$X = \bigcap_{\ell\geq 1} G_\ell, \qquad G_\ell = G_\ell^1 \times G_{\ell-1}^2 \times \ldots \times G_1^\ell \times W_{\ell+1} \times \ldots \times W_j \times \ldots,$$

$$G_\ell = \bigcup_{m_1,\ldots,m_\ell\geq 1} K_{\ell\, m_1}^1 \times K_{\ell-1\, m_2}^2 \times \ldots \times K_{1\, m_\ell}^\ell \times W_{\ell+1} \times \ldots \times W_j \times \ldots$$

where each term in the union is closed in $W = \prod W_j$. □

(10.9) Definition. *A space E is said to be K-analytic if E is a continuous image of a $K_{\sigma\delta}$ space X.*

(10.10) Proposition. *Let Ω be a topological space and let $E_1, E_2, \ldots$ be K-analytic subsets of Ω. Then $\bigcup E_j$ and $\bigcap E_j$ are K-analytic.*

Proof. Let $f_j : X_j \to E_j$ be a continuous map from a $K_{\sigma\delta}$ space onto E_j. Set $X = \coprod X_j$ and $f = \coprod f_j : X \to \Omega$. Then X is a $K_{\sigma\delta}$ space, f is continuous and $f(X) = \bigcup E_j$. Now set

$$X = \big\{x = (x_1, x_2, \ldots) \in \prod X_j\,;\ f_1(x_1) = f_2(x_2) = \cdots\big\},$$
$$f : X \to \Omega, \qquad f(x) = f_1(x_1) = f_2(x_2) = \cdots.$$

Then X is closed in $\prod X_j$, so X is a $K_{\sigma\delta}$ space by 10.8 (a,c) and $f(X) = \bigcap E_j$. □

(10.11) Corollary. *Let Ω be a separable locally compact space. Then all Borel subsets of Ω are K-analytic.*

Proof. Any open or closed open set in Ω is a countable union of compact subsets, hence K-analytic. On the other hand, Prop. 10.10 shows that

$$\mathcal{A} = \{E \subset \Omega\,;\ E \text{ and } \Omega \setminus E \text{ are } K\text{-analytic}\}$$

is a σ-algebra. Since $\mathcal{A}$ contains all open sets in E, $\mathcal{A}$ must also contain all Borel subsets. □

Before going further, we need a simple lemma.

(10.12) Lemma. *Let E be a relatively compact K-analytic subset of a topological space Ω. There exists a compact space T, a continuous map $g : T \to \Omega$ and a $F_{\sigma\delta}$ subset $Y \subset T$ such that $g(Y) = E$.*

Proof. There is a compact space W, a $F_{\sigma\delta}$ subset $X \subset W$ and a continuous map $f : X \to E$ onto E. Let

$$Y = \{(x, f(x))\,;\, x \in X\} \subset X \times E$$

be the graph of f and $T = \overline{Y}$ the closure of Y in the compact space $\overline{X} \times \overline{E}$. As f is continuous, Y is closed in $X \times \overline{E}$, thus $Y = T \cap (X \times \overline{E})$. Now, X is a $F_{\sigma\delta}$ subset of $\overline{X}$, so $X \times \overline{E}$ is a $F_{\sigma\delta}$ subset of $\overline{X} \times \overline{E}$ and Y is a $F_{\sigma\delta}$ subset of T. Finally E is the image of Y by the second projection $g : T \to \overline{E}$. □

(10.13) Choquet's capacitability theorem. *Let Ω be a K_σ space and let $\overline{c}$ be a capacity on Ω. Then every K-analytic subset $E \subset \Omega$ satisfies*

$$\overline{c}(E) = \sup_{K \text{ compact} \subset E} \overline{c}(K).$$

Proof. As Ω is an increasing union of compact sets L_j, axiom 10.1 (b) implies $\overline{c}(E) = \lim_{j\to+\infty} \overline{c}(E \cap L_j)$; we may therefore assume that E is relatively compact in Ω. Then Lemma 10.12 shows that there is a $F_{\sigma\delta}$ subset Y in a compact space T and a continuous map $g : T \to \Omega$ such that $g(Y) = E$. It is immediate to check that the set function $g^\star\overline{c}$ on T defined by $g^\star\overline{c}(E) = \overline{c}(g(E))$ is a capacity. Hence we are reduced to proving the theorem when Ω is a compact space and E is a $F_{\sigma\delta}$ subset of Ω. We then write

$$E = \bigcap_{\ell \geq 1} G_\ell, \qquad G_\ell = \bigcup_{m \geq 1} K_{\ell m}$$

where $K_{\ell m}$ is a closed subset of Ω. Without loss of generality, we can arrange that $K_{\ell m}$ is increasing in m. Fix $\lambda < \overline{c}(E)$. Then

$$E = G_1 \cap \bigcap_{\ell \geq 2} G_\ell = \bigcup_{m \geq 1} (K_{1m} \cap \bigcap_{\ell \geq 2} G_\ell)$$

and axiom (b) implies that exists a subset $E_1 = K_{1m_1} \cap \bigcap_{\ell \geq 2} G_\ell$ of E such that $\overline{c}(E_1) > \lambda$. By induction, there is a decreasing sequence $E \supset E_1 \supset \ldots \supset E_s$ with

$$E_s = K_{1m_1} \cap \ldots \cap K_{sm_s} \cap \bigcap_{\ell \geq s+1} G_\ell$$

and $\overline{c}(E_s) > \lambda$. Set $K = \bigcap K_{sm_s} = \bigcap E_s \subset E$. Axiom 10.1 (c) implies

$$\overline{c}(K) = \lim_{s \to +\infty} \overline{c}\big(K_{1m_s} \cap \ldots \cap K_{sm_s}\big) \geq \lim_{s \to +\infty} \overline{c}(E_s) \geq \lambda$$

and the theorem is proved. □

(10.14) Corollary. *Let c be an outer regular precapacity on a separable locally compact space Ω. Then c is also inner regular and every K-analytic subset of Ω is c-capacitable.*

Proof. By Prop. 10.4, we know that $c^\star$ is a capacity. Choquet's theorem 10.13 implies that $c^\star(E) = \sup_{K \text{ compact} \subset E} c^\star(K)$ for every K-analytic set E in Ω. By the outer regularity we have $c^\star(K) = c(K)$, hence $c^\star(E) = c_\star(E)$ and c is inner regular. □

11. Monge-Ampère Capacities and Quasicontinuity

Let Ω be a bounded open subset of $\mathbb{C}^n$. We denote by $P(\Omega)$ the set of all plurisubharmonic functions that are $\not\equiv -\infty$ on each connected component of Ω. The following fundamental definition has been introduced in [B-T2].

(11.1) Definition. *For every Borel subset $E \subset \Omega$, we set*

$$c(E,\Omega) = \sup\Big\{\int_E (dd^c u)^n\ ;\ u \in P(\Omega),\ 0 \le u \le 1\Big\}.$$

The Chern-Levine-Nirenberg inequalities show that $c(E,\Omega) < +\infty$ as soon as $E \subset\subset \Omega$. If $\Omega \subset B(z_0,R)$, we can choose $u(z) = R^{-2}|z-z_0|^2$ and we obtain therefore

$$(11.2) \qquad c(E,\Omega) \ge \frac{2^n n!}{\pi^n R^{2n}}\,\lambda(E)$$

where λ is the Lebesgue measure. As a special case of Example 10.5, we see that $c(\bullet,\Omega)$ is the upper envelope of the family of measures $\mu_u = (dd^c u)^n$, $u \in P(\Omega)$, $0 \le u \le 1$. In particular $c(\bullet,\Omega)$ is a subadditive and inner regular precapacity; it is also outer regular, but this fact is non trivial and will be proved only in § 14. The set function $c(\bullet,\Omega)$, resp. $c^\star(\bullet,\Omega)$, is called the *relative Monge-Ampère precapacity, resp. capacity,* of Ω. We first compare capacities associated to different open sets Ω.

(11.3) Proposition. *Let $\Omega_1 \subset \Omega_2 \subset\subset \mathbb{C}^n$. Then*

(a) $c(E,\Omega_1) \ge c(E,\Omega_2)$ *for all Borel subsets $E \subset \Omega_1$.*

(b) *Let $\omega \subset\subset \Omega_1$. There exists a constant $A > 0$ such that for all Borel subsets $E \subset \omega$ we have $c(E,\Omega_1) \le A\,c(E,\Omega_2)$.*

Proof. Since every plurisubharmonic function $u \in P(\Omega_2)$ with $0 \le u \le 1$ induces a plurisubharmonic function in $P(\Omega_1)$ with the same property, (a) is clear.

(b) Use a finite covering of $\overline{\omega}$ by open balls contained in Ω_1 and cut E into pieces. The proof is then reduced to the case when $\omega \subset\subset \Omega_1$ are concentric balls, say $\Omega_1 = B(0,r)$ and $\omega = B(0,r-\varepsilon)$. For every $u \in P(\Omega_1)$ such that $0 \le u \le 1$, set

$$\widetilde{u}(z) = \begin{cases} \max\{u(z), \lambda(|z|^2 - r^2) + 2\} & \text{on } \ \Omega_1, \\ \lambda(|z|^2 - r^2) + 2 & \text{on } \ \Omega_2 \setminus \Omega_1. \end{cases}$$

Choose λ so large that $\lambda((r-\varepsilon)^2 - r^2) \le -2$. Then $\widetilde{u} \in P(\Omega_2)$ and $\widetilde{u} = u$ on ω. Moreover $0 \le \widetilde{u} \le M$ for some constant $M > 0$, thus for $E \subset \omega$ we get

$$\int_E (dd^c u)^n = \int_E (dd^c \widetilde{u})^n \le M^n c(E,\Omega_2).$$

Therefore $c(E,\Omega_1) \le M^n c(E,\Omega_2)$. □

As a consequence of Prop. 11.3, it is in general harmless to shrink the domain Ω when capacities have to be estimated.

(11.4) Proposition. *Let K be a compact subset of Ω and $\omega \subset\subset \Omega$ a neighborhood of K. There is a constant $A > 0$ such that for every $v \in P(\Omega)$*

$$c\big(K \cap \{v < -m\}, \Omega\big) \le A\|v\|_{L^1(\overline{\omega})} \cdot \frac{1}{m}.$$

Proof. For every $u \in P(\Omega),\ 0 \le u \le 1$, Prop. 1.11 implies

$$\int_{K\cap\{v<-m\}} (dd^c u)^n \le \frac{1}{m}\int_K |v|(dd^c u)^n \le \frac{1}{m} C_{K,\overline{\omega}}\|v\|_{L^1(\overline{\omega})}. \qquad \square$$

(11.5) Definition. *A set $P \subset \Omega$ is said to be globally pluripolar in Ω if there exists $v \in P(\Omega)$ such that $P \subset \{v = -\infty\}$.*

(11.6) Corollary. *If P is pluripolar in Ω, then*

$$c^\star(P, \Omega) = 0.$$

Proof. Write $P \subset \{v = -\infty\}$ and $\Omega = \bigcup_{j\ge1} \Omega_j$ with $\Omega_j \subset\subset \Omega$. Prop. 11.4 shows that there is an open set $G_j = \Omega_j \cap \{v < -m_j\}$ with $c(G_j, \Omega) < \varepsilon 2^{-j}$. Then $\{v = -\infty\} \subset G = \bigcup G_j$ and $c(G, \Omega) < \varepsilon$. $\square$

(11.7) Proposition. *Let $v_k,\ v \in P(\Omega)$ be locally bounded plurisubharmonic functions such that (v_k) decreases to v. Then for every compact subset $K \subset \Omega$ and every $\delta > 0$*

$$\lim_{k\to+\infty} c\big(K \cap \{v_k > v + \delta\}, \Omega\big) = 0.$$

Proof. It is sufficient to show that

$$\sup_{u\in P(\Omega), 0\le u\le 1} \int_K (v_k - v)(dd^c u)^n$$

tends to 0, because this supremum is larger than $\delta.c(K\cap\{v_k > v+\delta\}, \Omega)$. By cutting K into pieces and modifying v, v_k, u with the max construction, we may assume that $K \subset \Omega = B(0, r)$ are concentric balls and that all functions v, v_k, u are equal to $\lambda(|z|^2 - r^2) + 2)$ on the corona $\Omega \setminus \omega$, $\omega = B(0, r - \varepsilon)$. An integration by parts yields

$$\int_\Omega (v_k - v)(dd^c u)^n = -\int_\Omega d(v_k - v) \wedge d^c u \wedge (dd^c u)^{n-1}.$$

The Cauchy-Schwarz inequality implies that this integral is bounded by

$$A\Big(\int_\Omega d(v_k - v) \wedge d^c(v_k - v) \wedge (dd^c u)^{n-1}\Big)^{1/2}$$

where

$$A^2 = \int_\Omega du \wedge d^c u \wedge (dd^c u)^{n-1} \le \int_\Omega dd^c(u^2) \wedge (dd^c u)^{n-1}$$

and this last integral depends only on the constants λ, r. Another integration by parts yields

$$\begin{aligned}\int_\Omega d(v_k - v) \wedge d^c(v_k - v) \wedge (dd^c u)^{n-1} &= \int_\Omega (v_k - v) dd^c(v - v_k) \wedge (dd^c u)^{n-1} \\ &\le \int_\Omega (v_k - v) dd^c v \wedge (dd^c u)^{n-1}.\end{aligned}$$

We have thus replaced one factor $dd^c u$ by $dd^c v$ in the integral. Repeating the argument $(n-1)$ times we get

$$\int_\Omega (v_k - v)(dd^c u)^n \le C\Big(\int_\Omega (v_k - v)(dd^c v)^n\Big)^{1/2^n}$$

and the last integral converges to 0 by the bounded convergence theorem. □

(11.8) Theorem (quasicontinuity of plurisubharmonic functions)**.** *Let Ω be a bounded open set in $\mathbb{C}^n$ and $v \in P(\Omega)$. Then for each $\varepsilon > 0$, there is an open subset G of Ω such that $c(G, \Omega) < \varepsilon$ and v is continuous on $\Omega \setminus G$.*

Proof. Let $\omega \subset\subset \Omega$ be arbitrary. We first show that there exists $G \subset \omega$ such that $c(G, \Omega) < \varepsilon$ and v continuous on $\omega \setminus G$. For $m > 0$ large enough, the set $G_0 = \omega \cap \{v < -m\}$ has capacity $< \varepsilon/2$ by Prop. 11.4. On $\omega \setminus G_0$ we have $v \ge -m$, thus $\widetilde{v} = \max\{v, -m\}$ coincides with v there and $\widetilde{v}$ is locally bounded on Ω. Let (v_k) be a sequence of smooth plurisubharmonic functions which decrease to $\widetilde{v}$ in a neighborhood of $\overline{\omega}$. For each $\ell \ge 1$, Prop. 11.7 shows that there is an index $k(\ell)$ and an open set

$$G_{k(\ell)} = \omega \cap \{v_{k(\ell)} > \widetilde{v} + 1/\ell\}$$

such that $c(G_{k(\ell)}, \Omega) < \varepsilon 2^{-\ell-1}$. Then $G = G_0 \cup \bigcup G_{k(\ell)}$ has capacity $c(G, \Omega) < \varepsilon$ by subadditivity and $(v_{k(\ell)})$ converges uniformly to $\widetilde{v} = v$ on $\omega \setminus G$. Hence v is continuous on $\omega \setminus G$. Now, take an increasing sequence

$\omega_1 \subset \omega_2 \subset \ldots$ with $\bigcup \omega_j = \Omega$ and $G_j \subset \omega_j$ such that $c(G_j, \Omega) < \varepsilon 2^{-j}$ and v continuous on $\omega_j \setminus G_j$. The set $G = \bigcup G_j$ satisfies all requirements. □

As an example of application, we prove an interesting inequality for the Monge-Ampère operator.

(11.9) Proposition. *Let u, v be locally bounded plurisubharmonic functions on Ω. Then we have an inequality of measures*

$$(dd^c \max\{u, v\})^n \geq \mathbb{1}_{\{u \geq v\}}(dd^c u)^n + \mathbb{1}_{\{u < v\}}(dd^c v)^n.$$

Proof. It is enough to check that

$$\int_K (dd^c \max\{u, v\})^n \geq \int_K (dd^c u)^n$$

for every compact set $K \subset \{u \geq v\}$; the other term is then obtained by reversing the roles of u and v. By shrinking Ω, adding and multiplying with constants, we may assume that $0 \leq u, v \leq 1$ and that u, v have regularizations $u_\varepsilon = u \star \rho_\varepsilon$, $v_\varepsilon = v \star \rho_\varepsilon$ with $0 \leq u_\varepsilon,\ v_\varepsilon \leq 1$ on Ω. Let $G \subset \Omega$ be an open set of small capacity such that u, v are continuous on $\Omega \setminus G$. By Dini's lemma, u_ε, v_ε converge uniformly to u, v on $\Omega \setminus G$. Hence for any $\delta > 0$, we can find an arbitrarily small neighborhood L of K such that $u_\varepsilon > v_\varepsilon - \delta$ on $L \setminus G$ for ε small enough. As $(dd^c u_\varepsilon)^n$ converges weakly to $(dd^c u)^n$ on Ω, we get

$$\begin{aligned}
\int_K (dd^c u)^n &\leq \liminf_{\varepsilon \to 0} \int_L (dd^c u_\varepsilon)^n \\
&\leq \liminf_{\varepsilon \to 0} \Big(\int_G (dd^c u_\varepsilon)^n + \int_{L \setminus G} (dd^c u_\varepsilon)^n \Big) \\
&\leq c(G, \Omega) + \liminf_{\varepsilon \to 0} \int_{L \setminus G} (dd^c \max\{u_\varepsilon + \delta, v_\varepsilon\})^n.
\end{aligned}$$

Observe that $\max\{u_\varepsilon + \delta, v_\varepsilon\}$ coincides with $u_\varepsilon + \delta$ on a neighborhood of $L \setminus G$. By weak convergence again, we get

$$\int_K (dd^c u)^n \leq c(G, \Omega) + \int_{L \setminus G} (dd^c \max\{u + \delta, v\})^n.$$

By taking L very close to K and $c(G, \Omega)$ arbitrarily small, this implies

$$\int_K (dd^c u)^n \leq \int_K (dd^c \max\{u + \delta, v\})^n$$

and the desired conclusion follows by letting δ tend to 0. $\square$

12. Upper Envelopes and the Dirichlet Problem

Let (u_α) be a family of upper semi-continuous functions on Ω which is locally bounded from above. Then the upper envelope

$$u = \sup_\alpha u_\alpha(z)$$

need not be upper semi-continuous, so we consider its "upper semi-continuous regularization"

$$u^\star(z) = \lim_{\varepsilon\to 0} \sup_{B(z,\varepsilon)} u \geq u(z).$$

It is easy to check that $u^\star$ is upper semi-continuous and that $u^\star$ is the smallest upper semi-continuous function $\geq u$.

Let $B(z_j,\varepsilon_j)$ be a countable basis of the topology of Ω. For each j, let (z_{jk}) be a sequence in $B(z_j,\varepsilon_j)$ such that

$$\sup_k u(z_{jk}) = \sup_{B(z_j,\varepsilon_j)} u,$$

and for each (j,k), let $\alpha(j,k,\ell)$ be a sequence of indices α such that $u(z_{jk}) = \sup_\ell u_{\alpha(j,k,\ell)}(z_{jk})$. Set

$$v = \sup_{j,k,\ell} u_{\alpha(j,k,\ell)}.$$

Then $v \leq u$ and $v^\star \leq u^\star$. On the other hand

$$\sup_{B(z_j,\varepsilon_j)} v \geq \sup_k v(z_{jk}) \geq \sup_{k,\ell} u_{\alpha(j,k,\ell)}(z_{jk}) = \sup_k u(z_{jk}) = \sup_{B(z_j,\varepsilon_j)} u.$$

As every ball $B(z,\varepsilon)$ is a union of balls $B(z_j,\varepsilon_j)$, we easily conclude that $v^\star \geq u^\star$, hence $v^\star = u^\star$. Therefore:

(12.1) Choquet's lemma. *Every family (u_α) has a countable subfamily $(u_{\alpha(j)})$ whose upper envelope v satisfies $v \leq u \leq u^\star = v^\star$.*

(12.2) Proposition. *If all u_α are plurisubharmonic, then $u^\star$ is plurisubharmonic and equal almost everywhere to u.*

Proof. By Choquet's lemma we may assume that (u_α) is countable. Then $u = \sup u_\alpha$ is a Borel function. For every $(z_0,a) \in \Omega \times \mathbb{C}^n$, u_α satisfies the mean value inequality on circles, hence

$$u(z_0) = \sup u_\alpha(z_0) \leq \sup \int_0^{2\pi} u_\alpha(z_0 + ae^{\mathrm{i}\theta})\frac{d\theta}{2\pi} \leq \int_0^{2\pi} u(z_0 + ae^{\mathrm{i}\theta})\frac{d\theta}{2\pi}.$$

It follows easily that each convolution $u \star \rho_\varepsilon$ also satisfies the mean value inequality, thus $u\star\rho_\varepsilon$ is smooth and plurisubharmonic. Therefore $(u\star\rho_\varepsilon)\star\rho_\eta$ is increasing in η. Letting ε tends to 0, we see that $u \star \rho_\eta$ in increasing in η. Since $u \star \rho_\varepsilon$ is smooth and $u \star \rho_\varepsilon \geq u$ by the mean value inequality, we also have $u \star \rho_\varepsilon \geq u^\star$. By the upper semi-continuity we get $\lim_{\varepsilon\to 0} u \star \rho_\varepsilon = u^\star$, in particular $u^\star$ is plurisubharmonic and coincides almost everywhere with the $L^1_{\rm loc}$ limit u. □

In the sequel, we need the fundamental result of Bedford-Taylor [B-T1] on the solution of the Dirichlet problem for complex Monge-Ampère equations.

(12.3) Theorem. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smooth strongly pseudoconvex domain and let $f \in C^0(\partial\Omega)$ be a continuous function on the boundary. Then*

$$u(z) = \sup\{v(z)\,;\ v \in P(\Omega) \cap C^0(\overline{\Omega}),\ v \leq f \text{ on } \partial\Omega\}$$

is continuous on $\overline{\Omega}$ and plurisubharmonic on Ω, and solves the Dirichlet problem

$$(dd^c u)^n = 0 \text{ on } \Omega, \quad u = f \text{ on } \partial\Omega.$$

Proof. The main difficulty is to obtain sufficient regularity of u when f is smooth, so as to be able to analyze the local convexity of u at any point. The proof given in [B-T1] consists of three steps.

Step 1. The upper envelope u is continuous on $\overline{\Omega}$ and $u = f$ on $\partial\Omega$.

Let $g \in C^2(\overline{\Omega})$ be an approximate extension of f such that $|g - f| < \varepsilon$ on $\partial\Omega$ and let $\psi < 0$ be a smooth strongly plurisubharmonic exhaustion of Ω. Then $g - \varepsilon + A\psi$ is plurisubharmonic for $A > 0$ large enough and $g - \varepsilon + A\psi = g - \varepsilon \leq f$ on $\partial\Omega$, hence $g - \varepsilon + A\psi \leq u$ on $\overline{\Omega}$. Similarly, for all $v \in P(\Omega) \cap C^0(\overline{\Omega})$ with $v \leq f$ on $\partial\Omega$, the function $v - g - \varepsilon + A\psi$ equals $v - g - \varepsilon \leq 0$ on $\partial\Omega$ and is plurisubharmonic for A large, thus $v - g - \varepsilon + A\psi \leq 0$ on $\overline{\Omega}$ by the maximum principle. Therefore we get $u \leq g + \varepsilon - A\psi$; as ε tends to 0, we see that $u = f$ on $\partial\Omega$ and that u is continuous at every point of $\partial\Omega$. Since $g + \varepsilon + A\psi = g + \varepsilon > f$ on $\partial\Omega$, there exists $\delta > 0$ such that $u^\star < g + \varepsilon + A\psi$ on $\Omega \setminus \Omega_\delta$, where $\Omega_\delta = \{\psi < -\delta\}$. For $\eta > 0$ small enough, the regularizations of $u^\star$ satisfy $u^\star \star \rho_\eta < g + \varepsilon + A\psi$ on a neighborhood of $\partial\Omega_\delta$. Then we let

$$v_\varepsilon = \begin{cases} \max\{u^\star \star \rho_\eta - 2\varepsilon, g - \varepsilon + A\psi\} & \text{on } \ \Omega_\delta \\ g - \varepsilon + A\psi & \text{on } \ \overline{\Omega} \setminus \Omega_\delta. \end{cases}$$

It is clear that v_ε is plurisubharmonic and continuous on $\overline{\Omega}$ and we have $v_\varepsilon = g - \varepsilon \le f$ on $\partial\Omega$, thus $v_\varepsilon \le u$ on $\overline{\Omega}$. We get therefore $u^\star \star \rho_\eta \le u + 2\varepsilon$ on Ω_δ. As $u \le u^\star \le u^\star \star \rho_\eta$, we see that $u^\star \star \rho_\eta$ converges uniformly to u on every compact subset of Ω. Hence u is plurisubharmonic and continuous on $\overline{\Omega}$.

Step 2. If $\Omega = B = B(0,1) \subset \mathbb{C}^n$ and $f \in C^{1+\mathrm{lip}}(\partial B)$, then $u \in C^{1+\mathrm{lip}}(B)$.

Here $C^{1+\mathrm{lip}}$ denotes the space of functions admitting Lipschitz continuous first derivatives; such functions have locally bounded second derivatives almost everywhere by the Lebesgue differentiability theorem. Since we are going to obtain uniform estimates in terms of $||f||_{C^2(\partial B)}$, it is enough to consider the case when $f \in C^2(\partial B)$, thanks to a regularization argument. The C^0 estimate $||u||_{C^0(\overline{B})} \le ||f||_{C^0(\partial B)}$ is clear by the maximum principle. Now, there is a C^2 extension $\widehat{f}$ of f to $2\overline{B}$ such that $||\widehat{f}||_{C^2(2\overline{B})} \le C||f||_{C^2(\partial B)}$. After adding or subtracting a sufficiently large multiple of $1-|z|^2$ to $\widehat{f}$, we get a plurisubharmonic extension f' of f to $2\overline{B}$ and a plurisuperharmonic extension f'' satisfying similar estimates. Then $f' \le u \le f''$ on B, the second equality being a consequence of the maximum principle applied to the plurisubharmonic function $u - f''$. We get a plurisubharmonic extension $\widehat{u}$ of u to $2\overline{B}$ by setting $\widehat{u} = u$ on B and $\widehat{u} = f'$ on $2\overline{B} \setminus B$. Then $\widehat{u} \le \max\{f', f''\}$ on $2\overline{B}$ and for all $z \in \partial B$, $|h| < 1$ we get

$$\begin{aligned}\widehat{u}(z+h) &\le f(z) + \max\left\{||f'||_{C^1(2\overline{B})}, ||f''||_{C^1(2\overline{B})}\right\}|h| \\ &\le f(z) + C'||f||_{C^2(\partial B)}|h|.\end{aligned}$$

The definition of u as an upper envelope implies

$$\widehat{u}(z+h) - C'||f||_{C^2(\partial B)}|h| \le u(z) \quad \text{ on } \ B.$$

By changing h into $-h$, we conclude that $|u(z+h) - u(z)| \le C'||f||_{C^2(\partial B)}|h|$ for $z \in B$ and $|h|$ small, thus $||u||_{C^1(\overline{B})} \le C'||f||_{C^2(\partial B)}$.

In order to get the estimate for second derivatives, the idea is to move u and f by automorphisms of B instead of ordinary translations (this is the reason why we have to assume $\Omega = B$). Let $a = \lambda\zeta \in B$ with $\zeta \in \partial B$ and $\lambda \in \mathbb{R}$, $|\lambda| < 1$. We define an analytic map Φ_a on B by

$$\begin{aligned}\Phi_a(z) &= \frac{P_a(z) - a + (1-|a|^2)^{1/2}Q_a(z)}{1 - \langle z, a\rangle} \\ &= \frac{(\langle z,\zeta\rangle - \lambda)\zeta + (1-\lambda^2)^{1/2}(z - \langle z,\zeta\rangle\zeta)}{1 - \lambda\langle z,\zeta\rangle},\end{aligned}$$

where $\langle \bullet, \bullet \rangle$ denotes the usual hermitian inner product on $\mathbb{C}^n$, P_a the orthogonal projection of $\mathbb{C}^n$ onto $\mathbb{C}\{a\}$ and $Q_a = \mathrm{Id} - P_a$. Easy computations show that

$$1 - |\Phi_a(z)|^2 = \frac{(1-\lambda^2)(1-|z|^2)}{|1-\lambda\langle z, \zeta\rangle|^2},$$

from which it follows that Φ_a is an automorphism of B (see e.g. [Rud]). Moreover

$$\Phi_a(z) = \frac{z - a + O(|a|^2)}{1 - \langle z, a\rangle} = z - a + \langle z, a\rangle z + O(|a|^2)$$

where $O(|a|^2)$ is uniform with respect to $(z, \zeta) \in \overline{B} \times \partial B$ when $|a|$ tends to 0. Then $v = u \circ \Phi_a + u \circ \Phi_{-a}$ is plurisubharmonic on B and

$$v = f \circ \Phi_a + f \circ \Phi_{-a} \leq 2f + K|a|^2 \quad \text{on } \partial B$$

where $K = \text{const}\, ||f||_{C^2(\partial B)}$. As above, we conclude that $\frac{1}{2}(v - K|a|^2) \leq u$ on B, hence with $h = a - \langle z, a\rangle z$ we get

$$u(z-h) + u(z+h) \leq v(z) + C'||u||_{C^1(\overline{B})}|a|^2 \leq 2u(z) + C''||f||_{C^2(\partial B)}|a|^2.$$

The inverse linear map $h \mapsto a$ has norm $\leq (1-|z|^2)^{-1}$, thus we finally get

$$u(z-h) + u(z+h) - 2u(z) \leq C''(1-|z|^2)^{-2}||f||_{C^2(\partial B)}|h|^2.$$

By taking a convolution with a regularizing kernel ρ_ε we infer

$$u_\varepsilon(z-h) + u_\varepsilon(z+h) - 2u_\varepsilon(z) \leq C''\big(1 - (|z|+\varepsilon)^2\big)^{-2}||f||_{C^2(\partial B)}|h|^2$$

with $u_\varepsilon = u \star \rho_\varepsilon$. A Taylor expansion of degree two of u_ε at z gives

$$D^2u_\varepsilon(z) \cdot h^2 \leq C''\big(1 - (|z|+\varepsilon)^2\big)^{-2}||f||_{C^2(\partial B)}|h|^2.$$

As u_ε is plurisubharmonic, u_ε has a semi-positive complex Hessian, that is, $D^2u_\varepsilon(z) \cdot h^2 + D^2u_\varepsilon(z) \cdot (ih)^2 \geq 0$. This implies

$$D^2u_\varepsilon(z) \cdot h^2 \geq -D^2u_\varepsilon(z) \cdot (ih)^2 \geq -C''\big(1 - (|z|+\varepsilon)^2\big)^{-2}||f||_{C^2(\partial B)}|h|^2,$$

thus $|D^2u_\varepsilon(z)| \leq C''\big(1 - (|z|+\varepsilon)^2\big)^{-2}||f||_{C^2(\partial B)}$. By taking the limit as ε tends to 0, we infer that the distribution D^2u has a L^∞_{loc} density such that $|D^2u(z)| \leq C''(1-|z|^2)^{-2}||f||_{C^2(\partial B)}$, in particular $u \in C^{1+\mathrm{lip}}(B)$. By exercising a little more care in the estimates, one can show in fact that $|D^2u(z)| \leq C(1-|z|^2)^{-1}||f||_{C^2(\partial B)}$ (see A. Dufresnoy [Duf]).

Step 3. The upper envelope u satisfies $(dd^cu)^n = 0$ on Ω.

We first prove the result under the additional assumption $u \in C^{1+\mathrm{lip}}(\Omega)$, which we know to be true if $\Omega = B$ and $f \in C^{1+\mathrm{lip}}(\partial\Omega)$. Then $D^2 u$ has second partial derivatives almost everywhere. As $D^2 u_\varepsilon$ converges to $D^2 u$ almost everywhere by Lebesgue's theorem, it is immediately seen by Th. 1.7 (b) that the Monge-Ampère current $(dd^c u)^n$ defined in § 1 coincides with the corresponding L^∞_{loc} form of type (n,n) obtained by a pointwise computation. The plurisubharmonicity of u implies $\det(\partial^2 u/\partial z_j \partial\overline{z}_k) \geq 0$. If the determinant were not equal to 0 almost everywhere, there would exist a point $z_0 \in \Omega$, say $z_0 = 0$ for simplicity, at which u would have second derivatives and such that $\det(\partial^2 u/\partial z_j\partial\overline{z}_k(0)) > 0$. Then the Taylor expansion of u at z_0 would give

$$u(z) = \operatorname{Re} P(z) + \sum c_{jk} z_j \overline{z}_k + o(|z|^2)$$

where P is a holomorphic polynomial of degree 2 and (c_{jk}) is a positive definite hermitian matrix. Hence we would have $u > \operatorname{Re} P + \varepsilon$ on a small sphere $S(0,r)$ with $\overline{B}(0,r) \subset \Omega$. The function

$$v = \begin{cases} \max\{u, \operatorname{Re} P + \varepsilon\} & \text{on} \quad B(0,r) \\ u & \text{on} \quad \overline{\Omega} \setminus B(0,r) \end{cases}$$

is then continuous on $\overline{\Omega}$ and plurisubharmonic, and satisfies

$$v = u \leq f \quad \text{on } \partial\Omega.$$

By the definition of u, we thus have $u \geq v$ on $\overline{\Omega}$. This is a contradiction because

$$v(0) > \operatorname{Re} P(0) = u(0).$$

Therefore we must have $(dd^c u)^n = 0$, as desired.

We first get rid of the additional assumption $u \in C^{1+\mathrm{lip}}(\Omega)$ when $\Omega = B$. Supposing only $f \in C^0(\partial B)$, we select a decreasing of functions $f_\nu \in C^2(\partial B)$ converging to f. It is then easy to see that the sequence of associated envelopes u_ν is decreasing and converges uniformly to u, with $||u_\nu - u||_{C^0(B)} \leq ||f_\nu - f||_{C^0(B)}$. Theorem 1.7 implies therefore $(dd^c u)^n = 0$. Before finishing the proof for arbitrary strongly pseudoconvex domains Ω, we infer the following:

(12.5) Corollary. *Fix a ball $\overline{B}(z_0,r) \subset \Omega$ and let $g \in P(\Omega)$ be locally bounded. There exists a function $\widetilde{g} \in P(\Omega)$ such that $\widetilde{g} \geq g$ on Ω, $\widetilde{g} = g$ on $\Omega \setminus B(0,r)$ and $(dd^c\widetilde{g})^n = 0$ on $B(z_0,r)$. Moreover, for $g_1 \leq g_2$ we have $\widetilde{g}_1 \leq \widetilde{g}_2$.*

Proof. Assume first that $g \in C^0(\Omega)$. By Th. 12.3 applied on $B(z_0,r)$, there exists a function u, plurisubharmonic and continuous on $\overline{B}(z_0,r)$, with $u = g$ on $S(z_0,r)$ and $(dd^c u)^n = 0$ on $B(z_0,r)$. Set

$$\widetilde{g} = \begin{cases} u & \text{on} \quad B(z_0, r) \\ g & \text{on} \quad \Omega \setminus B(z_0, r). \end{cases}$$

By definition of u, we have $\widetilde{g} = u \geq g$ on $B(z_0, r)$. Moreover, $\widetilde{g}$ is the decreasing limit of the plurisubharmonic functions

$$g_k = \begin{cases} \max\{u, g + \frac{1}{k}\} & \text{on} \quad B(z_0, r) \\ g + \frac{1}{k} & \text{near} \quad \Omega \setminus B(z_0, r) \end{cases}$$

hence $\widetilde{g}$ is plurisubharmonic. Also clearly, for $g_1 \leq g_2$ we have $u_1 \leq u_2$, hence $\widetilde{g}_1 \leq \widetilde{g}_2$. For an arbitrary locally bounded function $g \in P(\Omega)$, write g as a decreasing limit of smooth plurisubharmonic functions $g_k = g \star \rho_{1/k}$ and set $\widetilde{g} = \lim_{k \to +\infty} \downarrow \widetilde{g}_k$. Then $\widetilde{g}$ has all required properties. □

End of proof of Step 3 in Theorem 12.3. Apply Cor. 12.5 to $g = u$ on an arbitrary ball $\overline{B}(z_0, r) \subset \Omega$. Then we get a continuous plurisubharmonic function $\widetilde{u} \geq u$ with the same boundary values as u on $\partial\Omega$, so we must have $\widetilde{u} = u$. In particular $(dd^c u)^n = (dd^c \widetilde{u})^n = 0$ on $B(z_0, r)$. □

13. Extremal Functions and Negligible Sets

To study further properties of complex potential theory, it is necessary to make a much deeper study of upper envelopes.

(13.1) Definition. *A negligible set in an open set $\Omega \subset \mathbb{C}^n$ is a set of the form*

$$N = \{z \in \Omega\,;\ u(z) < u^\star(z)\}$$

where u is the upper envelope of a family (u_α) of plurisubharmonic functions which is locally bounded from above on Ω, and where $u^\star$ is the upper semicontinuous regularization of u.

(13.2) Proposition. *If $\Omega \subset \mathbb{C}^n$ is pseudoconvex, every pluripolar set $P = \{v = -\infty\}$ in Ω is negligible.*

Proof. Let $w \in P(\Omega) \cap C^\infty(\Omega)$ be such that $w \geq v$ and let $u_\alpha = (1-\alpha)v + \alpha w$, $\alpha \in]0, 1[$. Then u_α is increasing in α and $u = \sup_\alpha u_\alpha$ satisfies

$$\begin{aligned} u &= -\infty && \text{on} \ \{v = -\infty\}, \\ u &= w && \text{on} \ \{v > -\infty\}. \end{aligned}$$

Hence $u^\star = w$ and $\{u < u^\star\} = \{v = -\infty\}$. □

Next we consider the extremal function associated to a subset E of Ω :

(13.3) $$u_E(z) = \sup\{v(z); v \in P(\Omega),\ v \leq -1 \text{ on } E,\ v \leq 0 \text{ on } \Omega\}.$$

Proposition 12.2 implies $u_E^\star \in P(\Omega)$ and $-1 \leq u_E^\star \leq 0$. We prove the following three fundamental results by a simultaneous induction on n.

(13.4) Proposition. *Let $u, u_j \in P(\Omega)$ be locally bounded functions such that u_j increases to u almost everywhere. Then the measure $(dd^c u_j)^n$ converges weakly to $(dd^c u)^n$ on Ω.*

(13.5) Proposition. *Let Ω be a strongly pseudoconvex smooth open set in $\mathbb{C}^n$. If $K \subset \Omega$ is compact, then*

(a) $(dd^c u_K^\star)^n = 0$ *on* $\Omega \setminus K$.

(b) $c(K, \Omega) = \int_K (dd^c u_K^\star)^n = \int_\Omega (dd^c u_K^\star)^n$.

(13.6) Proposition. *If a Borel set $N \subset \Omega$ is negligible, then $c(N, \Omega) = 0$.*

The proof is made in three inductive steps.

Step 1: (13.4) in $\mathbb{C}^n \Rightarrow$ (13.5) in $\mathbb{C}^n$.
Step 2: (13.5) in $\mathbb{C}^n \Rightarrow$ (13.6) in $\mathbb{C}^n$.
Step 3: (13.4) and (13.6) in $\mathbb{C}^n \Rightarrow$ (13.4) in $\mathbb{C}^{n+1}$.

In the case $n = 1$, Prop. 13.4 is a well-known fact of distribution theory: u_j converges to u in $L^1_{\mathrm{loc}}(\Omega)$, thus $dd^c u_j$ converges weakly to $dd^c u$. By the inductive argument, Prop. 13.4, 13.5, 13.6 hold in all dimensions.

Proof of Step 1. By Choquet's lemma, there is a sequence of functions $v_j \in P(\Omega)$ such that $v_j \leq 0$ on Ω, $v_j \leq -1$ on K and $v^\star = u_K^\star$. If we replace v_j by $\max\{-1, v_1, \ldots, v_j\}$, we see that we may assume $v_j \geq -1$ for all j and v_j increasing. Then fix an arbitrary ball $B(z_0, r) \subset \Omega \setminus K$ and consider the increasing sequence $\widetilde{v}_j$ given by Cor. 12.5. We still have $\widetilde{v}_j \leq 0$ on Ω and $\widetilde{v}_j \leq -1$ on K, thus $v_j \leq \widetilde{v}_j \leq u_K$ and $\widetilde{v} = \lim \widetilde{v}_j$ satisfies $v^\star = \widetilde{v}^\star = u_K^\star$, in particular $\lim \widetilde{v}_j = \lim v_j = u_K^\star$ almost everywhere. Since $(dd^c \widetilde{v}_j)^n = 0$ on $B(z_0, r)$, we conclude by 13.4 that $(dd^c u_K^\star)^n = 0$ on $B(z_0, r)$ and 13.5 (a) is proved.

To prove 13.5 (b), we first observe that $-1 \leq u_K^\star \leq 0$ on Ω, hence $c(K, \Omega) \geq \int_K (dd^c u_K^\star)^n$ by definition of the capacity. If $\psi < 0$ is a smooth strictly plurisubharmonic exhaustion function of Ω, we have $A\psi \leq -1$ on K

for A large enough. We can clearly assume $v_j \geq A\psi$ on Ω ; otherwise replace v_j by $\max\{v_j, A\psi\}$. Now, let $w \in P(\Omega)$ be such that $0 \leq w \leq 1$ and set

$$w' = (1-\varepsilon)w - 1 + \varepsilon/2, \qquad w_j = \max\{w', v_j\}.$$

Since $-1+\varepsilon/2 \leq w' \leq -\varepsilon/2$ on Ω, we have $w_j = v_j$ as soon as $A\psi > -\varepsilon/2$, whereas $w_j = w' \geq -1+\varepsilon/2 > v_j$ on a neighborhood of K. Hence for $\delta > 0$ small enough Stokes' theorem implies

$$\int_{\Omega_\delta} (dd^c v_j)^n = \int_{\Omega_\delta} (dd^c w_j)^n \geq \int_K (dd^c w_j)^n = (1-\varepsilon)^n \int_K (dd^c w)^n.$$

By 13.4 $(dd^c v_j)^n$ converges weakly to $(dd^c u_K^\star)^n$ and we get

$$\limsup_{j \to +\infty} \int_{\Omega_\delta} (dd^c v_j)^n \leq \int_{\overline{\Omega}_\delta} (dd^c u_K^\star)^n = \int_K (dd^c u_K^\star)^n.$$

Therefore $\int_K (dd^c w)^n \leq \int_K (dd^c u_K^\star)^n$ and $c(K, \Omega) \leq \int_K (dd^c u_K^\star)^n$. □

Proof of Step 2. Let $N = \{v < v^\star\}$ with $v = \sup v_\alpha$. By Choquet's lemma, we may assume that v_α is an increasing sequence of plurisubharmonic functions. The theorem of quasicontinuity shows that there exists an open set $G \subset \Omega$ such that all functions v_α and $v^\star$ are continuous on $\Omega \setminus G$ and $c(G, \Omega) < \varepsilon$. Write

$$N \subset G \cup (N \cap (\Omega \setminus G)) = G \cup \bigcup_{\delta,\lambda,\mu \in \mathbb{Q}} K_{\delta\lambda_\mu}$$

where $\delta > 0$, $\lambda < \mu$ and

$$K_{\delta\lambda\mu} = \big\{z \in \overline{\Omega}_\delta \setminus G\,;\ v(z) \leq \lambda < \mu \leq v^\star(z)\big\}.$$

As $v^\star$ is continuous and v lower semi-continuous on $\Omega \setminus G$, we see that $K_{\delta\lambda\mu}$ is compact. We only have to prove that $c(K_{\delta\lambda\mu}, \Omega) = 0$. Set $K = K_{\delta\lambda\mu}$ for simplicity and take an open set $\omega \subset\subset \Omega$. By subtracting a large constant, we may assume $v^\star \leq 0$ on $\overline{\omega}$.

Multiplying by another constant, we may set $\lambda = -1$. Then all v_α satisfy $v_\alpha \leq 0$ on ω and $v_\alpha \leq v \leq -1$ on K. We infer that the extremal function u_K on ω satisfies $u_K \geq v$, $u_K^\star \geq v^\star$, in particular $u_K^\star \geq \mu > -1$ on K. By Prop. 11.9 we obtain

$$c(K, \omega) = \int_K (dd^c u_K^\star)^n \leq \int_K (dd^c \max\{u_K^\star, \mu\})^n \leq |\mu|^n c(K, \omega)$$

because $-1 \leq |\mu|^{-1} \max\{u_K^\star, \mu\} \leq 0$. As $|\mu| < 1$, we infer that $c(K, \omega) = 0$, hence $c(K, \Omega) = 0$. □

Proof of Step 3. We have to show that if $\Omega \subset \mathbb{C}^{n+1}$,

$$\lim_{j\to+\infty} \int_\Omega \chi(dd^c u_j)^{n+1} = \int_\Omega \chi(dd^c u)^{n+1}$$

for all test functions $\chi \in C_0^\infty(\Omega)$. That is,

$$\lim_{j\to+\infty} \int_\Omega u_j(dd^c u_j)^n \wedge \gamma = \int_\Omega u(dd^c u)^n \wedge \gamma \tag{13.7}$$

with $\gamma = dd^c\chi$. As all $(1,1)$-forms γ can be written as linear combinations of forms of the type $\mathrm{i}\alpha \wedge \overline{\alpha}$, $\alpha \in \Lambda^{1,0}(\mathbb{C}^n)^\star$, it is sufficient, after a change of coordinates, to consider forms of the type $\gamma = \frac{\mathrm{i}}{2}\chi(z)dz_{n+1} \wedge d\overline{z}_{n+1}$ with $\chi \in C_0^\infty(\Omega)$. In this case, for any locally bounded plurisubharmonic function v on Ω, the Fubini theorem yields

$$\int_\Omega v(dd^c v)^n \wedge \gamma = \int_{\mathbb{C}} d\lambda(z_{n+1}) \int_{\Omega(z_{n+1})} \chi(\bullet, z_{n+1})(dd^c v(\bullet, z_{n+1}))^n$$

where $\Omega(z_{n+1}) = \{z \in \mathbb{C}^n; (z, z_{n+1}) \in \Omega\}$ and $f(\bullet, z_{n+1})$ denotes the function $z \mapsto f(z, z_{n+1})$ on $\Omega(z_{n+1})$. Indeed, the result is clearly true if v is smooth. The general case follows by taking smooth plurisubharmonic functions v_j decreasing to v. The convergence of both terms in the equality is guaranteed by Th. 1.7 (a), combined with 1.3 and the bounded convergence theorem for the right hand side.

In order to prove (13.7), we thus have to show

$$\lim_{j\to+\infty} \int_\omega \chi u_j(dd^c u_j)^n = \int_\omega \chi u(dd^c u)^n \tag{13.8}$$

for $\omega \subset \mathbb{C}^n$, $\chi \in C_0^\infty(\omega)$ and $u_j \in P(\omega) \cap L^\infty_{\mathrm{loc}}(\omega)$ increasing to $u \in P(\omega)$ almost everywhere. To prove (13.8), we can clearly assume $0 \le \chi \le 1$ and $0 \le u_j \le u \le 1$ on Ω. By our inductive hypothesis 13.4, $(dd^c u_j)^n$ converges weakly to $(dd^c u)^n$. As $u_j \le u \le u_\varepsilon = u \star \rho_\varepsilon$, we get

$$\begin{aligned}\limsup_{j\to+\infty} \int_\omega \chi u_j(dd^c u_j)^n &\le \lim_{\varepsilon\to 0}\lim_{j\to+\infty} \int_\omega \chi u_\varepsilon(dd^c u_j)^n \\ &= \lim_{\varepsilon\to 0} \int_\omega \chi u_\varepsilon(dd^c u)^n = \int_\omega \chi u(dd^c u)^n.\end{aligned}$$

To prove the other inequality, let $\varepsilon > 0$ and choose an open set $G \subset \omega$ such that $c(G,\omega) < \varepsilon$ and u, u_j are all continuous on $\omega \setminus G$. Let $v = \sup u_j$. Then $v^\star = u$ because $v^\star$ and u are plurisubharmonic and coincide almost everywhere. Let $\widetilde{u}_j$ be a continuous extension of $u_j|_{\omega\setminus G}$ to ω such that $0 \le \widetilde{u}_j \le 1$. For $j \ge k$ we have $u_j \ge u_k$, hence

$$\int_\omega \chi u_j (dd^c u_j)^n \geq \int_{\omega\setminus G} \chi \widetilde{u}_k (dd^c u_j)^n$$
$$\geq \int_\omega \chi \widetilde{u}_k (dd^c u_j)^n - \int_G (dd^c u_j)^n.$$

The last integral on the right is $\leq c(G,\omega) < \varepsilon$. Taking the limit as j tends to $+\infty$, we obtain

$$\liminf_{j\to+\infty} \int_\omega \chi u_j (dd^c u_j)^n \geq \int_\omega \chi \widetilde{u}_k (dd^c u)^n - \varepsilon$$
$$\geq \int_\omega \chi u_k (dd^c u)^n - 2\varepsilon.$$

The second term ε comes from $\int_G (dd^c u)^n \leq c(G,\omega) < \varepsilon$. Now let $k \to +\infty$ and $\varepsilon \to 0$ to get

$$\liminf_{j\to+\infty} \int_\omega \chi u_j (dd^c u_j)^n \geq \int_\omega \chi v (dd^c u)^n.$$

Moreover, the Borel set $N = \{v < u = v^\star\}$ is negligible and the inductive hypothesis 13.6 implies $c(N,\omega) = 0$. Therefore

$$\int_\omega \chi(u-v)(dd^c u)^n \leq \int_N (dd^c u)^n = 0$$

and the proof is complete. □

(13.9) Theorem. *For each $j = 1,\ldots,q$, let u_j^k be an increasing sequence of locally bounded plurisubharmonic functions such that u_j^k converges almost everywhere to $u_j \in P(\Omega)$. Then*

(a) $dd^c u_1^k \wedge \ldots \wedge dd^c u_q^k \to dd^c u_1 \wedge \ldots \wedge dd^c u_q$ *weakly.*

(b) $u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \to u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q$ *weakly.*

Proof. (a) Without loss of generality, we may assume $q = n$, otherwise we complete with additional stationary sequences $u_{q+1}^k = u_{q+1}, \ldots, u_n^k = u_n$ where $u_{q+1}, \ldots, u_n$ are chosen arbitrarily in $P(\Omega) \cap C^\infty(\Omega)$. Now apply Prop. 13.4 to $u_k = \lambda_1 u_1^k + \cdots + \lambda_n u_n^k$, $\lambda_j > 0$, and consider the coefficient of $\lambda_1 \ldots \lambda_n$ in $(dd^c u_k)^n$.

(b) Same proof as for (13.8). □

(13.10) Example. Let $\Omega = B(0,R)$ and $K = \overline{B}(0,r)$. The corresponding extremal function u_K is

$$u_K(z) = \Big(\log\frac{R}{r}\Big)^{-1}\max\Big\{\log\frac{|z|}{R}, \log\frac{r}{R}\Big\}.$$

In fact, for any $v \in P(\Omega)$ with $v \le -1$ on K and $v \ge 0$ on Ω, the convexity of $\log\rho \mapsto \sup_{B(0,\rho)} v$ shows that $v \le u_K$. Formula 13.5 (b) then gives

$$c(K,\Omega) = \int_\Omega (dd^c u_K)^n = \Big(\log\frac{R}{r}\Big)^{-n}$$

where $(dd^c u_K)^n$ is a unitary invariant measure supported on the sphere $S(0,r)$ (see example (4.3)). □

Next we quote a few elementary properties of the extremal functions $u_E^\star$ for arbitrary sets $E \subset \Omega$.

(13.11) Properties.

(a) *if* $E_1 \subset E_2 \subset \Omega$, *then* $u_{E_1}^\star \ge u_{E_2}^\star$.

(b) *if* $E \subset \Omega_1 \subset \Omega_2$, *then* $u_{E,\Omega_1}^\star \ge u_{E,\Omega_2}^\star$.

(c) *if* $E \subset \Omega$, *then* $u_E^\star = u_E = -1$ *on* E^0 *and* $(dd^c u_E^\star)^n = 0$ *on* $\Omega \setminus \overline{E}$; *hence* $(dd^c u_E^\star)^n$ *is supported by* ∂E.

(d) *one has* $u_E^\star \equiv 0$ *if and only if there exists* $v \in P(\Omega)$, $v \le 0$ *such that* $E \subset \{v = -\infty\}$.

(e) *if* $E \subset\subset \Omega$ *and if* Ω *is strongly pseudoconvex with exhaustion* $\psi < 0$, *then* $u_E^\star \ge A\psi$ *for some* $A > 0$.

Proof. (a), (b) are obvious from Def. (13.3); (e) is true as soon as $A\psi \le -1$ on E; the equality $(dd^c u_E^\star)^n = 0$ on $\Omega \setminus \overline{E}$ in (c) is proved exactly in the same way as 13.5 (a) in Step 1.

(d) If $E \subset \{v = -\infty\}$, $v \in P(\Omega)$, $v \le 0$, then for every $\varepsilon > 0$ we have $\varepsilon v \le u_E$, hence $u_E = 0$ on $\Omega \setminus \{v = -\infty\}$ and $u_E^\star = 0$.

Conversely, Choquet's lemma shows that there is an increasing sequence $v_j \in P(\Omega)$, $-1 \le v_j \le u_E$, converging almost everywhere to $u_E^\star$. If $u_E^\star = 0$, we can extract a subsequence in such a way that $\int_\Omega |v_j| d\lambda < 2^{-j}$. As $v_j \le 0$ and $v_j \le -1$ on E, the function $v = \sum v_j$ is plurisubharmonic ≤ 0 and $v = -\infty$ on E. □

(13.12) Proposition. *Let* $\Omega \subset\subset \mathbb{C}^n$ *and let* $K_1 \supset K_2 \supset \ldots$, $K = \bigcap K_j$ *be compact subsets of* Ω. *Then*

(a) $\big(\lim_{j\to+\infty}\uparrow u^\star_{K_j}\big)^\star = u^\star_K$.

(b) $\lim c(K_j,\Omega) = c(K,\Omega)$.

(c) $c^\star(K,\Omega) = c(K,\Omega)$.

Proof. We have $\lim\uparrow u^\star_{K_j} \le u^\star_K$ by 13.11 (a). On the other hand, let $v\in P(\Omega)$ be such that $v\le 0$ on Ω and $v\le -1$ on K. For every $\varepsilon>0$ the open set $\{v<-1+\varepsilon\}$ is a neighborhood of K, thus $K\subset\{v<-1+\varepsilon\}$ for j large. We obtain therefore $v-\varepsilon\le u^\star_{K_j}$ and $u_K=\sup\{v\}\le \lim u^\star_{K_j}$, whence equality (a). Property (b) follows now from Prop. 13.4 and 13.5 (b), and (c) is a consequence of (b) when the K_j's are neighborhoods of K. □

(13.13) Corollary. *Every negligible set $N\subset\Omega$ satisfies $c^\star(N,\Omega)=0$.*

Proof. By Choquet's lemma every negligible set is contained in a Borel negligible set $N=\{v<v^\star\}$ with $v=\sup v_j$. In Step 2 of the proof of Prop. 13.4, 13.5 and 13.6, we already showed that $N\subset G\cup\bigcup K_{\delta\lambda\mu}$ with G open, $c(G,\Omega)<\varepsilon$ and $c(K_{\delta\lambda\mu},\Omega)=0$. By definition $c^\star(G,\Omega)=c(G,\Omega)$, and by 13.12 (c) we also have $c^\star(K_{\delta\lambda\mu},\Omega)=0$, therefore $c^\star(N,\Omega)<\varepsilon$ for every $\varepsilon>0$. □

We finally extend the basic formula 13.5 (b) to arbitrary relatively compact subsets $E\subset\subset\Omega$. We first need a lemma.

(13.14) Lemma. *Let $\Omega\subset\subset\mathbb{C}^n$ and let $u,v\in P(\Omega)$ be locally bounded plurisubharmonic functions such that $u\le v\le 0$ and $\lim_{z\to\partial\Omega} u(z)=0$. Then*

$$\int_\Omega (dd^cv)^n \le \int_\Omega (dd^cu)^n.$$

Moreover $\int_\Omega (dd^cu)^n = 0$ if and only if $u=0$.

Proof. As $\max\{u+\varepsilon,v\}=u+\varepsilon$ near $\partial\Omega$, we get

$$\int_\Omega (dd^cu)^n = \int_\Omega (dd^c\max\{u+\varepsilon,v\})^n.$$

Let ε tend to 0, and observe that the integrand on the right hand side converges weakly to $(dd^cv)^n$ by Th. 1.7. The asserted inequality follows.

Now, assume that $u(z_0)<0$ at some point. Then

$$v(z)=\max\{u(z),\varepsilon^2|z|^2-\varepsilon\}$$

coincides with u near $\partial\Omega$ and with $\varepsilon^2|z|^2 - \varepsilon$ on a neighborhood ω of z_0. We get therefore

$$\int_\Omega (dd^c u)^n = \int_\Omega (dd^c v)^n \geq \int_\omega (dd^c v)^n > 0. \qquad \square$$

(13.15) Proposition. *Let $\Omega \subset\subset \mathbb{C}^n$ be strongly pseudoconvex. If $E \subset\subset \Omega$ is an arbitrary subset, then*

$$c^\star(E, \Omega) = \int_\Omega (dd^c u_E^\star)^n.$$

Proof. We first show that

$$c(G, \Omega) = \int_{\overline{G}} (dd^c u_G^\star)^n = \int_\Omega (dd^c u_G^\star)^n \tag{13.16}$$

for every open set $G \subset\subset \Omega$. Let $K_1 \subset K_2 \subset \ldots$ be compact subsets of G with $K_j \subset K_{j+1}^0$ and $\bigcup K_j = G$. Then $u_{K_j}^\star = -1$ on $K_j^0 \supset K_{j-1}$ and $\lim \downarrow u_{K_j}^\star = -1$ on G. Therefore $u_G^\star \leq \lim u_{K_j}^\star \leq u_G \leq u_G^\star$ and Th. 1.7, 13.5 (b), 13.11 (c) imply the above equality (13.16).

Now, let $E \subset\subset \Omega$ be given and let $\psi < 0$ be a strictly plurisubharmonic exhaustion function on Ω. For every open set $G \supset E$ with $G \subset\subset \Omega$, we have $u_G^\star \geq A\psi$ and $u_E^\star \geq u_G^\star$ by 13.11. Lemma 13.14 implies

$$\int_\Omega (dd^c u_E^\star)^n \leq \int_\Omega (dd^c u_G^\star)^n = c(G, \Omega),$$

thus $\int_\Omega (dd^c u_E^\star)^n \leq c^\star(E, \Omega)$.

Conversely, Choquet's lemma shows that there exists an increasing sequence $v_j \in P(\Omega)$ with $-1 \leq v_j \leq 0$, $v_j \geq A\psi$ on Ω and $\lim v_j = u_E$ almost everywhere. If

$$G_j = \{z \in \Omega; (1 + 1/j)v_j(z) < -1\}$$

then $G_j \supset E$, G_j is decreasing and $(1 + 1/j)v_j \leq u_{G_j}$. Thus $\lim \uparrow u_{G_j}^\star = u_E^\star$ almost everywhere and Th. 13.4 gives

$$\lim_{j \to +\infty} \int_\Omega (dd^c u_{G_j}^\star)^n = \int_\Omega (dd^c u_E^\star)^n. \qquad \square$$

(13.17) Corollary. *Let $\Omega \subset\subset \mathbb{C}^n$ be strongly pseudoconvex. If $E \subset\subset \Omega$, then $c^\star(E, \Omega) = 0$ if and only if $u_E^\star = 0$.* $\square$

14. Outer Regularity of Monge-Ampère Capacities and Characterization of Pluripolar and Negligible Sets

We first check that the outer Monge-Ampère capacity $c^\star$ is actually a capacity.

(14.1) Theorem. *Let $\Omega \subset\subset \mathbb{C}^n$ be strongly pseudoconvex. Then the outer capacity $c^\star(\bullet, \Omega)$ is a capacity in the sense of Def.* 10.1.

Proof. Axiom 10.1 (a) is clear, and 10.1 (c) is a consequence of 13.12. To prove 10.1 (b), we only have to show that $c^\star(\bigcup E_j, \Omega) \le \lim_{j\to+\infty} c^\star(E_j, \Omega)$. It is no loss of generality to assume that $E_j \subset\subset \Omega$. Let N_j be the negligible set $N_j = \{u_{E_j} < u^\star_{E_j}\}$ and G_0 an open subset of Ω with $G_0 \supset \bigcup N_j$ and $c(G_0, \Omega) < \varepsilon$ (such an open set exists by Cor. 13.13). Consider the open sets defined by $V_j = \{u^\star_{E_j} < -1 + \eta\}$ and $G_j = G_0 \cup V_j \supset E_j$. Then $(1-\eta)^{-1} u^\star_{E_j} \le u^\star_{V_j} \le 0$ and Lemma 13.14 implies

$$\begin{aligned} c(G_j, \Omega) &\le \varepsilon + c(V_j, \Omega) = \varepsilon + \int_\Omega (dd^c u^\star_{V_j})^n \\ &\le \varepsilon + (1-\eta)^{-n} \int_\Omega (dd^c u^\star_{E_j})^n = \varepsilon + (1-\eta)^{-n} c^\star(E_j, \Omega), \end{aligned}$$

thanks to Prop. 13.15. Further $E_j \subset G_j$ and $G_1 \subset G_2 \subset \ldots$ since $u^\star_{E_j}$ is decreasing. Thus $G = \bigcup G_j \supset E = \bigcup E_j$ and

$$c(G, \Omega) = \lim_{j\to+\infty} c(G_j, \Omega) \le \varepsilon + (1-\eta)^{-n} \lim_{j\to+\infty} c^\star(E_j, \Omega).$$

Letting $\varepsilon, \eta \to 0$ we get the desired inequality

$$c^\star(E, \Omega) \le \lim_{j\to+\infty} c^\star(E_j, \Omega). \qquad \square$$

(14.2) Corollary. *If $\Omega \subset\subset \mathbb{C}^n$ is strongly pseudoconvex, then $c(\bullet, \Omega)$ is outer regular and every K-analytic subset* (*in particular every Borel subset*) *of Ω is capacitable.*

Proof. Choquet's capacitability theorem combined with 13.12 (c) implies that every K-analytic subset $E \subset \Omega$ satisfies

$$c^\star(E, \Omega) = \sup_{K \text{ compact} \subset E} c^\star(K, \Omega) = \sup_{K \text{ compact} \subset E} c(K, \Omega) = c_\star(E, \Omega). \qquad \square$$

Now, we prove an important result due to Josefson [Jo]. A set P in $\mathbb{C}^n$ is said to be *locally pluripolar* if for each $z \in P$ there is an open neighborhood Ω of z and $v \in P(\Omega)$ such that $P \cap \Omega \subset \{v = -\infty\}$.

(14.3) Theorem (Josefson)**.** *If $P \subset \mathbb{C}^n$ is locally pluripolar, there exists $v \in P(\mathbb{C}^n)$ with $P \subset \{v = -\infty\}$, i.e. P is globally pluripolar in $\mathbb{C}^n$. Moreover, v can be taken to have logarithmic growth at infinity, such that $v(z) \le \log_+ |z|$.*

Proof. By the definition of locally pluripolar, we can find sets P_j, Ω_j with Ω_j open and strictly pseudoconvex, $P_j \subset\subset \Omega_j \subset\subset \mathbb{C}^n$, $\bigcup_{j\ge 1} P_j = P$ and such that P_j is contained in the $-\infty$ poles of a single plurisubharmonic function in Ω_j. By 13.11 (d) and 13.15, we have $c^\star(P_j, \Omega_j) = 0$.

Let B_k be the ball of center 0 and radius e^{2^k} in $\mathbb{C}^n$ and let $j(k)$ be a sequence of integers such that each integer is repeated infinitely many times and $\Omega_{j(k)} \subset B_k$. By the comparison result (11.3) we have $c^\star(P_{j(k)}, B_{k+1}) = 0$, hence the extremal function $u^\star_{P_{j(k)}}$ in B_{k+1} is zero and we can find $v_k \in P(B_{k+1})$ with $-1 \le v_k \le 0$, $v_k = -1$ on $P_{j(k)}$ and $\int_{B_k} |v_k| d\lambda < 2^{-k}$. Now set

$$\widetilde{v}_k(z) = \begin{cases} v_k(z) & \text{on } B_k \\ \max\{v_k(z), 2^{-k}\log|z| - 2\} & \text{on } B_{k+1} \setminus B_k \\ 2^{-k}\log|z| - 2 & \text{on } \mathbb{C}^n \setminus B_{k+1}. \end{cases}$$

As $\widetilde{v}_k \le 0$ on B_k and $\int_{B_k} |\widetilde{v}_k| d\lambda < 2^{-k}$, the series $v = \sum \widetilde{v}_k$ converges to a global plurisubharmonic function $v \not\equiv -\infty$ on $\mathbb{C}^n$. Moreover $\widetilde{v}_k = -1$ on $P_{j(k)}$ and each P_j is repeated infinitely many times, therefore

$$v = -\infty \quad \text{on} \quad \bigcup P_j = P.$$

Finally, we have $v(z) \le \sum 2^{-k} \log_+ |z| \le \log_+ |z|$ (the logarithmic growth estimate was first obtained by H. El Mir [E-M1] and J. Siciak [Sic]).

(14.4) Corollary. *Let $\Omega \subset \mathbb{C}^n$ and $P \subset \Omega$. Then P is pluripolar in Ω if and only if $c^\star(P, \Omega) = 0$.*

Proof. That P is pluripolar implies $c^\star(P, \Omega) = 0$ was proved in Cor. 11.6. Conversely, if $c^\star(P, \Omega) = 0$ then $c^\star(P \cap \omega', \omega) = 0$ for all concentric balls $\omega' \subset\subset \omega \subset\subset \Omega$ and Cor. 13.17 combined with 13.11 (d) shows that $P \cap \omega'$ is pluripolar in ω. Josefson's theorem implies that P is globally pluripolar in $\mathbb{C}^n$. □

(14.5) Corollary. *Negligible sets are the same as pluripolar sets.*

Proof. We have already seen that pluripolar sets are negligible (by Prop. 13.2 applied in $\mathbb{C}^n$). Conversely, a negligible set $N \subset \Omega$ satisfies $c^\star(N, \Omega) = 0$ by Cor. 13.13, hence N is pluripolar. □

15. Siciak Extremal Functions and Alexander Capacity

We work here on the whole space $\mathbb{C}^n$ rather than on a bounded open subset Ω. In this case, the relevant class of plurisubharmonic functions to consider is the set $P_{\log}(\mathbb{C}^n)$ of plurisubharmonic functions v with logarithmic growth at infinity, i.e. such that

$$v(z) \leq \log_+ |z| + C \tag{15.1}$$

for some real constant C. Let E be a bounded subset of $\mathbb{C}^n$. We consider the global extremal function introduced by Siciak [Sic]:

$$U_E(z) = \sup\{v(z)\,;\ v \in P_{\log}(\mathbb{C}^n),\ v \leq 0 \text{ on } E\}. \tag{15.2}$$

(15.3) Theorem. *Let E be a bounded subset of $\mathbb{C}^n$. We have $U_E^\star \equiv +\infty$ if and only if E is pluripolar. Otherwise, $U_E^\star \in P_{\log}(\mathbb{C}^n)$ and $U_E^\star$ satisfies an inequality*

$$\log_+(|z|/R) \leq U_E^\star(z) \leq \log_+ |z| + M$$

for suitable constants $M, R > 0$. Moreover $U_E^\star = 0$ on E^0,

$$(dd^c U_E^\star) = 0 \text{ on } \mathbb{C}^n \setminus \overline{E}, \qquad \int_{\overline{E}} (dd^c U_E^\star)^n = 1.$$

Proof. If E is pluripolar, Th. 14.3 shows that there exists $v \in P_{\log}(\mathbb{C}^n)$ such that $v = -\infty$ on E. Then $v + C \leq 0$ on E for every $C > 0$ and we get $U_E = +\infty$ on $\mathbb{C}^n \setminus v^{-1}(-\infty)$, thus $U_E^\star \equiv +\infty$. Conversely, if $U_E^\star \equiv +\infty$, there is a point z_0 in the unit ball $B \subset \mathbb{C}^n$ such that $U_E(z_0) = +\infty$. Therefore, there is a sequence of functions $v_k \in P_{\log}(\mathbb{C}^n)$ with $v_k \leq 0$ on E and $M_k = \sup_B v_k \to +\infty$. By taking a suitable subsequence we may assume $M_k \geq 2^{k+1}$. We claim that $w = \sum_{k \in \mathbb{N}} 2^{-k-1}(v_k - M_k)$ satisfies $w = -\infty$ on E (obvious), $w \in P_{\log}(\mathbb{C}^n)$ and $w \not\equiv -\infty$; this will imply that E is actually pluripolar. In fact, if v is plurisubharmonic on $\mathbb{C}^n$, we have $\sup_{B(0,r)} v = \chi(\log r)$ where χ is a convex increasing function, and the condition $v \in P_{\log}(\mathbb{C}^n)$ implies $\chi' \leq 1$ on $\mathbb{R}$; therefore

$$\chi(\log r) \le \chi(\log r_0) + \log r/r_0 \quad \text{for } r \ge r_0.$$

In our case, by taking $v = v_k - M_k$ and $r_0 = 1$, we find $v_k(z) - M_k \le \log_+ |z|$, thus $w(z) \le \log_+ |z|$ and $w \in P_{\log}(\mathbb{C}^n)$. Moreover, by the maximum principle, there is a point $z_k \in \partial B$ at which $v_k(z_k) - M_k = 0$. The Harnack inequality for subharmonic functions applied to the nonpositive function $v = v_k - M_k - \log 2$ on $B(0,2) \subset \mathbb{R}^{2n}$ shows that

$$\begin{aligned} 0 &\ge \int_{S(0,2)} (v_k(z) - M_k - \log 2)\, d\sigma(z) \\ &= \int_{S(0,2)} v(z)\, d\sigma(z) \ge -C\, v(z_k) = -C \log 2 \end{aligned}$$

where $C = (1+1/2)^{2n}/(1-1/4) > 0$ and $d\sigma$ is the unit invariant measure on the sphere. Hence $\int_{S(0,2)} w(z)\, d\sigma(z) > -\infty$, $w \not\equiv -\infty$, and E is pluripolar.

Now, assume that E is not pluripolar. The above arguments show that there must exist a uniform upper bound $\sup_B v \le M$ for all functions $v \in P_{\log}(\mathbb{C}^n)$ with $v \le 0$ on E, thus $v(z) \le \log_+ |z| + M$ and $U_E^\star(z) = (\sup\{v\})^\star \le \log_+ |z| + M$. This implies that $U_E^\star \in P_{\log}(\mathbb{C}^n)$. On the other hand, E is contained in a ball $\overline{B}(0,R)$ so $\log_+ |z|/R \le 0$ on E and we get $U_E(z) \ge \log_+ |z|/R$. The equality $(dd^c U_E^\star)^n = 0$ on $\mathbb{C}^n \setminus \overline{E}$ is verified exactly in the same way as 12.8 (a). The value of the integral of $(dd^c U_E^\star)^n$ over $\overline{E}$ is obtained by the following lemma. □

(15.4) Lemma. *Let $v \in P_{\log}(\mathbb{C}^n)$ be such that*

$$\log_+ |z| - C_1 \le v(z) \le \log_+ |z| + C_2$$

for some constants. Then $\int_{\mathbb{C}^n} (dd^c v)^n = 1$.

Proof. It is sufficient to check that

$$\int_{\mathbb{C}^n} (dd^c v_1)^n \le \int_{\mathbb{C}^n} (dd^c v_2)^n$$

when v_1, v_2 are two such functions. Indeed, we have

$$\int_{\mathbb{C}^n} (dd^c \log_+ |z|)^n = \int_{\mathbb{C}^n} (dd^c \log |z|)^n = 1$$

by Stokes' theorem and remark 3.10, and we only have to choose $v_1(z)$ or $v_2(z) = \log_+ |z|$ and the other function equal to v. To prove the inequality, fix $r, \varepsilon > 0$ and choose $C > 0$ large enough so that $(1-\varepsilon)v_1 > v_2 - C$ on $B(0,r)$. As the function $u = \max\{(1-\varepsilon)v_1, v_2 - C\}$ is equal to $v_2 - C$ for $|z| = R$ large, we get

$$(1-\varepsilon)^n \int_{B(0,r)} (dd^c v_1)^n = \int_{B(0,r)} (dd^c u)^n \leq \int_{B(0,R)} (dd^c u)^n = \int_{B(0,R)} (dd^c v_2)^n$$

and the expected inequality follows as $\varepsilon \to 0$ and $r \to +\infty$. □

(15.5) Theorem. *Let $E, E_1, E_2, \ldots \subset B(0,R) \subset \mathbb{C}^n$.*

(a) *If $E_1 \subset E_2$, then $U^\star_{E_1} \geq U^\star_{E_2}$.*

(b) *If $E_1 \subset E_2 \subset \ldots$ and $E = \bigcup E_j$, then $U^\star_E = \lim \downarrow U^\star_{E_j}$.*

(c) *If $K_1 \supset K_2 \supset \ldots$ and $K = \bigcap K_j$, then $U^\star_K = (\lim \uparrow U^\star_{K_j})^\star$.*

(d) *For every set E, there exists a decreasing sequence of open sets $G_j \supset E$ such that $U^\star_E = (\lim \uparrow U^\star_{G_j})^\star$.*

Proof. (a) is obvious and the proof of (c) is similar to that of 13.12 (a).

(d) By Choquet's lemma, there is an increasing sequence $v_j \in P_{\log}(\mathbb{C}^n)$ with $U^\star_E = (\lim v_j)^\star$ and $v_j(z) \geq \log_+ |z|/R$. Set $G_j = \{v_j < 1/j\}$ and observe that $U^\star_{G_j} \geq v_j - 1/j$.

(b) Set $v = \lim \downarrow U^\star_{E_j}$. Then $v \in P_{\log}(\mathbb{C}^n)$ and $v = 0$ on E, except on the negligible set $N = \bigcup \{U_{E_j} < U^\star_{E_j}\}$. By Josefson's theorem 14.3, there exists $w \in P(\mathbb{C}^n)$ such that $N \subset \{w = -\infty\}$ and $w(z) \leq \log_+(|z|/R)$, in particular $w \leq 0$ on E. We set

$$v_j(z) = \Big(1 - \frac{1}{j}\Big) v(z) + \frac{1}{j} w(z).$$

Then $v_j \in P_{\log}(\mathbb{C}^n)$ and $v_j \leq 0$ everywhere on E. Therefore

$$U^\star_E \geq U_E \geq v_j = \Big(1 - \frac{1}{j}\Big) v + \frac{1}{j} w \quad \text{on } \mathbb{C}^n$$

and letting $j \to +\infty$ we get $U^\star_E \geq v$. The converse inequality $U^\star_E \leq \lim \downarrow U^\star_{E_j}$ is clear. □

Now, we show that the extremal function of a compact set can be computed in terms of polynomials. We denote by $\mathcal{P}_d$ the space of polynomials of degree $\leq d$ in $\mathbb{C}[z_1, \ldots, z_n]$.

(15.6) Theorem. *Let K be a compact subset of $\mathbb{C}^n$. Then*

$$U_K(z) = \sup\Big\{\frac{1}{d}\log|P(z)|\,;\, d \ge 1,\ P \in \mathcal{P}_d,\ \|P\|_{L^\infty(K)} \le 1\Big\}.$$

Proof. For any of the polynomials P involved in the above formula, we clearly have $\frac{1}{d}\log|P| \in P_{\log}(\mathbb{C}^n)$ and this function is ≤ 0 on K. Hence

$$\frac{1}{d}\log|P| \le U_K.$$

Conversely, fix a point $z_0 \in \mathbb{C}^n$ and a real number $a < U_K(z_0)$. Then there exists $v \in P_{\log}(\mathbb{C}^n)$ such that $v \le 0$ on K and $v(z_0) > a$. Replacing v by $v \star \rho_\delta - \varepsilon$ with $\delta \ll \varepsilon \ll 1$, we may assume that $v \in P_{\log}(\mathbb{C}^n) \cap C^\infty(\mathbb{C}^n)$, $v < 0$ on K and $v(z_0) > a$. Choose a ball $B(z_0, r)$ on which $v > a$, a smooth function χ with compact support in $B(z_0, r)$ such that $\chi = 1$ on $B(z_0, r/2)$ and apply Hörmander's L^2 estimates to the closed $(0,1)$-form $d''\chi$ and to the weight

$$\varphi(z) = 2dv(z) + 2n\log|z - z_0| + \varepsilon\log(1 + |z|^2).$$

We find a solution f of $d''f = d''\chi$ such that

$$\begin{aligned}\int_{\mathbb{C}^n} &|f|^2 e^{-2dv}|z - z_0|^{-2n}(1 + |z|^2)^{-\varepsilon} d\lambda \\ &\le \int_{B(z_0,r)} |d''\chi|^2 e^{-2dv}|z - z_0|^{-2n}(1 + |z|^2)^{2-\varepsilon} d\lambda \le C_1 e^{-2da}.\end{aligned}$$

We thus have $f(z_0) = 0$ and $F = \chi - f$ is a holomorphic function on $\mathbb{C}^n$ such that $F(z_0) = 1$. In addition we get

$$\int_{\mathbb{C}^n} |F|^2 e^{-2dv}(1 + |z|^2)^{-2n-2\varepsilon} d\lambda \le C_2 e^{-2da}$$

where $C_1, C_2 > 0$ are constants independent of d. As $v(z) \le \log_+|z| + C_3$, it follows that $F \in \mathcal{P}_d$. Moreover, since $v < 0$ on a neighborhood of K, the mean value inequality applied to the subharmonic function $|F|^2$ gives

$$\sup_K |F|^2 \le C_4 e^{-2da}.$$

The polynomial $P = C_4^{-1/2} e^{da} F \in \mathcal{P}_d$ is such that $\|P\|_K = 1$ and we have $\log|P(z_0)| \ge da - C_5$, whence

$$\sup\Big\{\frac{1}{d}\log|P(z_0)|\,;\, d \ge 1,\ P \in \mathcal{P}_d,\ \|P\|_{L^\infty(K)} \le 1\Big\} \ge a.$$

As a was an arbitrary number $< U_K(z_0)$, the proof is complete. □

Now, we introduce a few concepts related to extremal polynomials. Let B be the unit ball of $\mathbb{C}^n$ and K a compact subset of B. The Chebishev constants $M_d(K)$ are defined by

$$M_d(K) = \inf\big\{\|P\|_{L^\infty(K)}\ ;\ P \in \mathcal{P}_d,\ \|P\|_{L^\infty(B)} = 1\big\}. \tag{15.7}$$

It is clear that $M_d(K) \le 1$ and that $M_d(K)$ satisfies

$$M_{d+d'}(K) \le M_d(K) M_{d'}(K).$$

The *Alexander capacity* is defined by

$$T(K) = \inf_{d\ge 1} M_d(K)^{1/d}. \tag{15.8}$$

It is easy to see that we have in fact $T(K) = \lim_{d\to+\infty} M_d(K)^{1/d}$: for any integer $\delta \ge 1$, write $\delta = qd + r$ with $0 \le r < d$ and observe that

$$M_\delta(K)^{1/\delta} \le M_{qd}(K)^{1/(qd+r)} \le M_d(K)^{q/(qd+r)}\ ;$$

letting $\delta \to +\infty$ with d fixed, we get

$$T(K) \le \liminf_{\delta\to+\infty} M_\delta(K)^{1/\delta} \le \limsup_{\delta\to+\infty} M_\delta(K)^{1/\delta} \le M_d(K)^{1/d},$$

whence the equality. Now, for an arbitrary subset $E \subset B$, we set

$$T_\star(E) = \sup_{K\subset E} T(K)\,, \qquad T^\star(E) = \inf_{G\ \text{open}\ \supset E} T_\star(G). \tag{15.9}$$

(15.10) Siciak's theorem. *For every set $E \subset B$,*

$$T^\star(E) = \exp(-\sup_B U_E^\star).$$

Proof. The main step is to show that the equality holds for compact subsets $K \subset B$, i.e. that

$$T(K) = \exp(-\sup_B U_K^\star). \tag{15.11}$$

Indeed, it is clear that $\sup_B U_K^\star = \sup_B U_K$ and Th. 15.6 gives

$$\begin{aligned}\sup_B U_K &= \sup\Big\{\frac{1}{d}\log\|P\|_{L^\infty(B)}\ ;\ d\ge 1,\ P\in\mathcal{P}_d,\ \|P\|_{L^\infty(K)} = 1\Big\}\\ &= \sup\Big\{-\frac{1}{d}\log\|P\|_{L^\infty(K)}\ ;\ d\ge 1,\ P\in\mathcal{P}_d,\ \|P\|_{L^\infty(B)} = 1\Big\}\end{aligned}$$

after an obvious rescaling argument $P \mapsto \alpha P$. Taking the exponential, we get

$$\exp(-\sup_B U_K^\star) = \inf_{d\geq 1} \inf \left\{ \|P\|_{L^\infty(K)}^{1/d} \;;\; P \in \mathcal{P}_d,\ \|P\|_{L^\infty(B)} = 1 \right\}$$
$$= \inf_{d\geq 1} M_d(K)^{1/d} = T(K).$$

Next, let G be an open subset of B and K_j an increasing sequence of compact sets such that $G = \bigcup K_j$ and $T_\star(G) = \lim T(K_j)$. Then 15.5 (b) implies $U_G^\star = \lim \downarrow U_{K_j}^\star$, hence

$$\lim_{j\to+\infty} \sup_B U_{K_j}^\star = \sup_B U_G^\star = \sup_{\overline{B}} U_G^\star$$

by Dini's lemma. Taking the limit in (15.11), we get

$$T_\star(G) = \exp(-\sup_B U_G^\star).$$

Finally, 15.5 (d) shows that there exists a decreasing sequence of open sets $G_j \supset E$ such that $U_E^\star = (\lim \uparrow U_{G_j}^\star)^\star$. We may take G_j so small that $T^\star(E) = \lim T_\star(G_j)$. Theorem 15.9 follows. □

(15.12) Corollary. *The set function $T^\star$ is a capacity in the sense of* Def. 10.1 *and we have $T^\star(E) = T_\star(E)$ for every K-analytic set $E \subset B$.*

Proof. Axioms 10.1 (a,b,c) are immediate consequences of properties 15.5 (a,b,c) respectively. In addition, formulas 15.10 and 15.11 show that we have $T^\star(K) = T(K)$ for every compact set $K \subset B$. The last statement is then a consequence of Choquet's capacitability theorem. □

To conclude this section, we show that $1/|\log T^\star|$ is not very far from being subadditive. We need a lemma.

(15.13) Lemma. *For every $P \in \mathcal{P}_d$, one has*

$$\log \|P\|_{L^\infty(B)} - c_n d \leq \int_{\partial B} \log |P(z)| d\sigma(z) \leq \log \|P\|_{L^\infty(B)}$$

where $d\sigma$ is the unit invariant measure on the sphere and c_n a constant such that $c_n \sim \log(2n)$ as $n \to +\infty$.

Proof. Without loss of generality, we may assume that $\|P\|_{L^\infty(B)} = 1$. Since $\frac{1}{d} \log |P| \in P_{\log}(\mathbb{C}^n)$, the logarithmic convexity property already used implies that

$$\sup_{B(0,r)} \frac{1}{d} \log |P| \geq \log r \quad \text{for } r < 1.$$

The Harnack inequality for the Poisson kernel now implies

$$\sup_{B(0,r)} \log |P| \leq \frac{1-r^2}{(1+r)^{2n}} \int_{\partial B} \log |P| d\sigma,$$

$$\int_{\partial B} \log |P| d\sigma \geq \frac{1+r^{2n}}{1-r^2} \log r.d.$$

The lemma follows with

$$c_n = \inf_{r \in]0,1[} \frac{(1+r)^{2n}}{1-r^2} \log \frac{1}{r} \, ;$$

the infimum is attained approximately for $r = 1/(2n \log 2n)$. □

(15.14) Corollary. *For $P_j \in \mathcal{P}_{d_j}$, $1 \leq j \leq N$,*

$$\|P_1 \ldots P_N\|_{L^\infty(B)} \geq e^{-c_n(d_1+\cdots+d_N)} \|P_1\|_{L^\infty(B)} \ldots \|P_N\|_{L^\infty(B)}.$$

Proof. Apply Lemma 15.13 to each P_j and observe that

$$\int_{\partial B} \log |P_1 \ldots P_N| d\sigma = \sum_{1 \leq j \leq N} \int_{\partial B} \log |P_j| d\sigma.$$ □

(15.15) Theorem. *For any set $E = \bigcup_{j \geq 1} E_j$, one has*

$$\frac{1}{c_n - \log T^\star(E)} \leq \sum_{j \geq 1} \frac{1}{|\log T^\star(E_j)|}.$$

Proof. It is sufficient to check the inequality for a finite union $K = \bigcup K_j$ of compacts sets $K_j \subset B$, $1 \leq j \leq N$. Select $P_j \in \mathcal{P}_{d_j}$ such that

$$\|P_j\|_{L^\infty(B)} = 1, \qquad \|P_j\|_{L^\infty(K_j)} = M_{d_j}(K_j),$$

and set $P = P_1 \ldots P_N$, $d = d_1 + \cdots + d_N$. Then Cor. 15.14 shows that $\|P_j\|_{L^\infty(B)} \geq e^{-c_n d}$, thus

$$M_d(K) \leq e^{c_n d} \|P\|_{L^\infty(K)}.$$

If $z \in K$ is in K_j, then $|P(z)| \leq |P_j(z)| \leq \|P_j\|_{L^\infty(K_j)}$ because all other factors are ≤ 1. Thus

$$M_d(K) \le e^{c_n d} \max\{\|P_j\|_{L^\infty(K_j)}\},$$

$$T(K) \le M_d(K)^{1/d} \le e^{c_n} \max\{M_{d_j}(K)^{1/d_j \cdot d_j/d}\}.$$

Take $d_j = [k\alpha_j]$ with arbitrary $\alpha_j > 0$ and let $k \to +\infty$. It follows that

$$T(K) \le e^{c_n} \max\{T(K_j)^{\alpha_j/\alpha}\}$$

where $\alpha = \sum \alpha_j$. The inequality asserted in Th. 15.15 is obtained for the special choice $\alpha_j = 1/|\log T(K_j)| > 0$ which makes all terms in $\max\{\ldots\}$ equal. □

16. Comparison of Capacities and El Mir's Theorem

We first prove a comparison theorem for the capacities $c(\bullet, \Omega)$ and T, due to Alexander and Taylor [A-T].

(16.1) Theorem. *Let K be a compact subset of the unit ball $B \subset \mathbb{C}^n$. Then*

(a) $T(K) \le \exp(-c(K, B)^{-1/n})$.

For each $r < 1$, there is a constant $A(r)$ such that

(b) $T(K) \ge \exp(-A(r)\, c(K, B)^{-1})$ *when* $K \subset B(0, r)$.

(16.2) *Remark.* Both set functions $c^\star(\bullet, \Omega)$ and $T^\star$ are capacities in the sense of 10.1. Hence, the estimates of the theorem also hold for all K-analytic sets, in particular all Borel sets.

(16.3) *Remark.* The inequalities are sharp, at least as far as the exponents on $c(K, B)$ are concerned. For if $K = \overline{B}(0, \varepsilon)$, then it is easy to check that $T(K) = \varepsilon$ and example 13.10 gives $c(K, B) = (\log 1/\varepsilon)^{-n}$. Hence, equality holds in 16.1 (a). On the other hand, if K is a small polydisc

$$K = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \;;\; |z_1| \le \delta,\, |z_j| \le 1/n,\, 1 < j \le n\}$$

and $\delta \le 1/n$, then $T(K) \le \delta$, while $c(K, B) \ge C(\log 1/\delta)^{-1}$. To check this last inequality (which shows the optimality of 16.1 (b)), put

$$u(z) = \Big(\log \frac{1}{\delta}\Big)^{-1} \log_+ \frac{|z_1|}{\delta} + (\log n)^{-1} \sum_{j=2}^{n} \log_+(n|z_j|).$$

Then $u = 0$ and $u \le n$ on B, hence

$$c(K,B) \ge \int_B \Big(\frac{1}{n} dd^c u\Big)^n = \frac{n!}{n^n} (\log n)^{-(n-1)} (\log 1/\delta)^{-1}$$

because all measures $dd^c \log_+(|z_j|/r)$ have total mass 1 in $\mathbb{C}$.

Proof of Theorem 16.1. If K is pluripolar then $U_K^\star \equiv +\infty$, hence $c(K,B) = T(K) = 0$ and the inequalities are satisfied. We thus assume that K is not pluripolar.

Proof of 16.1 (a). Set $M = \sup_B U_K^\star$; then $T(K) = e^{-M}$ by Siciak's theorem. Since $u = U_K^\star/M \in P(B)$ and $0 \le u \le 1$ on B, we get

$$c(K,B) \ge M^{-n} \int_K (dd^c U_K^\star)^n = M^{-n}$$

by Th. 15.3. This inequality is equivalent to 16.1 (a). □

Proof of 16.1 (b). Let $u_K^\star$ be the extremal function for K relative to the ball $B' = B(0,e) \supset\supset B$. For any $v \in P_{\log}(\mathbb{C}^n)$ such that $v \le 0$ on K, we have $v \le U_K^\star \le M+1$ on B', hence the function

$$w = \frac{v - M - 1}{M+1}$$

satisfies $w \le 0$ on B' and $w \le -1$ on K. We infer $w \le u_K^\star$; by taking the supremum over all choices of v, we get

$$u_K^\star \ge \frac{U_K^\star - M - 1}{M+1}.$$

Now, there is a point $z_0 \in \overline{B}$ such that $U_K^\star(z_0) = M$, thus

$$u_K^\star(z_0) \ge -\frac{1}{M+1}.$$

As $u_K^\star \le 0$ on B', the mean value and Harnack inequalities show that

$$u_K^\star(z_0) \le C_1 \int_{B'} u_K^\star d\lambda \Longrightarrow \|u_K^\star\|_{L^1(B')} \le -\frac{1}{C_1} u_K^\star(z_0) \le \frac{C_2}{M}.$$

The Chern-Levine-Nirenberg inequalities 1.3 and 1.4 (a) imply now

$$c(K,B') = \int_B (dd^c u_K^\star)^n \le C_3 \|u_K^\star\|_{L^1(B')} \, \|u_K^\star\|^{n-1}_{L^\infty(B')} \le \frac{C_4}{M}.$$

As $K \subset B(0,r) \subset\subset B$, Prop. 11.3 (b) gives

$$c(K,B) \le C_5(r)c(K,B') \le A(r)M^{-1}$$

and inequality 16.1 (b) follows. □

We now prove El Mir's theorem [E-M1]. This result is an effective version of Josefson's theorem: given a plurisubharmonic function in the ball, a subextension can be found with prescribed singularities of poles and slow growth at infinity.

(16.4) El Mir's theorem. *Let $v \in P(B)$ with $v \le -1$, $\varepsilon \in]0, 1/n[$ and $r < 1$. Then there exists $u \in P_{\log}(\mathbb{C}^n)$ such that $u \le -|v|^{\frac{1}{n}-\varepsilon}$ on $B(0,r)$.*

Proof. For $t \ge 1$, set $G_t = \{z \in B(0,r); v(z) < -t\}$ and let $U_t^\star \in P_{\log}(\mathbb{C}^n)$ be the Siciak extremal function of G_t. Since G_t is open, we have $U_t^\star = 0$ on G_t. We set $M(t) = \sup_B U_t^\star$ and

$$u(z) = \varepsilon^{-1} \int_1^{+\infty} t^{-1-\varepsilon}(U_t^\star(z) - M(t))dt.$$

Proposition 11.4 shows that $c(G_t, B) \le C_1/t$, therefore

$$M(t) = -\log T^\star(G_t) \ge c(G_t, B)^{-1/n} \ge C_2 t^{1/n}$$

by inequality 16.1 (a). As $U_t^\star - M(t) \le 0$ on B, we get $U_t^\star(z) - M(t) \le \log_+ |z|$ by logarithmic convexity, thus

$$u(z) \le \log_+ |z|.$$

For $z \in B(0,r)$ we have $U_t^\star(z) = 0$ as soon as $G_t \ni z$, i.e. $t < -v(z)$. Hence

$$u(z) \le -\varepsilon^{-1} \int_1^{|v(z)|} t^{-1-\varepsilon} M(t)dt \le -C_3 \int_1^{|v(z)|} t^{-1-\varepsilon+\frac{1}{n}} dt = -C_4 |v(z)|^{\frac{1}{n}-\varepsilon}.$$

Starting if necessary with a smaller value of ε and subtracting a constant to u, we can actually get

$$u \le -|v|^{\frac{1}{n}-\varepsilon} \quad \text{on } B(0,r).$$

It remains to check that u is not identically $-\infty$. By logarithmic convexity again, we have

$$\sup_{\overline{B}(0,1/2)} U_t^\star \ge M(t) - \log 2$$

and there exists $z_0 \in S(0;1/2)$ such that $U_t^\star(z_0) - M(t) \ge -\log 2$. The Harnack inequality shows that

$$\frac{1-1/4}{(1+1/2)^{2n}}\int_{\partial B}\big(U_t^\star(z)-M(t)\big)d\sigma(z)\ge U_t^\star(z_0)-M(t)\ge -\log 2$$

and integration with respect to t yields

$$\int_{\partial B}u(z)d\sigma(z)\ge -4/3(3/2)^{2n}\log 2>-\infty. \qquad \square$$

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