On the cohomology of pseudoeffective line bundles

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Dedicated to Professor Yum-Tong Siu on the occasion of his 70th birthday

Abstract. The goal of this survey is to present various results concerning the cohomology of pseudoeffective line bundles on compact Kähler manifolds, and related properties of their multiplier ideal sheaves. In case the curvature is strictly positive, the prototype is the well known Nadel vanishing theorem, which is itself a generalized analytic version of the fundamental Kawamata-Viehweg vanishing theorem of algebraic geometry. We are interested here in the case where the curvature is merely semipositive in the sense of currents, and the base manifold is not necessarily projective. In this situation, one can still obtain interesting information on cohomology, e.g. a Hard Lefschetz theorem with pseudoeffective coefficients, in the form of a surjectivity statement for the Lefschetz map. More recently, Junyan Cao, in his PhD thesis defended in Grenoble, obtained a general Kähler vanishing theorem that depends on the concept of numerical dimension of a given pseudoeffective line bundle. The proof of these results depends in a crucial way on a general approximation result for closed $(1,1)$-currents, based on the use of Bergman kernels, and the related intersection theory of currents. As an application, we discuss a structure theorem for compact Kähler threefolds without nontrivial subvarieties, following a joint work with F. Campana and M. Verbitsky. We hope that these notes will serve as a useful guide to the more detailed and more technical papers in the litterature; in some cases, we provide here substantially simplified proofs and unifying viewpoints. Some parts – especially Subsections 3.1 and 3.2 – raise new open questions.

Key-words. Closed positive current, plurisubharmonic function, Ohsawa-Takegoshi extension theorem, curvature current, pseudoeffective line bundle, Bergman approximation, multiplier ideal sheaf, Nadel vanishing theorem, hard Lefschetz theorem, intersection theory, numerical dimension, openness conjecture, simple Kähler manifold, complex torus

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0. Introduction and statement of the main results

Let X be a compact Kähler n-dimensional manifold, equipped with a Kähler metric, i.e. a positive definite Hermitian $(1,1)$-form $\omega = \mathrm{i}\sum_{1\le j,k\le n}\omega_{jk}(z)\,dz_j\wedge d\overline{z}_k$ such that $d\omega = 0$. By definition a holomorphic line bundle L on X is said to be *pseudoeffective* if there exists a singular hermitian metric h on L, given by $h(z) = e^{-\varphi(z)}$ with respect to a local trivialization $L_{|U}\simeq U\times\mathbb{C}$, such that the curvature form

$$(0.1)\qquad \mathrm{i}\,\Theta_{L,h} := \mathrm{i}\,\partial\overline{\partial}\varphi$$

is (semi)positive in the sense of currents, i.e. φ is locally integrable and $i\Theta_{L,h}\ge 0$: in other words, the weight function φ is plurisubharmonic (psh) on the corresponding trivializing open set U. A basic concept is the notion of *multiplier ideal sheaf*, introduced in [Nad89].

0.2. Definition. *To any psh function φ on an open subset U of a complex manifold X, one associates the "multiplier ideal sheaf" $\mathcal{I}(\varphi)\subset\mathcal{O}_{X|U}$ of germs of holomorphic functions $f\in\mathcal{O}_{X,x}$, $x\in U$, such that $|f|^2e^{-\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x. We also define the global multiplier ideal sheaf $\mathcal{I}(h)\subset\mathcal{O}_X$ of* *a hermitian metric h on $L \in \mathrm{Pic}(X)$ to be equal to $\mathcal{I}(\varphi)$ on any open subset U where $L_{|U}$ is trivial and $h = e^{-\varphi}$. In such a definition, we may in fact assume $\mathrm{i}\,\Theta_{L,h} \ge -C\omega$, i.e. locally $\varphi = psh + C^\infty$, we say in that case that φ is* quasi-psh.

Let us observe that a multiplier ideal sheaf $\mathcal{I}(\varphi)$ is left unmodified by adding a smooth function to φ, so, for such purposes, the additional C^∞ terms are irrelevant in quasi-psh functions. A crucial and well-known fact is that the ideal sheaves $\mathcal{I}(\varphi) \subset \mathcal{O}_{X|U}$ and $\mathcal{I}(h) \subset \mathcal{O}_X$ are always *coherent analytic sheaves*; when $U \subset X$ is a coordinate open ball, this can be shown by observing that $\mathcal{I}(\varphi)$ coincides with the locally stationary limit $\mathcal{J} = \lim\uparrow_{N\to+\infty}\mathcal{J}_N$ of the increasing sequence of coherent ideals $\mathcal{J}_N = (g_j)_{0\le j<N}$ associated with a Hilbert basis $(g_j)_{j\in\mathbb{N}}$ of the Hilbert space of holomorphic functions $f \in \mathcal{O}_X(U)$ such that $\int_U |f|^2 e^{-\varphi} dV_\omega < +\infty$. The proof is a consequence of Hörmander's L^2 estimates applied to weights of the form

$$\psi(z) = \varphi(z) + (n+k)\log|z-x|^2.$$

This easily shows that $\mathcal{I}(\varphi)_x + \mathfrak{m}_x^k = \mathcal{J}_x + \mathfrak{m}_x^k$, and one then concludes that $\mathcal{I}(\varphi)_x = \mathcal{J}_x$ by the Krull lemma. When X is *projective algebraic*, Serre's GAGA theorem implies that $\mathcal{I}(h)$ is in fact a *coherent algebraic sheaf*, in spite of the fact that φ may have very "wild" analytic singularities – they might e.g. be everywhere dense in X in the Euclidean topology. Therefore, in some sense, the multiplier ideal sheaf is a powerful tool to extract algebraic (or at least analytic) data from arbitrary singularities of psh functions. In this context, assuming strict positivity of the curvature, one has the following well-known fundamental vanishing theorem.

0.3. Theorem. (Nadel Vanishing Theorem, [Nad89], [Dem93b]) *Let (X,ω) be a compact Kähler n-dimensional manifold, and let L be a holomorphic line bundle over X equipped with a singular Hermitian metric h. Assume that $\mathrm{i}\,\Theta_{L,h} \ge \varepsilon\omega$ for some $\varepsilon > 0$ on X. Then*

$$H^q\big(X, \mathcal{O}(K_X \otimes L) \otimes \mathcal{I}(h)\big) = 0 \qquad \text{for all } q \ge 1,$$

where $K_X = \Omega_X^n = \Lambda^n T_X^*$ denotes the canonical line bundle.

The proof follows from an application of Hörmander's L^2 estimates with singular weights, themselves derived from the Bochner-Kodaira identity (see [Hör66], [Dem82], [Dem92]). One should observe that the strict positivity assumption implies L to be big, hence X must be projective, since every compact manifold that is Kähler and Moishezon is also projective (cf. [Moi66], [Pet86], [Pet98a]). However, when relaxing the strict positivity assumption, one can enter the world of general compact Kähler manifolds, and their study is one of our main goals.

In many cases, one has to assume that the psh functions involved have milder singularities. We say that a psh or quasi-psh function φ has *analytic singularities* if locally on the domain of definition U of φ one can write

$$\varphi(z) = c\log\sum_{j=1}^{N}|g_j|^2 + O(1) \tag{0.4}$$

where the g_j's are holomorphic functions, $c \in \mathbb{R}_+$ and $O(1)$ means a locally bounded remainder term. Assumption (0.4) implies that the set of poles $Z = \varphi^{-1}(-\infty)$ is an analytic set, locally defined as $Z = \bigcap g_j^{-1}(0)$, and that φ is locally bounded on $U \smallsetminus Z$. We also refer to this situation by saying that φ has *logarithmic poles.* In general, one introduces the following comparison relations for psh or quasi-psh functions φ and hermitian metrics $h = e^{-\varphi}$; a more flexible comparison relation will be introduced in Section 3 .

0.5. Definition. *Let φ_1, φ_2 be psh functions on an open subset U of a complex manifold X. We say that*

(a) *φ_1 has less singularities than φ_2, and write $\varphi_1 \preccurlyeq \varphi_2$, if for every point $x \in U$, there exists a neighborhood V of x and a constant $C \geq 0$ such that $\varphi_1 \geq \varphi_2 - C$ on V.*

(b) *φ_1 and φ_2 have equivalent singularities, and write $\varphi_1 \sim \varphi_2$, if locally near any point of U we have $\varphi_1 - C \leq \varphi_2 \leq \varphi_1 + C$.*

Similarly, given a pair of hermitian metrics h_1, h_2 on a line bundle $L \to X$,

(a') *we say that h_1 is less singular than h_2, and write $h_1 \preccurlyeq h_2$, if locally there exists a constant $C > 0$ such that $h_1 \leq Ch_2$.*

(b') *we say that h_1, h_2 have equivalent singularities, and write $h_1 \sim h_2$, if locally there exists a constant $C > 0$ such that $C^{-1}h_2 \leq h_1 \leq Ch_2$.*

(*of course when h_1 and h_2 are defined on a compact manifold X, the constant C can be taken global on X in* (a') *and* (b')).

Now, if L is a pseudoeffective line bundle, it was observed in [Dem00] that there always exist a unique equivalence class $h_{\min}$ of singular hermitian metrics with minimal singularities, such that $\mathrm{i}\,\Theta_{L,h_{\min}} \geq 0$ (by this we mean that $h_{\min}$ is unique up to equivalence of singularities). In fact, if h_∞ is a smooth metric on L, one can define the corresponding weight $\varphi_{\min}$ of $h_{\min}$ as an upper envelope

$$\varphi_{\min}(z) = \sup\big\{\varphi(z)\,;\ \mathrm{i}\,\Theta_{L,h_\infty} + \mathrm{i}\,\partial\overline{\partial}\varphi \geq 0,\ \varphi \leq 0 \text{ on } X\big\}, \tag{0.6}$$

and put $h_{\min} = h_\infty e^{-\varphi_{\min}}$. In general, $h_{\min}$ need not have analytic singularities.

An important fact is that one can approximate arbitrary psh functions by psh functions with analytic singularities. The appropriate technique consists of using an asymptotic Bergman kernel procedure (cf. [Dem92] and Section 1). If φ is a holomorphic function on a ball $B \subset \mathbb{C}^n$, one puts

$$\varphi_m(z) = \frac{1}{2m}\log\sum_{\ell\in\mathbb{N}} |g_{m,\ell}(z)|^2$$

where $(g_{m,\ell})_{\ell\in\mathbb{N}}$ is a Hilbert basis of the space $\mathcal{H}(B, m\varphi)$ of L^2 holomorphic functions on B such that $\int_B |f|^2 e^{-2m\varphi} dV < +\infty$. When $T = \alpha + dd^c\varphi$ is a closed $(1,1)$-current on X in the same cohomology class as a smooth $(1,1)$-form α and φ is a quasi-psh potential on X, a sequence of global approximations T_m can be produced by taking a finite covering of X by coordinate balls (B_j). A partition of unity argument allows to glue the local approximations $\varphi_{m,j}$ of φ on B_j into a global potential φ_m, and one sets $T_m = \alpha + dd^c\varphi_m$. These currents T_m converge weakly to T, are smooth in the complement $X \smallsetminus Z_m$ of an increasing family of analytic subsets $Z_m \subset X$, and their singularities approach those of T. More precisely, the Lelong numbers $\nu(T_m, z)$ converge uniformly to those of T, and whenever $T \geq 0$, it is possible to produce a current T_m that only suffers a small loss of positivity, namely $T_m \geq -\varepsilon_m\omega$ where $\lim_{m\to+\infty} \varepsilon_m = 0$. These considerations lead in a natural way to the concept of *numerical dimension* of a closed positive $(1,1)$-current T. We define

$$\mathrm{nd}(T) = \max\big\{p = 0, 1, \ldots, n\,;\ \limsup_{m\to+\infty} \int_{X\smallsetminus Z_m} (T_m + \varepsilon_m\omega)^p \wedge \omega^{n-p} > 0\big\}. \tag{0.7}$$

One can easily show (see Section 3) that the right hand side of (0.7) does not depend on the sequence (T_m), provided that the singularities approach those of T (we call this an "asymptotically equisingular approximation").

These concepts are very useful to study cohomology groups with values in pseudoeffective line bundles (L, h). Without assuming any strict positivity of the curvature, one can obtain at least a hard Lefschetz theorem with coefficients in L. The technique is based on a use of harmonic forms with respect to suitable "equisingular approximations" φ_m of the weight φ of h (in that case we demand that $\mathcal{I}(\varphi_m) = \mathcal{I}(\varphi)$ for all m); the main idea is to work with complete Kähler metrics in the open complements $X \smallsetminus Z_m$ where φ_m is smooth, and to apply a variant of the Bochner formula on these sets. More details can be found in Section 2 and in [DPS01].

0.8. Theorem. ([DPS01]) *Let (L, h) be a pseudo-effective line bundle on a compact Kähler manifold (X, ω) of dimension n, let $\Theta_{L,h} \geq 0$ be its curvature current and $\mathcal{I}(h)$ the associated multiplier ideal sheaf. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism*

$$\Phi^q_{\omega,h} : H^0(X, \Omega^{n-q}_X \otimes L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h)).$$

The special case when L is nef is due to Takegoshi [Tak97]. An even more special case is when L is semipositive, i.e. possesses a smooth metric with semipositive curvature. In that case the multiple ideal sheaf $\mathcal{I}(h)$ coincides with $\mathcal{O}_X$ and we get the following consequence already observed by Enoki [Eno93] and Mourougane [Mou95].

0.9. Corollary. *Let (L, h) be a semipositive line bundle on a compact Kähler manifold (X, ω) of dimension n. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism*

$$\Phi^q_\omega : H^0(X, \Omega^{n-q}_X \otimes L) \longrightarrow H^q(X, \Omega^n_X \otimes L).$$

It should be observed that although all objects involved in Th. 0.8 are algebraic when X is a projective manifold, there is no known algebraic proof of the statement; it is not even clear how to define algebraically $\mathcal{I}(h)$ for the case when $h = h_{min}$ is a metric with minimal singularity. However, even in the special circumstance when L is nef, the multiplier ideal sheaf is crucially needed.

Our next statement is taken from the PhD thesis of Junyan Cao [JC13]. The proof is a combination of our Bergman regularization techniques, together with an argument of Ch. Mourougane [Mou95] relying on a use of the Calabi-Yau theorem for Monge-Ampère equations.

0.10. Theorem. ([JC13], [JC14]) *Let (L, h) be a pseudoeffective line bundle on a compact Kähler n-dimensional manifold X. Then*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}_+(h)) = 0 \qquad \textit{for every} \;\; q \geq n - \mathrm{nd}(L, h) + 1,$$

where $\mathrm{nd}(L, h) := \mathrm{nd}(\mathrm{i}\,\Theta_{L,h})$ *and $\mathcal{I}_+(h)$ is the upper semicontinuous regularization of the multiplier ideal sheaf, i.e.*

$$\mathcal{I}_+(h) = \lim_{\varepsilon \to 0} \mathcal{I}(h^{1+\varepsilon}). \tag{0.11}$$

In general $\mathcal{I}_+(h) \subset \mathcal{I}(h)$ and it is clear that the equality holds when h has analytic singularities (this can be easily seen via Hironaka's desingularization theorem [Hir64]). The question whether it is always true that $\mathcal{I}_+(h) = \mathcal{I}(h)$ was possibly first raised in ([Dem00], Remark 15.2.2), and then in [DP02], following the proof of the semicontinuity theorem for psh singularities in [DK01]. Actually, the equality is easy to show in dimension 1, and it follows from the work of Favre and Jonsson [FJ05] in dimension 2. Bo Berndtsson [Bern13] recently showed that $\mathcal{I}(h)_x = \mathcal{O}_{X,x}$ implies $\mathcal{I}_+(h)_x = \mathcal{I}(h)_x = \mathcal{O}_{X,x}$ in arbitrary dimension. Finally, during the Fall 2013, Qi'an Guan and Xiangyu Zhou showed that the equality $\mathcal{I}_+(h) = \mathcal{I}(h)$ holds in all cases in the most general situation, cf. [GZ13]. Therefore, thanks to [GZ13], the above Theorem 0.10 could also be stated with $\mathcal{I}(h)$ in place of $\mathcal{I}_+(h)$. As a final geometric application of this circle of ideas, we present the following result which was recently obtained in [CDV13].

0.12. Theorem. ([CDV13]) *Let X be a compact Kähler threefold that is "strongly simple" in the sense that X has no nontrivial analytic subvariety. Then the Albanese morphism $\alpha : X \to \mathrm{Alb}(X)$ is a biholomorphism, and therefore X is biholomorphic to a 3-dimensional complex torus $\mathbb{C}^3/\Lambda$.*

1. Approximation of psh functions and of closed (1,1)-currents

We first recall here the basic result on the approximation of psh functions by psh functions with analytic singularities. The main idea is taken from [Dem92] and relies on the Ohsawa-Takegoshi extension theorem, For other applications to algebraic geometry, see [Dem93b] and Demailly-Kollár [DK01]. Let φ be a psh function on an open set $\Omega \subset \mathbb{C}^n$. Recall that the Lelong number of φ at a point $x_0 \in \Omega$ is defined to be

$$\nu(\varphi, x_0) = \liminf_{z \to x_0} \frac{\varphi(z)}{\log|z - x_0|} = \lim_{r \to 0_+} \frac{\sup_{B(x_0,r)} \varphi}{\log r}. \tag{1.1}$$

In particular, if $\varphi = \log|f|$ with $f \in \mathcal{O}(\Omega)$, then $\nu(\varphi, x_0)$ is equal to the vanishing order

$$\mathrm{ord}_{x_0}(f) = \sup\{k \in \mathbb{N}\,; D^\alpha f(x_0) = 0,\ \forall|\alpha| < k\}.$$

1.2. Theorem. *Let φ be a plurisubharmonic function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi} d\lambda < +\infty$ and let $\varphi_m = \frac{1}{2m} \log \sum |g_{m,\ell}|^2$ where $(g_{m,\ell})$ is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that*

(a) $\varphi(z) - \dfrac{C_1}{m} \le \varphi_m(z) \le \sup\limits_{|\zeta - z| < r} \varphi(\zeta) + \dfrac{1}{m} \log \dfrac{C_2}{r^n}$ *for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$ and*

(b) $\nu(\varphi, z) - \dfrac{n}{m} \le \nu(\varphi_m, z) \le \nu(\varphi, z)$ *for every $z \in \Omega$.*

Proof. (a) Note that $\sum |g_{m,\ell}(z)|^2$ is the square of the norm of the evaluation linear form $\mathrm{ev}_z : f \mapsto f(z)$ on $\mathcal{H}_\Omega(m\varphi)$, since $g_{m,\ell}(z) = \mathrm{ev}_z(g_{m,\ell})$ is the ℓ-th coordinate of ev_z in the orthonormal basis $(g_{m,\ell})$. In other words, we have

$$\sum |g_{m,\ell}(z)|^2 = \sup_{f \in B(1)} |f(z)|^2$$

where $B(1)$ is the unit ball of $\mathcal{H}_\Omega(m\varphi)$ (The sum is called the *Bergman kernel* associated with $\mathcal{H}_\Omega(m\varphi)$). As φ is locally bounded from above, the L^2 topology is actually stronger than the topology of uniform convergence on compact subsets of Ω. It follows that the series $\sum |g_{m,\ell}|^2$ converges uniformly on Ω and that its sum is real analytic. Moreover, by what we just explained, we have

$$\varphi_m(z) = \sup_{f\in B(1)} \frac{1}{2m}\log|f(z)|^2 = \sup_{f\in B(1)} \frac{1}{m}\log|f(z)|.$$

For $z_0 \in \Omega$ and $r < d(z_0, \partial\Omega)$, the mean value inequality applied to the psh function $|f|^2$ implies

$$\begin{aligned}|f(z_0)|^2 &\le \frac{1}{\pi^n r^{2n}/n!}\int_{|z-z_0|<r}|f(z)|^2 d\lambda(z)\\ &\le \frac{1}{\pi^n r^{2n}/n!}\exp\Big(2m\sup_{|z-z_0|<r}\varphi(z)\Big)\int_\Omega |f|^2 e^{-2m\varphi}d\lambda.\end{aligned}$$

If we take the supremum over all $f \in B(1)$ we get

$$\varphi_m(z_0) \le \sup_{|z-z_0|<r}\varphi(z) + \frac{1}{2m}\log\frac{1}{\pi^n r^{2n}/n!}$$

and the second inequality in (a) is proved – as we see, this is an easy consequence of the mean value inequality. Conversely, the Ohsawa-Takegoshi extension theorem ([OT87]) applied to the 0-dimensional subvariety $\{z_0\} \subset \Omega$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on Ω such that $f(z_0) = a$ and

$$\int_\Omega |f|^2 e^{-2m\varphi}d\lambda \le C_3|a|^2 e^{-2m\varphi(z_0)},$$

where C_3 only depends on n and $\operatorname{diam}\Omega$. We fix a such that the right hand side is 1. Then $\|f\| \le 1$ and so we get

$$\varphi_m(z_0) \ge \frac{1}{m}\log|f(z_0)| = \frac{1}{m}\log|a| = \varphi(z) - \log\frac{C_3}{m}.$$

The inequalities given in (a) are thus proved. Taking $r = 1/m$, we find that

$$\lim_{m\to+\infty}\sup_{|\zeta-z|<1/m}\varphi(\zeta) = \varphi(z)$$

by the upper semicontinuity of φ, and therefore $\lim\varphi_m(z) = \varphi(z)$, since $\lim\frac{1}{m}\log(C_2m^n) = 0$.

(b) The above estimates imply

$$\sup_{|z-z_0|<r}\varphi(z) - \frac{C_1}{m} \le \sup_{|z-z_0|<r}\varphi_m(z) \le \sup_{|z-z_0|<2r}\varphi(z) + \frac{1}{m}\log\frac{C_2}{r^n}.$$

After dividing by $\log r < 0$ when $r \to 0$, we infer

$$\frac{\sup_{|z-z_0|<2r}\varphi(z) + \frac{1}{m}\log\frac{C_2}{r^n}}{\log r} \le \frac{\sup_{|z-z_0|<r}\varphi_m(z)}{\log r} \le \frac{\sup_{|z-z_0|<r}\varphi(z) - \frac{C_1}{m}}{\log r},$$

and from this and definition (1.1), it follows immediately that

$$\nu(\varphi, x) - \frac{n}{m} \le \nu(\varphi_m, z) \le \nu(\varphi, z). \qquad \square$$

Theorem 1.2 implies in a straightforward manner the deep result of [Siu74] on the analyticity of the Lelong number upperlevel sets.

1.3. Corollary. [Siu74] *Let φ be a plurisubharmonic function on a complex manifold X. Then, for every $c > 0$, the Lelong number upperlevel set*

$$E_c(\varphi) = \big\{z \in X\,;\ \nu(\varphi, z) \geq c\big\}$$

is an analytic subset of X.

Proof. Since analyticity is a local property, it is enough to consider the case of a psh function φ on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. The inequalities obtained in Theorem 13.2 (b) imply that

$$E_c(\varphi) = \bigcap_{m \geq m_0} E_{c-n/m}(\varphi_m).$$

Now, it is clear that $E_c(\varphi_m)$ is the analytic set defined by the equations $g^{(\alpha)}_{m,\ell}(z) = 0$ for all multi-indices α such that $|\alpha| < mc$. Thus $E_c(\varphi)$ is analytic as a (countable) intersection of analytic sets. □

1.4. Remark. It has been observed by Dano Kim [Kim13] that the functions φ_m produced by Th. 1.2 do not in general satisfy $\varphi_{m+1} \not\succcurlyeq \varphi_m$, in other words their singularities may not always increase monotonically to those of φ. Thanks to the subbadditivity result of [DEL00], this is however the case for any subsequence φ_{m_k} such that m_k divides m_{k+1}, e.g. $m_k = 2^k$ or $m_k = k!$ (we will refer to such a sequence below as being a "*multiplicative sequence*"). In that case, a use of the Ohsawa-Takegoshi theorem on the diagonal of $\Omega \times \Omega$ shows that one can obtain $\varphi_{m_{k+1}} \leq \varphi_{m_k}$ (after possibly replacing φ_{m_k} by $\varphi_{m_k} + C/m_k$ with C large enough), see [DPS01].

Our next goal is to study the regularization process more globally, i.e. on a compact complex manifold X. For this, we have to take care of cohomology class. It is convenient to introduce $d^n = \frac{\mathrm{i}}{4\pi}(\overline{\partial} - \partial)$, so that $dd^c = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}$. Let T be a closed $(1,1)$-current on X. We assume that T is *quasi-positive*, i.e. that there exists a $(1,1)$-form γ with continuous coefficients such that $T \geq \gamma$; observe that a function φ is quasi-psh iff its complex Hessian is bounded below by a $(1,1)$-form with continuous or locally bounded coefficients, that is, if $dd^c\varphi$ is quasi-positive. The case of positive currents ($\gamma = 0$) is of course the most important.

1.5. Lemma. *There exists a smooth closed $(1,1)$-form α representing the same $\partial\overline{\partial}$-cohomology class as T and an* quasi-psh *function φ on X such that $T = \alpha + dd^c\varphi$.*

Proof. Select an open covering (B_j) of X by coordinate balls such that $T = dd^c\varphi_j$ over B_j, and construct a global function $\varphi = \sum \theta_j\varphi_j$ by means of a partition of unity $\{\theta_j\}$ subordinate to B_j. Now, we observe that $\varphi - \varphi_k$ is smooth on B_k because all differences $\varphi_j - \varphi_k$ are smooth in the intersections $B_j \cap B_k$ and we can write $\varphi - \varphi_k = \sum \theta_j(\varphi_j - \varphi_k)$. Therefore $\alpha := T - dd^c\varphi$ is smooth. □

Thanks to Lemma 1.5 , the problem of approximating a quasi-positive closed $(1,1)$-current is reduced to approximating a quasi-psh function. In this way, we get

1.6. Theorem. *Let $T = \alpha + dd^c\varphi$ be a quasi-positive closed $(1,1)$-current on a compact Hermitian manifold (X, ω) such that $T \geq \gamma$ for some continuous $(1,1)$-form γ. Then there exists a sequence of quasi-positive currents $T_m = \alpha + dd^c\varphi_m$ whose local potentials have the* *same singularities as $1/2m$ times a logarithm of a sum of squares of holomorphic functions and a decreasing sequence $\varepsilon_m > 0$ converging to 0, such that*

(a) *T_m converges weakly to T,*

(b) $\nu(T,x) - \dfrac{n}{m} \le \nu(T_m,x) \le \nu(T,x)$ *for every $x \in X$;*

(c) $T_m \ge \gamma - \varepsilon_m\omega$.

We say that our currents T_m are approximations of T with analytic singularities (possessing logarithmic poles). Moreover, for any multiplicative subsequence m_k, one can arrange that $T_{m_k} = \alpha + dd^c\varphi_{m_k}$ where (φ_{m_k}) is a non-increasing sequence of potentials.

Proof. We just briefly sketch the idea – essentially a partition of unity argument – and refer to [Dem92] for the details. Let us write $T = \alpha + dd^c\varphi$ with α smooth, according to Lemma 1.5 . After replacing T with $T - \alpha$ and γ with $\gamma - \alpha$, we can assume without loss of generality that $\{T\} = 0$, i.e. that $T = dd^c\varphi$ with a quasi-psh function φ on X such that $dd^c\varphi \ge \gamma$. Now, for $\varepsilon > 0$ small, we select a finite covering $(B_j)_{1\le j\le N(\varepsilon)}$ of X by coordinate balls on which there exists an ε-approximation of γ as

$$\sum_{1\le\ell\le n} \lambda_{j,\ell}\,\mathrm{i}\,dz^j_\ell \wedge d\bar z^j_\ell \le \gamma_{|B_j} \le \sum_{1\le\ell\le n} (\lambda_{j,\ell}+\varepsilon)\,\mathrm{i}\,dz^j_\ell \wedge d\bar z^j_\ell$$

in terms of holomorphic coordinates $(z^j_\ell)_{1\le\ell\le n}$ on B_j (for this we just diagonalize $\gamma(a_j)$ at the center a_j of B_j, and take the radius of B_j small enough). By construction $\psi_{j,\varepsilon}(z) = \varphi(z) - \sum_{1\le\ell\le n}\lambda_{j,\ell}|z^j_\ell|^2$ is psh on B_ℓ, and we can thus obtain approximations $\psi_{j,\varepsilon,m}$ of ψ_j by the Bergman kernel process applied on each ball B_j. The idea is to define a global approximation of φ by putting

$$\varphi_{\varepsilon,m}(x) = \frac{1}{m}\log\Bigg(\sum_{1\le j\le N(\varepsilon)} \theta_{j,\varepsilon}(x)\,\exp\Big(m\Big(\psi_{j,\varepsilon,m}(x) + \sum_{1\le\ell\le n}(\lambda_{j,\ell}-\varepsilon)|z^j_\ell|^2\Big)\Big)\Bigg)$$

where $(\theta_{j,\varepsilon})_{1\le j\le N(\varepsilon)}$ is a partition of unity subordinate to the B_j's. If we take $\varepsilon = \varepsilon_m$ and $\varphi_m = \varphi_{\varepsilon_m,m}$ where ε_m decays very slowly, then it is not hard to check that $T_m = dd^c\varphi_m$ satisfies the required estimates; it is essentially enough to observe that the derivatives of $\theta_{j,\varepsilon}$ are "killed" by the factor $\frac{1}{m}$ when $m \gg \frac{1}{\varepsilon}$. □

We need a variant of Th. 1.6 providing more "equisingularity" in the sense that the multiplier ideal sheaves are preserved. A priori, this can be done a priori at the expense of accepting more complicated singularities, which can no longer be guaranteed to be logarithmic poles; a posteriori, using the deep result of [GZ13] on the strong openness conjecture, it would be possible to do so, but we indicate here a way of bypassing that difficult result.

1.7. Theorem. *Let $T = \alpha + dd^c\varphi$ be a closed $(1,1)$-current on a compact Hermitian manifold (X,ω), where α is a smooth closed $(1,1)$-form and φ a quasi-psh function. Let γ be a continuous real $(1,1)$-form such that $T \ge \gamma$. Then one can write $\varphi = \lim_{m\to+\infty}\widetilde\varphi_m$ where*

(a) *$\widetilde\varphi_m$ is smooth in the complement $X \smallsetminus Z_m$ of an analytic set $Z_m \subset X$;*

(b) *$\{\widetilde\varphi_m\}$ is a non-increasing sequence, and $Z_m \subset Z_{m+1}$ for all m;*

(c) *$\int_X (e^{-\varphi} - e^{-\widetilde\varphi_m})dV_\omega$ is finite for every m and converges to 0 as $m \to +\infty$;*

(d) *("equisingularity") $\mathcal{I}(\widetilde\varphi_m) = \mathcal{I}(\varphi)$ for all m ;*

(e) $T_m = \alpha + dd^c\widetilde{\varphi}_m$ *satisfies* $T_m \ge \gamma - \varepsilon_m\omega$, *where* $\lim_{m\to+\infty}\varepsilon_m = 0$.

Proof. (A substantial simplication of the original proof in [DPS01].) As in the previous proof, we may assume that $\alpha = 0$ and $T = dd^c\varphi$, and after subtracting a constant to φ we can also achieve that $\varphi \le -1$ everywhere on X. For every germ $f \in \mathcal{O}_{X,x}$, (c) implies $\int_U |f|^2(e^{-\varphi} - e^{-\widetilde{\varphi}_m})dV_\omega < +\infty$ on some neighborhood U of x, hence the integrals $\int_U |f|^2e^{-\varphi}dV_\omega$ and $\int_U |f|^2e^{-\widetilde{\varphi}_m}dV_\omega$ are simultaneously convergent or divergent, and (d) follows trivially. We define

$$\widetilde{\varphi}_m(x) = \sup_{k\ge m}\,(1+2^{-k})\varphi_{p_k}$$

where (p_k) is a multiplicative sequence that grows fast enough, with $\varphi_{p_{k+1}} \le \varphi_{p_k} \le 0$ for all k. Clearly $\widetilde{\varphi}_m$ is a non-increasing sequence, and

$$\lim_{m\to+\infty}\widetilde{\varphi}_m(x) = \lim_{k\to+\infty}\varphi_{p_k}(x) = \varphi(x)$$

at every point $x \in X$. If Z_m is the set of poles of φ_{p_m}, it is easy to see that

$$\widetilde{\varphi}_m(x) = \lim_{\ell\to+\infty}\ \sup_{k\in[m,\ell]}\,(1+2^{-k})\varphi_{p_k}$$

converges uniformly on every compact subset of $X \smallsetminus Z_m$, since any new term $(1+2^{-\ell})\varphi_{p_\ell}$ may contribute to the sup only in case

$$\varphi_{p_\ell} \ge \frac{1+2^{-p_m}}{1+2^{-p_\ell}}\varphi_{p_m}\quad (\ge 2\varphi_{p_m}),$$

and the difference of that term with respect to the previous term $(1+2^{-(\ell-1)})\varphi_{p_{\ell-1}} \ge (1+2^{-(\ell-1)})\varphi_{p_\ell}$ is less than $2^{-\ell}|\varphi_{p_\ell}| \le 2^{1-\ell}|\varphi_{p_m}|$. Therefore $\widetilde{\varphi}_m$ is continuous on $X \smallsetminus Z_m$, and getting it to be smooth is only a matter of applying Richberg's approximation technique ([Ric68], [Dem12]). The only serious thing to prove is property (c). To achieve this, we observe that $\{\varphi < \widetilde{\varphi}_m\}$ is contained in the union $\bigcup_{k\ge m}\{\varphi < (1+2^{-k})\varphi_{p_k}\}$, therefore

$$\int_X \big(e^{-\varphi} - e^{-\widetilde{\varphi}_m}\big)dV_\omega \le \sum_{k=m}^{+\infty}\int_X \mathbf{1}_{\varphi<(1+2^{-k})\varphi_{p_k}}e^{-\varphi}dV_\omega \tag{1.8}$$

and

$$\begin{aligned}\int_X \mathbf{1}_{\varphi<(1+2^{-k})\varphi_{p_k}}e^{-\varphi}dV_\omega &= \int_X \mathbf{1}_{\varphi<(1+2^{-k})\varphi_{p_k}}\exp\big(2^k\varphi - (2^k+1)\varphi\big)dV_\omega\\ &\le \int_X \mathbf{1}_{\varphi<(1+2^{-k})\varphi_{p_k}}\exp\big((2^k+1)(\varphi_{p_k}-\varphi)\big)dV_\omega\\ &\le \int_X \mathbf{1}_{\varphi<(1+2^{-k})\varphi_{p_k}}\exp\big(2p_k(\varphi_{p_k}-\varphi)\big)dV_\omega \qquad (1.9)\end{aligned}$$

if we take $p_k > 2^{k-1}$ (notice that $\varphi_{p_k} - \varphi \ge 0$). Now, by Lemma 1.10 below, our integral (1.9) is finite. By Lebesgue's monotone convergence theorem, we have for k fixed

$$\lim_{p\to+\infty}\int_X \mathbf{1}_{\varphi<(1+2^{-k})\varphi_p}e^{-\varphi}dV_\omega = 0$$

as a decreasing limit, and we can take p_k so large that $\int_{\varphi<(1+2^{-k})\varphi_{p_k}} e^{-\varphi}dV_\omega \le 2^{-k}$. This ensures that property (c) holds true by (1.8). □

1.10. Lemma. *On a compact complex manifold, for any quasi-psh potential φ, the Bergman kernel procedure leads to quasi-psh potentials φ_m with analytic singularities such that*

$$\int_X e^{2m(\varphi_m-\varphi)}dV_\omega < +\infty.$$

Proof. By definition of φ_m in Th. 1.2 , $\exp(2m(\varphi_m))$ is (up to the irrelevant partition of unity procedure) equal to the Bergman kernel $\sum_{\ell\in\mathbb{N}}|g_{m,\ell}|^2$. By local uniform convergence and the Noetherian property, it has the same local vanishing behavior as a finite sum $\sum_{\ell\le N(m)}|g_{m,\ell}|^2$ with $N(m)$ sufficiently large. Since all terms $g_{m,\ell}$ have L^2 norm equal to 1 with respect to the weight $e^{-2m\varphi}$, our contention follows. □

1.11. Remark. A very slight variation of the proof would yield the improved condition

$$\text{(c')}\quad \forall\lambda\in\mathbb{R}_+,\quad \int_X(e^{-\lambda\varphi}-e^{-\lambda\widetilde{\varphi}_m})dV_\omega \le 2^{-m} \ \text{ for } m\ge m_0(\lambda),$$

and thus an equality $\mathcal{I}(\lambda\widetilde{\varphi}_m)=\mathcal{I}(\lambda\varphi)$ for $m\ge m_0(\lambda)$. We just need to replace estimate (1.8) by

$$\int_X\big(e^{-m\varphi}-e^{-m\widetilde{\varphi}_m}\big)dV_\omega \le \sum_{k=m}^{+\infty}\int_X \mathbf{1}_{\varphi<(1+2^{-k})\varphi_{p_k}}e^{-k\varphi}dV_\omega$$

and take p_k so large that $2p_k\ge k(2^k+1)$ and $\int_{\varphi<(1+2^{-k})\varphi_{p_k}} e^{-k\varphi}dV_\omega \le 2^{-k-1}$. □

We also quote the following very simple consequence of Lemma 1.10 , which will be needed a bit later. Since φ_m is less singular than φ, we have of course an inclusion $\mathcal{I}(\lambda\varphi)\subset\mathcal{I}(\lambda\varphi_m)$ for all $\lambda\in\mathbb{R}_+$. Conversely :

1.12. Corollary. *For every pair of positive real numbers $\lambda'>\lambda>0$, we have an inclusion of multiplier ideals*

$$\mathcal{I}(\lambda'\varphi_m)\subset\mathcal{I}(\lambda\varphi) \qquad \textit{as soon as}\ \ m\ge\Big\lceil\frac{1}{2}\frac{\lambda\lambda'}{\lambda'-\lambda}\Big\rceil.$$

Proof. If $f\in\mathcal{O}_{X,x}$ and U is a sufficiently small neighborhood of x, the Hölder inequality for conjugate exponents $p,q>1$ yields

$$\int_U|f|^2e^{-\lambda\varphi}dV_\omega \le \Big(\int_U|f|^2e^{-\lambda'\varphi_m}dV_\omega\Big)^{1/p}\Big(\int_U|f|^2e^{\frac{q}{p}\lambda'\varphi_m-q\lambda\varphi}dV_\omega\Big)^{1/q}.$$

Therefore, if $f\in\mathcal{I}(\lambda'\varphi_m)_x$, we infer that $f\in\mathcal{I}(\lambda\varphi)_x$ as soon as the integral $\int_X e^{\frac{q}{p}\lambda'\varphi_m-q\lambda\varphi}dV_\omega$ is convergent. If we select $p\in\,]1,\lambda'/\lambda]$, this is implied by the condition $\int_X e^{q\lambda(\varphi_m-\varphi)}dV_\omega<+\infty$. If we further take $q\lambda=2m_0$ to be an even integer so large that

$$p=\frac{q}{q-1}=\frac{2m_0/\lambda}{2m_0/\lambda-1}\le\frac{\lambda'}{\lambda},\qquad \text{e.g.}\ \ m_0=m_0(\lambda,\lambda')=\Big\lceil\frac{1}{2}\frac{\lambda\lambda'}{\lambda'-\lambda}\Big\rceil,$$

then we indeed have $\int_X e^{2m_0(\varphi_m-\varphi)}dV_\omega\le\int_X e^{2m(\varphi_m-\varphi)}dV_\omega<+\infty$ for $m\ge m_0(\lambda,\lambda')$, thanks to Lemma 1.10 . □

2. Hard Lefschetz theorem for pseudoeffective line bundles

2.1. A variant of the Bochner formula

We first recall a variation of the Bochner formula that is required in the proof of the Hard Lefschetz Theorem with values in a positively curved (an therefore non flat) line bundle (L, h). Here the base manifold is a Kähler (non necessarily compact) manifold (Y, ω). We denote by $|\ \ | = |\ \ |_{\omega,h}$ the pointwise Hermitian norm on $\Lambda^{p,q}T^*_Y \otimes L$ associated with ω and h, and by $\|\ \ \| = \|\ \ \|_{\omega,h}$ the global L^2 norm

$$\|u\|^2 = \int_Y |u|^2 dV_\omega \qquad \text{where} \quad dV_\omega = \frac{\omega^n}{n!}$$

We consider the $\overline{\partial}$ operator acting on (p, q)-forms with values in L, its adjoint $\overline{\partial}^*_h$ with respect to h and the complex Laplace-Beltrami operator $\Delta''_h = \overline{\partial}\overline{\partial}^*_h + \overline{\partial}^*_h\overline{\partial}$. Let v be a smooth $(n-q, 0)$-form with compact support in Y. Then $u = \omega^q \wedge v$ satisfies

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}^*_h u\|^2 = \|\overline{\partial}v\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j\in J} \lambda_j\Big)|u_{IJ}|^2 \tag{2.1.1}$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are the curvature eigenvalues of $\Theta_{L,h}$ expressed in an orthonormal frame $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ (at some fixed point $x_0 \in Y$), in such a way that

$$\omega_{x_0} = \mathrm{i} \sum_{1\leq j\leq n} dz_j \wedge d\overline{z}_j, \qquad (\Theta_{L,h})_{x_0} = dd^c\varphi_{x_0} = \mathrm{i} \sum_{1\leq j\leq n} \lambda_j dz_j \wedge d\overline{z}_j.$$

Formula (2.1.1) follows from the more or less straightforward identity

$$(\overline{\partial}^*_\varphi\, \overline{\partial} + \overline{\partial}\, \overline{\partial}^*_\varphi)(v \wedge \omega^q) - (\overline{\partial}^*_\varphi\, \overline{\partial}v) \wedge \omega^q = q\ \mathrm{i}\,\partial\overline{\partial}\varphi \wedge \omega^{q-1} \wedge v,$$

by taking the inner product with $u = \omega^q \wedge v$ and integrating by parts in the left hand side (we leave the easy details to the reader). Our formula is thus established when v is smooth and compactly supported. In general, we have:

2.1.2. Proposition. *Let (Y, ω) be a* complete *Kähler manifold and (L, h) a smooth Hermitian line bundle such that the curvature possesses a uniform lower bound $\Theta_{L,h} \geq -C\omega$. For every measurable $(n - q, 0)$-form v with L^2 coefficients and values in L such that $u = \omega^q \wedge v$ has differentials $\overline{\partial}u$, $\overline{\partial}^* u$ also in L^2, we have*

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}^*_h u\|^2 = \|\overline{\partial}v\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j\in J} \lambda_j\Big)|u_{IJ}|^2$$

(*here, all differentials are computed in the sense of distributions*).

Proof. Since (Y, ω) is assumed to be complete, there exists a sequence of smooth forms v_ν with compact support in Y (obtained by truncating v and taking the convolution with a regularizing kernel) such that $v_\nu \to v$ in L^2 and such that $u_\nu = \omega^q \wedge v_\nu$ satisfies $u_\nu \to u$, $\overline{\partial}u_\nu \to \overline{\partial}u$, $\overline{\partial}^* u_\nu \to \overline{\partial}^* u$ in L^2. By the curvature assumption, the final integral in the right hand side of (2.1.1) must be under control (i.e. the integrand becomes nonnegative if we add a term $C\|u\|^2$ on both sides, $C \gg 0$). We thus get the equality by passing to the limit and using Lebesgue's monotone convergence theorem. $\square$

2.2. Proof of Theorem 0.8

Here X denotes a compact K ähler manifold equipped with a Kähler metric ω, and (L, h) is a pseudoeffective line bundle on X. To fix the ideas, we first indicate the proof in the much simpler case when (L, h) has a smooth metric h (so that $\mathcal{I}(h) = \mathcal{O}_X$), and then treat the general case.

2.2.1. *Special Case: (L, h) is Hermitian semipositive (with a smooth metric).*

Let $\{\beta\} \in H^q(X, \Omega_X^n \otimes L)$ be an arbitrary cohomology class. By standard L^2 Hodge theory, $\{\beta\}$ can be represented by a smooth harmonic $(0, q)$-form β with values in $\Omega_X^n \otimes L$. We can also view β as a (n, q)-form with values in L. The pointwise Lefschetz isomorphism produces a unique $(n-q, 0)$-form α such that $\beta = \omega^q \wedge \alpha$. Proposition 2.1.2 then yields

$$\|\overline{\partial}\alpha\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j\in J} \lambda_j\Big)|\alpha_{IJ}|^2 = \|\overline{\partial}\beta\|^2 + \|\overline{\partial}_h^*\beta\|^2 = 0,$$

and the curvature eigenvalues λ_j are nonnegative by our assumption. Hence $\overline{\partial}\alpha = 0$ and $\{\alpha\} \in H^0(X, \Omega_X^{n-q} \otimes L)$ is mapped to $\{\beta\}$ by $\Phi_{\omega,h}^q = \omega^q \wedge \bullet$.

2.2.2. *General Case.*

There are several difficulties. The first difficulty is that the metric h is no longer smooth and we cannot directly represent cohomology classes by harmonic forms. We circumvent this problem by smoothing the metric on an (analytic) Zariski open subset and by avoiding the remaining poles on the complement. However, some careful estimates have to be made in order to take the error terms into account.

Fix $\varepsilon = \varepsilon_\nu$ and let $h_\varepsilon = h_{\varepsilon_\nu}$ be an approximation of h, such that h_ε is smooth on $X \smallsetminus Z_\varepsilon$ (Z_ε being an analytic subset of X), $\Theta_{L,h_\varepsilon} \geq -\varepsilon\omega$, $h_\varepsilon \leq h$ and $\mathcal{I}(h_\varepsilon) = \mathcal{I}(h)$. This is possible by Th. 1.7 . Now, we can find a family

$$\omega_{\varepsilon,\delta} = \omega + \delta(\mathrm{i}\,\partial\overline{\partial}\psi_\varepsilon + \omega), \qquad \delta > 0$$

of *complete Kähler* metrics on $X \smallsetminus Z_\varepsilon$, where ψ_ε is a quasi-psh function on X with $\psi_\varepsilon = -\infty$ on Z_ε, ψ_ε on $X \smallsetminus Z_\varepsilon$ and $\mathrm{i}\,\partial\overline{\partial}\psi_\varepsilon + \omega \geq 0$ (see e.g. [Dem82], Théorème 1.5). By construction, $\omega_{\varepsilon,\delta} \geq \omega$ and $\lim_{\delta\to 0} \omega_{\varepsilon,\delta} = \omega$. We look at the L^2 Dolbeault complex $K_{\varepsilon,\delta}^{\bullet}$ of $(n, \bullet)$-forms on $X \smallsetminus Z_\varepsilon$, where the L^2 norms are induced by $\omega_{\varepsilon,\delta}$ on differential forms and by h_ε on elements in L. Specifically

$$K_{\varepsilon,\delta}^q = \Big\{u{:}X \smallsetminus Z_\varepsilon \to \Lambda^{n,q} T_X^* \otimes L;\ \int_{X\smallsetminus Z_\varepsilon} (|u|^2_{\Lambda^{n,q}\omega_{\varepsilon,\delta}\otimes h_\varepsilon} + |\overline{\partial}u|^2_{\Lambda^{n,q+1}\omega_{\varepsilon,\delta}\otimes h_\varepsilon}) dV_{\omega_{\varepsilon,\delta}} < \infty\Big\}.$$

Let $\mathcal{K}_{\varepsilon,\delta}^q$ be the corresponding sheaf of germs of locally L^2 sections on X (the local L^2 condition should hold on X, not only on $X \smallsetminus Z_\varepsilon$!). Then, for all $\varepsilon > 0$ and $\delta \geq 0$, $(\mathcal{K}_{\varepsilon,\delta}^q, \overline{\partial})$ is a resolution of the sheaf $\Omega_X^n \otimes L \otimes \mathcal{I}(h_\varepsilon) = \Omega_X^n \otimes L \otimes \mathcal{I}(h)$. This is because L^2 estimates hold locally on small Stein open sets, and the L^2 condition on $X \smallsetminus Z_\varepsilon$ forces holomorphic sections to extend across Z_ε ([Dem82], Lemma 6.9).

Let $\{\beta\} \in H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h))$ be a cohomology class represented by a smooth form with values in $\Omega_X^n \otimes L \otimes \mathcal{I}(h)$ (one can use a Čech cocycle and convert it to an element in the $\mathcal{C}^\infty$ Dolbeault complex by means of a partition of unity, thanks to the usual De Rham-Weil isomorphism, see also the final proof in Section 4 for more details). Then

$$\|\beta\|^2_{\varepsilon,\delta} \leq \|\beta\|^2 = \int_X |\beta|^2_{\Lambda^{n,q}\omega\otimes h} dV_\omega < +\infty.$$

The reason is that $|\beta|^2_{\Lambda^{n,q}\omega\otimes h}dV_\omega$ decreases as ω increases. This is just an easy calculation, shown by comparing two metrics ω, ω' which are expressed in diagonal form in suitable coordinates; the norm $|\beta|^2_{\Lambda^{n,q}\omega\otimes h}$ turns out to decrease faster than the volume dV_ω increases; see e.g. [Dem82], Lemma 3.2; a special case is $q = 0$, then $|\beta|^2_{\Lambda^{n,q}\omega\otimes h}dV_\omega = i^{n^2}\beta\wedge\overline{\beta}$ with the identification $L\otimes\overline{L}\simeq\mathbb{C}$ given by the metric h, hence the integrand is even independent of ω in that case.

By the proof of the De Rham-Weil isomorphism, the map $\alpha\mapsto\{\alpha\}$ from the cocycle space $Z^q(\mathcal{K}^\bullet_{\varepsilon,\delta})$ equipped with its L^2 topology, into $H^q(X,\Omega^n_X\otimes L\otimes\mathcal{I}(h))$ equipped with its finite vector space topology, is continuous. Also, Banach's open mapping theorem implies that the coboundary space $B^q(\mathcal{K}^\bullet_{\varepsilon,\delta})$ is closed in $Z^q(\mathcal{K}^\bullet_{\varepsilon,\delta})$. This is true for all $\delta\geq 0$ (the limit case $\delta = 0$ yields the strongest L^2 topology in bidegree (n,q)). Now, β is a $\overline{\partial}$-closed form in the Hilbert space defined by $\omega_{\varepsilon,\delta}$ on $X\smallsetminus Z_\varepsilon$, so there is a $\omega_{\varepsilon,\delta}$-harmonic form $u_{\varepsilon,\delta}$ in the same cohomology class as β, such that

$$\|u_{\varepsilon,\delta}\|_{\varepsilon,\delta}\leq\|\beta\|_{\varepsilon,\delta}. \tag{2.2.3}$$

Let $v_{\varepsilon,\delta}$ be the unique $(n-q,0)$-form such that $u_{\varepsilon,\delta} = v_{\varepsilon,\delta}\wedge\omega^q_{\varepsilon,\delta}$ ($v_{\varepsilon,\delta}$ exists by the pointwise Lefschetz isomorphism). Then

$$\|v_{\varepsilon,\delta}\|_{\varepsilon,\delta} = \|u_{\varepsilon,\delta}\|_{\varepsilon,\delta}\leq\|\beta\|_{\varepsilon,\delta}\leq\|\beta\|.$$

As $\sum_{j\in J}\lambda_j\geq -q\varepsilon$ by the assumption on Θ_{L,h_ε}, the Bochner formula yields

$$\|\overline{\partial}v_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\leq q\varepsilon\|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\leq q\varepsilon\|\beta\|^2.$$

These uniform bounds imply that there are subsequences $u_{\varepsilon,\delta_\nu}$ and $v_{\varepsilon,\delta_\nu}$ with $\delta_\nu\to 0$, possessing weak-L^2 limits $u_\varepsilon = \lim_{\nu\to+\infty}u_{\varepsilon,\delta_\nu}$ and $v_\varepsilon = \lim_{\nu\to+\infty}v_{\varepsilon,\delta_\nu}$. The limit $u_\varepsilon = \lim_{\nu\to+\infty}u_{\varepsilon,\delta_\nu}$ is with respect to $L^2(\omega) = L^2(\omega_{\varepsilon,0})$. To check this, notice that in bidegree $(n-q,0)$, the space $L^2(\omega)$ has the weakest topology of all spaces $L^2(\omega_{\varepsilon,\delta})$; indeed, an easy calculation made in ([Dem82], Lemma 3.2) yields

$$|f|^2_{\Lambda^{n-q,0}\omega\otimes h}dV_\omega\leq|f|^2_{\Lambda^{n-q,0}\omega_{\varepsilon,\delta}\otimes h}dV_{\omega_{\varepsilon,\delta}}\qquad\text{if } f \text{ is of type } (n-q,0).$$

On the other hand, the limit $v_\varepsilon = \lim_{\nu\to+\infty}v_{\varepsilon,\delta_\nu}$ takes place in all spaces $L^2(\omega_{\varepsilon,\delta})$, $\delta > 0$, since the topology gets stronger and stronger as $\delta\downarrow 0$ [possibly not in $L^2(\omega)$, though, because in bidegree (n,q) the topology of $L^2(\omega)$ might be strictly stronger than that of all spaces $L^2(\omega_{\varepsilon,\delta})$]. The above estimates yield

$$\begin{aligned}
&\|v_\varepsilon\|^2_{\varepsilon,0} = \int_X|v_\varepsilon|^2_{\Lambda^{n-q,0}\omega\otimes h_\varepsilon}dV_\omega\leq\|\beta\|^2,\\
&\|\overline{\partial}v_\varepsilon\|^2_{\varepsilon,0}\leq q\varepsilon\|\beta\|^2_{\varepsilon,0},
\end{aligned}$$

$$u_\varepsilon = \omega^q\wedge v_\varepsilon\equiv\beta\qquad\text{in}\ \ H^q(X,\Omega^n_X\otimes L\otimes\mathcal{I}(h_\varepsilon)).$$

Again, by arguing in a given Hilbert space $L^2(h_{\varepsilon_0})$, we find L^2 convergent subsequences $u_\varepsilon\to u$, $v_\varepsilon\to v$ as $\varepsilon\to 0$, and in this way get $\overline{\partial}v = 0$ and

$$\begin{aligned}
&\|v\|^2\leq\|\beta\|^2,\\
&u = \omega^q\wedge v\equiv\beta\qquad\text{in}\ \ H^q(X,\Omega^n_X\otimes L\otimes\mathcal{I}(h)).
\end{aligned}$$

Theorem 0.8 is proved. Notice that the equisingularity property $\mathcal{I}(h_\varepsilon)=\mathcal{I}(h)$ is crucial in the above proof, otherwise we could not infer that $u\equiv\beta$ from the fact that $u_\varepsilon\equiv\beta$. This is true only because all cohomology classes $\{u_\varepsilon\}$ lie in the same fixed cohomology group $H^q(X,\Omega_X^n\otimes L\otimes\mathcal{I}(h))$, whose topology is induced by the topology of $L^2(\omega)$ on $\overline\partial$-closed forms (e.g. through the De Rham-Weil isomorphism). □

2.2.4. Remark. In (2.2.3), the existence of a harmonic representative holds true only for $\omega_{\varepsilon,\delta}$, $\delta>0$, because we need to have a complete Kähler metric on $X\smallsetminus Z_\varepsilon$. The trick of employing $\omega_{\varepsilon,\delta}$ instead of a fixed metric ω, however, is not needed when Z_ε is (or can be taken to be) empty. This is the case if (L,h) is such that $\mathcal{I}(h)=\mathcal{O}_X$ and L is nef. Indeed, by definition, L is nef iff there exists a sequence of smooth metrics h_ν such that $\mathrm{i}\,\Theta_{L,h_\nu}\geq-\varepsilon_\nu\omega$, so we can take the φ_ν's to be everywhere smooth in Th. 1.7 . However, multiplier ideal sheaves are needed in the surjectivity statement even in case L is nef, as it may happen that $\mathcal{I}(h_{\min})\neq\mathcal{O}_X$ even then, and $h:=\lim h_\nu$ is anyway always more singular than $h_{\min}$. Let us recall a standard example (see [DPS94], [DPS01]). Let B be an elliptic curve and let V be the rank 2 vector bundle over B which is defined as the (unique) non split extension

$$0\to\mathcal{O}_B\to V\to\mathcal{O}_B\to 0.$$

In particular, the bundle V is numerically flat, i.e. $c_1(V)=0$, $c_2(V)=0$. We consider the ruled surface $X=\mathbb{P}(V)$. On that surface there is a unique section $C=\mathbb{P}(\mathcal{O}_B)\subset X$ with $C^2=0$ and

$$\mathcal{O}_X(C)=\mathcal{O}_{\mathbb{P}(V)}(1)$$

is a nef line bundle. One can check that $L=\mathcal{O}_{\mathbb{P}(V)}(3)$ leads to a *zero* Lefschetz map

$$\omega\wedge\bullet:\quad H^0(X,\Omega_X^1\otimes L)\longrightarrow H^1(X,K_X\otimes L)\simeq\mathbb{C},$$

so this is a counterexample to Cor. 0.9 in the nef case. Incidentally, this also shows (in a somewhat sophisticated way) that $\mathcal{O}_{\mathbb{P}(V)}(1)$ is nef but not semipositive, a fact that was first observed in [DPS94].

3. Numerical dimension of currents

A large part of this section borrows ideas from S. Boucksom's [Bou02], [Bou04] and Junyan Cao's [JC14] PhD theses. We try however to give here a slightly more formal exposition. The main difference with S. Boucksom's approach is that we insist on keeping track of singularities of currents and leaving them unchanged, instead of trying to minimize them in each cohomology class.

3.1. Monotone asymptotically equisingular approximations

Let X be a compact complex n-dimensional manifold. We consider the closed convex cone of *pseudoeffective classes*, namely the set $\mathcal{E}(X)$ of cohomology classes $\{\alpha\}\in H^{1,1}(X,\mathbb{R})$ containing a closed positive $(1,1)$-current $T=\alpha+dd^c\varphi$ (in the non Kähler case one should use Bott-Chern cohomology groups here, but we will be mostly concerned with the Kähler case in the sequel). We also introduce the set $\mathcal{S}(X)$ of singularity equivalence classes of closed positive $(1,1)$-currents $T=\alpha+dd^c\varphi$ (i.e., α being fixed, up to equivalence of singularities of the potentials φ, cf. Def. 0.5). Clearly, there is a fibration

$$\pi:\mathcal{S}(X)\to\mathcal{E}(X),\qquad T\mapsto\{\alpha\}\in\mathcal{E}(X)\subset H^{1,1}(X,\mathbb{R}).\tag{3.1.1}$$

We will denote by $\mathcal{S}_\alpha(X)$ the fiber $\pi^{-1}(\{\alpha\})$ of $\mathcal{S}(X)$ over a given cohomology class $\{\alpha\}\in\mathcal{E}(X)$. Observe that the base $\mathcal{E}(X)$ is a closed convex cone in a finite dimensional vector space, but in general the fiber $\mathcal{S}_\alpha(X)$ must be viewed as a very complicated infinite dimensional space : if we take e.g. $\{\alpha_1\}\in H^{1,1}(\mathbb{P}^n,\mathbb{R})$ to be the unit class $c_1(\mathcal{O}(1))$, then any current $T=\frac{1}{d}[H]$ where H_d is an irreducible hypersurface of degree d defines a point in $\mathcal{S}_{\alpha_1}(\mathbb{P}^n)$, and these points are all distinct. The set $\mathcal{S}(X)$ is nevertheless equipped in a natural way with an addition law $\mathcal{S}(X)\times\mathcal{S}(X)\to\mathcal{S}(X)$ that maps $\mathcal{S}_\alpha(X)+\mathcal{S}_\beta(X)$ into $\mathcal{S}_{\alpha+\beta}(X)$, a scalar multiplication $\mathbb{R}_+\times\mathcal{S}(X)\to\mathcal{S}(X)$ that takes $\lambda\cdot\mathcal{S}_\alpha(X)$ to the fiber $\mathcal{S}_{\lambda\alpha}(X)$. In this way, $\mathcal{S}(X)$ should be viewed as some sort of infinite dimensional convex cone. The fibers $\mathcal{S}_\alpha(X)$ also possess a partial ordering $\preccurlyeq$ (cf. Def. 0.5) such that $\forall j,\ S_j\preccurlyeq T_j\Rightarrow\sum S_j\preccurlyeq\sum T_j$, and a fiberwise "min" operation

$$\begin{aligned}&\min:\mathcal{S}_\alpha(X)\times\mathcal{S}_\alpha(X)\longrightarrow\mathcal{S}_\alpha(X),\\ &(T_1,T_2)=(\alpha+dd^c\varphi_1,\alpha+dd^c\varphi_2)\longmapsto T=\alpha+dd^c\max(\varphi_1,\varphi_2),\end{aligned}\tag{3.1.2}$$

with respect to which the addition is distributive, i.e.

$$\min(T_1+S,T_2+S)=\min(T_1,T_2)+S.$$

Notice that when $T_1=\frac{1}{d}[H_1]$, $T_2=\frac{1}{d}[H_2]$ are effective $\mathbb{Q}$-divisors, all these operations $+$, $\cdot$, $\min(\bullet)$ and the ordering $\preccurlyeq$ coincide with the usual ones known for divisors. Following Junyan Cao [JC14] (with slightly more restrictive requirements that do not produce much change in practice), we introduce

3.1.3. Definition. *Let $T=\alpha+dd^c\varphi$ be a closed positive $(1,1)$-current on X, where α is a smooth closed $(1,1)$-form and φ is a quasi-psh function on X. We say that the sequence of currents $T_k=\alpha+dd^c\psi_k$, $k\in\mathbb{N}$, is a "monotone asymptotically equisingular approximation of T by currents with analytic singularities" if the sequence of potentials (ψ_k) satisfies the following properties*:

(a) (monotonicity) *The sequence (ψ_k) is non-increasing and converges to φ at every point of X.*

(b) *The functions ψ_k have analytic singularities (and $\psi_k\preccurlyeq\psi_{k+1}$ by* (a)*).*

(c) (lower bound of positivity)

$$\alpha+dd^c\psi_k\geq-\varepsilon_k\cdot\omega\qquad\textit{with}\quad\lim_{k\to+\infty}\varepsilon_k=0$$

for any given smooth positive hermitian $(1,1)$-form ω on X.

(d) (asymptotic equisingularity) *For every pair of positive numbers $\lambda'>\lambda>0$, there exists an integer $k_0(\lambda,\lambda')\in\mathbb{N}$ such that*

$$\mathcal{I}(\lambda'\psi_k)\subset\mathcal{I}(\lambda\varphi)\qquad\textit{for}\ \ k\geq k_0(\lambda,\lambda').$$

3.1.4. Remark. Without loss of generality, one can always assume that the quasi-psh potentials $\varphi_k=c_k\log|g_k|^2+O(1)$ have rational coefficients $c_k\in\mathbb{Q}_+$. In fact, after subtracting constants, one can achieve that $\varphi\leq 0$ and $\psi_k\leq 0$ for all k. If the c_k are arbitrary nonnegative real numbers, one can always replace ψ_k by $\psi'_k=(1-\delta_k)\psi_k$ with a decreasing sequence $\delta_k\in\,]0,1[$ such that $\lim\delta_k=0$ and $(1-\delta_k)c_k\in\mathbb{Q}_+$. Then (a), (b), (d) are still valid, and (c) holds with $\varepsilon'_k=(1-\delta_k)\varepsilon_k+C\delta_k$ and C a constant such that $\alpha\geq-C\omega$. □

The fundamental observation is:

3.1.5. Theorem. *If $\psi_k := \varphi_{m_k}$ is the sequence of potentials obtained by the Bergman kernel approximation of $T = \alpha + dd^c\varphi$ given in the proof of Theorem 1.6 and (m_k) is a multiplicative sequence, then the ψ_k can be arranged to satisfy the positivity, monotonicity and asymptotic equisingularity properties of Definition 3.1.3 . Moreover, if we start with currents $T \preccurlyeq T'$ in the same cohomology class $\{\alpha\}$, we obtain corresponding approximations that satisfy $\psi_k \preccurlyeq \psi'_k$.*

Proof. By Cor. 1.12 , the asymptotic equisingularity property (d) in Def. 3.1.3 is satisfied for $m_k \geq \lceil \frac{1}{2}\frac{\lambda\lambda'}{\lambda'-\lambda}\rceil$. The other properties are already known or obvious, especially the coefficients $c_k = \frac{1}{m_k}$ are just inverses of integers in that case. □

The following proposition provides a precise comparison of analytic singularities of potentials when their multiplier ideal sheaves satisfy inclusion relations.

3.1.6. Proposition. *Let φ, ψ be quasi-psh functions with analytic singularities, let $c > 0$ be the constant such that φ can be expressed as $c\log\sum|g_j|^2 + O(1)$ with holomorphic functions g_j, and let $\lambda \in \mathbb{R}_+$. Denoting $t_+ := \max(t, 0)$, we have the implication*

$$\mathcal{I}(\psi) \subset \mathcal{I}(\lambda\varphi) \quad\Rightarrow\quad \int e^{\psi-\lambda\varphi}dV < +\infty \quad \textit{and} \quad \psi \succcurlyeq \tfrac{1}{c}\big(\lfloor\lambda c\rfloor - n\big)_+\varphi \quad \textit{(locally)}.$$

Proof. Since everything is local, we may assume that φ, ψ are psh functions on a small ball $B \subset \mathbb{C}^n$, and $\varphi(z) = c\log|g|^2 = c\log\sum_{1\leq j\leq N}|g_j(z)|^2$. If $(f_\ell)_{\ell\in\mathbb{N}}$ is a Hilbert basis of the space of L^2 holomorphic functions f with $\int_B |f|^2e^{-\psi}dV < +\infty$, the proof of Th. 1.2 yields $\psi \leq C + \log\sum|f_\ell|^2$ (and locally the singularity is achieved by a finite sum of f_ℓ's by the Noetherian property). After possibly shrinking B, the condition $\mathcal{I}(\psi) \subset \mathcal{I}(\lambda\varphi)$ implies

$$\int_B |f_\ell|^2e^{-\lambda\varphi}dV = \int_B |f_\ell|^2|g|^{-2\lambda c}dV < +\infty.$$

This already shows that $\int e^{\psi-\lambda\varphi}dV < +\infty$ locally. By openness of convergence exponents (one can use e.g. a log resolution of the ideal sheaf (f_ℓ, g_j) to see this), one gets

$$\int_B |f_\ell|^2|g|^{-2(\lfloor\lambda c\rfloor+\varepsilon)}dV < +\infty$$

for $\varepsilon > 0$ small enough. Now, if $\lfloor\lambda c\rfloor \geq n$, Skoda's division theorem [Sko72a] implies that each f_ℓ can be written $f_\ell = \sum h_{\ell,j}g_j$ where $h_{\ell,j}$ satisfies a similar estimate where the exponent of $|g|^{-2}$ is decreased by 1. An iteration of the Skoda division theorem for the $h_{\ell,j}$ yields $f_\ell \in (g_j)^k$ where $k = (\lfloor\lambda c\rfloor - n)_+$. Hence

$$\psi \leq C + \log\sum|f_\ell|^2 \leq C' + k\log|g|^2 \leq C' + \frac{k}{c}\varphi,$$

and our singularity comparison relation follows. □

3.1.7. Corollary. *If $T = \alpha + dd^c\varphi$ is a closed positive $(1,1)$-current and (ψ_k), (ψ'_k) are two monotone asymptotically equisingular approximations of φ with analytic singularities, then for every k and every $\varepsilon > 0$, there exists ℓ such that $(1-\varepsilon)\psi_k \preccurlyeq \psi'_\ell$ (and vice versa by exchanging the roles of (ψ_k) and (ψ'_k)).*

Proof. Let $c > 0$ be the constant occurring in the logarithmic poles of ψ_k (k being fixed). By condition (d) in Def. 3.1.3 , for $\lambda' > \lambda \gg 1$ we have $\mathcal{I}(\lambda'\psi'_\ell) \subset \mathcal{I}(\lambda\varphi) \subset \mathcal{I}(\lambda\psi_k)$ for $\ell \geq \ell_0(\lambda, \lambda')$ large enough. Proposition 3.1.6 implies the singularity estimate $\psi'_\ell \succcurlyeq \frac{1}{c\lambda'}(\lfloor c\lambda \rfloor - n)_+\psi_k$, and the final constant in front of ψ_k can be taken arbitrary close to 1. □

Our next observation is that the $\min(\bullet)$ procedure defined above for currents is well behaved in terms of asymptotic equisingular approximations.

3.1.8. Proposition. *Let $T = \alpha + dd^c\varphi$ and $T' = \alpha + dd^c\varphi'$ be closed positive $(1,1)$-currents in the same cohomology class $\{\alpha\}$. Let (ψ_k) and (ψ'_k) be respective monotone asymptotically equisingular approximations with analytic singularities and rational coefficients. Then $\max(\psi_k, \psi'_k)$ provides a monotone asymptotically equisingular approximation of $\min(T, T') = \alpha + dd^c\max(\varphi, \varphi')$ with analytic singularities and rational coefficients.*

Proof. If $\psi_k = c_k \log|g_k|^2 + O(1)$ and $\psi'_k = c'_k \log|g'_k|^2 + O(1)$, we can write $c_k = p_k/q_k$, $c'_k = p'_k/q'_k$ and

$$\max(\psi_k, \psi'_k) = \frac{1}{q_k q'_k}\log\big(|g_k|^{2p_k} + |g'_k|^{2p'_k}\big) + O(1),$$

hence $\max(\psi_k, \psi'_k)$ also has analytic singularities with rational coefficients (this would not be true with our definitions when the ratio c'_k/c_k is irrational, but of course we could just extend a little bit the definition of what we call analytic singularities, e.g. by allowing arbitrary positive real exponents, in order to avoid this extremely minor annoyance). It is well known that

$$\begin{aligned} &\alpha + dd^c\psi_k \geq -\varepsilon_k\omega, \ \ \alpha + dd^c\psi'_k \geq -\varepsilon'_k\omega \\ &\qquad\qquad\qquad \Rightarrow \ \ \alpha + dd^c\max(\psi_k, \psi'_k) \geq -\max(\varepsilon_k, \varepsilon'_k)\omega. \end{aligned}$$

Finally, if $\psi_{B,k}$ (resp. $\psi'_{B,k}$ and $\widetilde{\psi}_{B,k}$)) comes from the Bergman approximation of φ (resp. of φ' and $\widetilde{\varphi} := \max(\varphi, \varphi')$), we have

$$\widetilde{\varphi} \geq \varphi \ \ \Rightarrow \ \ \widetilde{\psi}_{B,k} \geq \psi_{B,k}, \qquad \widetilde{\varphi} \geq \varphi' \ \ \Rightarrow \ \ \widetilde{\psi}_{B,k} \geq \psi'_{B,k}$$

hence $\widetilde{\psi}_{B,k} \geq \max(\psi_{B,k}, \psi'_{B,k})$ and so $\widetilde{\psi}_{B,k} \preccurlyeq \max(\psi_{B,k}, \psi'_{B,k})$. However, for every $\epsilon > 0$, one has $(1-\varepsilon)\psi_{B_k} \preccurlyeq \psi_\ell$ and $(1-\varepsilon)\psi'_{B_k} \preccurlyeq \psi'_\ell$ for $\ell \geq \ell_0(k, \varepsilon)$ large, therefore $(1-\varepsilon)\widetilde{\psi}_{B,k} \preccurlyeq \max(\psi_\ell, \psi'_\ell)$. This shows that $\max(\psi_\ell, \psi'_\ell)$ has enough singularities (the "opposite" inequality $\max(\psi_\ell, \psi'_\ell) \geq \widetilde{\varphi} = \max(\varphi, \varphi')$, i.e. $\max(\psi_\ell, \psi'_\ell) \preccurlyeq \widetilde{\varphi}$, holds trivially). □

When we deal with sums of positive currents $T = \alpha + dd^c\varphi$ and $T' = \beta + dd^c\varphi'$ in cohomology classes $\{\alpha\}$, $\{\beta\} \in \mathcal{E}(X)$, the sum $\alpha + \beta + dd^c(\psi_{B,k} + \psi'_{B,k})$ of the Bergman approximations is less singular than what comes from the Bergman approximation of $\varphi + \varphi'$. This is a consequence of the fundamental "subadditivity" result $\mathcal{I}(\varphi + \varphi') \subset \mathcal{I}(\varphi)\mathcal{I}(\varphi')$ observed in [DEL00], itself a consequence of the Ohsawa-Takegoshi theorem. We do not know whether $\alpha + \beta + dd^c(\psi_{B,k} + \psi'_{B,k})$ might be asymptotically strictly less singular than the Bergman approximations of $\varphi + \varphi'$; this does not happen when φ or φ' have analytic singularities (or are sums of quasi-psh functions with analytic singularities and of functions with zero Lelong numbers), as one can show easily, but there might be a more subtle issue of a transcendental nature in general. This motivates the following formal definition.

3.1.9. Definition. *For each class $\{\alpha\} \in \mathcal{E}(X)$, we define $\widehat{\mathcal{S}}_\alpha(X)$ as a set of equivalence classes of sequences of quasi-positive currents $T_k = \alpha + dd^c\psi_k$ such that*

(a) $T_k = \alpha + dd^c\psi_k \geq -\varepsilon_k \cdot \omega$ *with* $\lim_{k\to+\infty}\varepsilon_k = 0$,

(b) *the functions* ψ_k *have analytic singularities and* $\psi_k \preccurlyeq \psi_{k+1}$ *for all* k*. We say that* (T_k) *is weakly less singular than* (T'_k) *in* $\widehat{\mathcal{S}}_\alpha(X)$*, and write* $(T_k) \preccurlyeq_W (T'_k)$*, if for every* $\varepsilon > 0$ *and* k*, there exists* ℓ *such that* $(1-\varepsilon)T_k \preccurlyeq T'_\ell$*. Finally, we write* $(T_k) \sim_W (T'_k)$ *when we have* $(T_k) \preccurlyeq_W (T'_k)$ *and* $(T'_k) \preccurlyeq_W (T_k)$*, and define* $\widehat{\mathcal{S}}_\alpha(X)$ *to be the quotient space by this equivalence relation.*

The set

$$\widehat{\mathcal{S}}(X) = \bigcup_{\{\alpha\}\in\mathcal{E}(X)} \widehat{\mathcal{S}}_\alpha(X) \tag{3.1.10}$$

is by construction a fiber space $\hat{\pi} : \widehat{\mathcal{S}}(X) \to \mathcal{E}(X)$, and, by fixing a multiplicative sequence such as $m_k = 2^k$, we find a natural "Bergman approximation map"

$$\mathbf{B} : \mathcal{S}(X) \to \widehat{\mathcal{S}}(X), \qquad T = \alpha + dd^c\varphi \longmapsto (T_{B,k}), \quad T_k = \alpha + dd^c\psi_{B,k}. \tag{3.1.11}$$

The set $\widehat{\mathcal{S}}(X)$ is equipped with a natural addition $(T_k) + (T'_k) = (T_k + T'_k)$, with a scalar multiplication $\lambda\cdot(T_k) = (\lambda T_k)$ for $\lambda \in \mathbb{R}_+$, as well as with the $\min(\bullet)$ operation $\min((T_k),(T'_k)) = (\min(T_k, T'_k))$ obtained by taking $\max(\psi_k, \psi'_k)$ of the corresponding potentials. As explained earlier, $\mathbf{B}$ is a morphism for the $\min(\bullet)$ operation, but it is unclear to us whether $\mathbf{B}$ is actually a morphism for addition ($\mathbf{B}$ is at least additive when all currents involved except one have analytic singularities, and these are dense in some sense, so things would be much nicer if there were no exception!)

For closed positive currents themselses, one could define weak equivalence of singularities by

$$T \preccurlyeq_W T' \iff_{\mathrm{def}} (T_{B,k}) \preccurlyeq_W (T'_{B,k}), \tag{3.1.12}$$

$$T \sim_W T' \iff T \preccurlyeq_W T' \text{ and } T' \preccurlyeq_W T, \tag{3.1.13}$$

but it is unclear at this point whether addition is compatible with $\preccurlyeq_W$ and $\sim_W$ on $\mathcal{S}(X)$, so the quotient space $\mathcal{S}(X)/\sim_W$ might be a little bit problematic. By the well-known result of Skoda [Sko72b], we have $\mathcal{I}(\varphi) = \mathcal{O}_X$ as soon as the Lelong numbers $\nu(\varphi, x)$ are less than 2 at every point $x \in X$, hence a quasi-psh function with zero Lelong numbers satisfies $\mathcal{I}(\lambda\varphi) = \mathcal{O}_X$ for every $\lambda > 0$. Such potentials are negligible (and indistinguishable from smooth potentials) in the above definition of $\sim_W$.

3.1.14. Remark. When X is projective algebraic and $\{\alpha\}$ belongs to the Neron-Severi space

$$\mathrm{NS}_{\mathbb{R}}(X) = (H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Z})/\mathrm{torsion}) \otimes_{\mathbb{Z}} \mathbb{R},$$

the fiber $\widehat{\mathcal{S}}_\alpha(X)$ is essentially an algebraic object. In fact, we could define $\widehat{\mathcal{S}}_\alpha(X)$ as the set of suitable equivalence classes of "formal limits" $\lim_{c_1(D)\to\{\alpha\}} \lim_{k\to+\infty} \frac{1}{k}\mathfrak{a}_k$ associated with sequences of graded ideals $\mathfrak{a}_k \subset H^0(X, \mathcal{O}_X(kD)$ satisfying the subadditive property $\mathfrak{a}_{k+\ell} \subset \mathfrak{a}_k\mathfrak{a}_\ell$, where D are big $\mathbb{Q}$-divisors whose first Chern classes $c_1(D)$ approximate $\{\alpha\} \in \mathrm{NS}_{\mathbb{R}}(X)$. Many related questions are discussed in the algebraic setting in Lazarfeld's book [Laz04]. It is nevertheless an interesting point, even in the projective case, that one can "extrapolate" these concepts to all transcendental classes, and get in this way a global space $\widehat{\mathcal{S}}(X)$ which looks well behaved, e.g. semicontinuous, under variation of the complex structure of X.

3.2. Intersection theory on $\mathcal{S}(X)$ and $\widehat{\mathcal{S}}(X)$

Let X be a compact Kähler n-dimensional manifold equipped with a Kähler metric ω. We consider closed positive $(1,1)$-currents $T_j = \alpha_j + dd^c\varphi_j$, $1 \le j \le p$. Let us first assume that the functions φ_j have analytic singularities, and let $Z \subset X$ be an analytic set such that the φ_j's are locally bounded on $X \smallsetminus Z$. The (p,p)-current

$$\Theta = \mathbf{1}_{X \smallsetminus Z} T_1 \wedge \ldots \wedge T_k$$

is well defined on $X \smallsetminus Z$, thanks to Bedford and Taylor [BT76], and it is a closed positive current there. By [BT76] such a current does not carry mass on any analytic set, so we can enlarge Z without changing the total mass of Θ. In fact, Θ extends as a closed positive current on the whole of X. To see this, let us take a simultaneous *log resolution* of the T_j's, i.e. a modification

$$\mu : \widehat{X} \to X$$

such that if $\varphi_j = c_j \log \sum_\ell |g_{j,\ell}|^2 + O(1)$, then the pull-back of the ideals $(g_{j,\ell})_\ell$, namely $\mu^*(g_{j,\ell})_\ell = (g_{j,\ell} \circ \mu)_\ell$ is a purely divisorial ideal sheaf $\mathcal{O}_{\widehat{X}}(-D_j)$ on $\widehat{X}$. Let $u_j = 0$ be a local holomorphic equation of the divisor D_j on $\widehat{X}$. Since $\log \sum_\ell |g_{j,\ell}|^2 = \log |u_j|^2 + \log \sum_\ell |g_{j,\ell}/uj|^2 = \log |u_j|^2 + v_j$, where $v_j \in C^\infty$ and $dd^c \log |u_j|^2 = [D_j]$ by the Lelong-Poincaré equation, we find

$$\mu^* T_j = \mu^* \alpha_j + dd^c(\varphi_j \circ \mu) = c_j [D_j] + \widehat{T}_j, \quad \text{where} \quad \widehat{T}_j = \mu^* \alpha_j + dd^c \widehat{\varphi}_j \tag{3.2.1}$$

and $\widehat{\varphi}_j$ is a locally bounded potential on $\widehat{X}$ such that $\widehat{T}_j \ge 0$. Now, if $E = \mu^{-1}(Z)$, we get

$$\mathbf{1}_{X \smallsetminus Z} T_1 \wedge \ldots \wedge T_p = \mu_*(\mathbf{1}_{\widehat{X} \smallsetminus E} \widehat{T}_1 \wedge \ldots \wedge \widehat{T}_p) = \mu_*(\widehat{T}_1 \wedge \ldots \wedge \widehat{T}_p). \tag{3.2.2}$$

Hence the right-hand side defines the desired extension of $\mathbf{1}_{X \smallsetminus Z} T_1 \wedge \ldots \wedge T_p$ to X as the direct image of a closed positive current on $\widehat{X}$ carrying no mass on E. An essential point is the following monotonicity lemma.

3.2.3. Lemma. *Assume that we have closed positive* $(1,1)$*-currents with analytic singularities* T_j, $T'_j \in \{\alpha_j\}$ *with* $T_j \preccurlyeq T'_j$, $1 \le j \le p$, *and let* $\gamma \ge 0$ *be a closed positive smooth* $(n-p, n-p)$*-form on* X. *If* Z *is an analytic set containing the poles of all* T_j *and* T'_j, *we have*

$$\int_X \mathbf{1}_{X \smallsetminus Z} T_1 \wedge \ldots \wedge T_p \wedge \gamma \ge \int_X \mathbf{1}_{X \smallsetminus Z} T'_1 \wedge \ldots \wedge T'_p \wedge \gamma.$$

Proof. We take a log-resolution $\mu : \widehat{X} \to X$ that works for all T_j and T'_j simultaneously. By (3.2.1) and (3.2.2), we have $\mu^* T_j = c_j[D_j] + \widehat{T}_j$ where $\widehat{T}_j \ge 0$ has a locally bounded potential on $\widehat{X}$, and

$$\int_X \mathbf{1}_{X \smallsetminus Z} T_1 \wedge \ldots \wedge T_p \wedge \gamma = \int_{\widehat{X}} \widehat{T}_1 \wedge \ldots \wedge \widehat{T}_p \wedge \mu^* \gamma.$$

There are of course similar formulas $\mu^* T'_j = c_j[D'_j] + \widehat{T}'_j$ for the T'_j's, and our assumption $T_j \preccurlyeq T'_j$ means that the corresponding divisors satisfy $c_j D_j \le c'_j D'_j$, hence $\Delta_j := c'_j D'_j - c_j D_j \ge 0$. In terms of cohomology, we have

$$\mu^*\{\alpha_j\} = \{\mu^* T_j\} = \{\widehat{T}_j\} + \{c_j D_j\} = \{\mu^* T'_j\} = \{\widehat{T}'_j\} + \{c'_j D'_j\},$$

hence $\{\widehat{T}_j\}=\{\widehat{T}'_j\}+\{\Delta_j\}$ in $H^2(\widehat{X},\mathbb{R})$. By Stokes' theorem, we conclude that

$$\begin{aligned}\int_{\widehat{X}}\widehat{T}_1\wedge\widehat{T}_2\wedge\ldots\wedge\widehat{T}_p\wedge\mu^*\gamma &= \int_{\widehat{X}}(\widehat{T}'_1+\{\Delta_1\})\wedge\widehat{T}_2\wedge\ldots\wedge\widehat{T}_p\wedge\mu^*\gamma\\ &\geq \int_{\widehat{X}}\widehat{T}'_1\wedge\widehat{T}_2\wedge\ldots\wedge\widehat{T}_p\wedge\mu^*\gamma\end{aligned}$$

thanks to the positivity of our currents $\widehat{T}_j$, $\widehat{T}'_j$ and the fact that the product of such currents with bounded potentials by the current of integration $[\Delta_j]$ is well defined and positive ([BT76]). By replacing successively all terms $\{\widehat{T}_j\}$ by $\{\widehat{T}'_j\}+\{\Delta_j\}$ we infer

$$\int_{\widehat{X}}\widehat{T}_1\wedge\ldots\wedge\widehat{T}_p\wedge\mu^*\gamma\geq\int_{\widehat{X}}\widehat{T}'_1\wedge\ldots\wedge\widehat{T}'_p\wedge\mu^*\gamma. \qquad \square$$

Now, assume that we have arbitrary closed positive $(1,1)$-currents $T_1,\ \ldots\ ,\ T_p$. For each of them, we take a sequence $T_{j,k}=\alpha_j+i\partial\overline{\partial}\psi_{j,k}$ of monotone asymptotically equisingular approximations by currents with analytic singularities, $T_{j,k}\geq-\varepsilon_{j,k}\omega$, $\lim_{k\to+\infty}\varepsilon_{j,k}=0$. We have $T_{j,k}\preccurlyeq T_{j,k+1}$, and we may also assume without loss of generality that $\varepsilon_{j,k}\geq\varepsilon_{j,k+1}>0$ for all j,k. Let Z_k be an analytic containing all poles of the $T_{j,k}$, $1\leq j\leq p$. It follows immediately from the above discussion and especially from Lemma 3.2.3 that the integrals

$$\int_X \mathbf{1}_{X\smallsetminus Z_k}(T_{1,k}+\varepsilon_{1,k}\omega)\wedge\ldots\wedge(T_{p,k}+\varepsilon_{p,k}\omega)\wedge\gamma\geq 0$$

are well defined and nonincreasing in k (the fact that $\varepsilon_{j,k}$ is non increasing even helps here). From this, we conclude

3.2.4. Theorem. *For every $p=1,\,2,\ldots,\,n$, there is a well defined p-fold intersection product*

$$\widehat{\mathcal{S}}(X)\times\cdots\times\widehat{\mathcal{S}}(X)\longrightarrow H^{p,p}_+(X,\mathbb{R})$$

which assigns to any p-tuple of equivalence classes of monotone sequences $(T_{j,k})$ in $\widehat{\mathcal{S}}(X)$, $1\leq j\leq p$, the limit cohomology class

$$\lim_{k\to+\infty}\big\{\mathbf{1}_{X\smallsetminus Z_k}(T_{1,k}+\varepsilon_{1,k}\omega)\wedge\ldots\wedge(T_{p,k}+\varepsilon_{p,k}\omega)\big\}\in H^{p,p}_+(X,\mathbb{R})$$

where $H^{p,p}_+(X,\mathbb{R})\subset H^{p,p}(X,\mathbb{R})$ denotes the cone of cohomology classes of closed positive (p,p)-currents. This product is additive and homogeneous in each argument in the space $\widehat{\mathcal{S}}(X)$.

3.2.5. Corollary. *By combining the above formal intersection product with the Bergman approximation operator $\mathbf{B}:\mathcal{S}(X)\to\widehat{\mathcal{S}}(X)$, we get an intersection product*

$$\mathcal{S}(X)\times\cdots\times\mathcal{S}(X)\longrightarrow H^{p,p}_+(X,\mathbb{R})\quad\textit{denoted}\quad(T_1,\ldots,T_p)\longmapsto\langle T_1,\ldots,T_p\rangle^+,$$

which is homogeneous in each argument (and additive as long as $\mathbf{B}$ is). It always satisfies at least the subadditivity property

$$\langle T'_1+T''_1,T_2,\ldots,T_p\rangle^+\leq\langle T'_1,T_2,\ldots,T_p\rangle^+ + \langle T''_1,T_2,\ldots,T_p\rangle^+.$$

Proof. The existence of a limit in cohomology is seen by fixing a dual basis $(\{\gamma_j\})$ of $H^{n-p,n-p}(X)$, using the Serre duality pairing

$$H^{p,p}(X,\mathbb{R})\times H^{n-p,n-p}(X)\to\mathbb{R},\qquad(\beta,\gamma)\mapsto\int_X\beta\wedge\gamma.$$

Since X is Kähler, we can take $\gamma_1=\omega^{n-p}$ and replace if necessary γ_j by $\gamma_j+C\omega^{n-p}$, $C\gg 1$, to get $\gamma_j\geq 0$ for all $j\geq 2$. Then the integrals

$$\int_X\mathbf{1}_{X\smallsetminus Z_k}(T_{1,k}+\varepsilon_{1,k}\omega)\wedge\ldots\wedge(T_{p,k}+\varepsilon_{p,k}\omega)\wedge\gamma_j\geq 0$$

are nonincreasing in k, and the limit must therefore exist by monotonicity. The subadditivity property on $\mathcal{S}(X)$ comes from Lemma 3.2.3 applied to the inequality

$$\mathbf{B}(T'+T'')\succcurlyeq_W\mathbf{B}(T')+\mathbf{B}(T'')$$

(itself a consequence of the multiplier ideal sheaf inclusion $\mathcal{I}(\varphi'+\varphi'')\subset\mathcal{I}(\varphi')\mathcal{I}(\varphi'')$). □

3.3. Kähler definition of the numerical dimension

Using the intersection product defined in Th. 3.2.4 , we can give a precise definition of the numerical dimension.

3.3.1. Definition. *Let (X,ω) be a compact Kähler n-dimensional manifold. We define the numerical dimension* $\mathrm{nd}(T)$ *of a closed positive $(1,1)$-current T on X to be the largest integer $p=0,1,\ldots,n$ such that $\langle T^p\rangle^+\neq 0$, i.e. $\int_X\langle T^p\rangle^+\wedge\omega^{n-p}>0$.*

Accordingly, if (L,h) be a pseudoeffective line bundle on X, we define its numerical dimension to be

$$\mathrm{nd}(L,h)=\mathrm{nd}(\mathrm{i}\,\Theta_{L,h}).\tag{3.3.2}$$

By the results of the preceding subsection, $\mathrm{nd}(L,h)$ depends only on the weak equivalence class of singularities of the metric h.

3.3.3. Remark. H. Tsuji [Tsu07] has defined a notion of numerical dimension by a more algebraic method:

3.3.4. Definition. *Let X be a projective variety and (L,h) a pseudo-effective line bundle. When V runs over all irreducible algebraic suvarieties of X, one defines*

$$\nu_{\mathrm{num}}(L,h)=\sup\Big\{p=\dim V\,;\ \limsup_{m\to\infty}\frac{h^0\big(\widetilde{V},\mu^*(L^{\otimes m})\otimes\mathcal{I}(\mu^*h^m)\big)}{m^p}>0\Big\}$$

where $\mu:\widetilde{V}\to V\subset X$ is an embedded desingularization of V in X.

Junyan Cao [JC14] has shown that $\nu_{\mathrm{num}}(L,h)$ coincides with $\mathrm{nd}(L,h)$ as defined in (3.3.2). The idea is to make a reduction to the “big” case $\mathrm{nd}(L,h)=\dim X$ and to use holomorphic Morse inequalities [Dem85b] in combination with a regularization procedure. We omit the rather technical details here.

3.3.5. Remark. If L is there is also a natural concept of numerical dimension $\mathrm{nd}(L)$ that does not depend on the choice of a metric h on L. One can set e.g.

$$\mathrm{nd}(L)=\max\Big\{p\in[0,n]\,;\ \exists c>0,\ \forall\varepsilon>0,\ \exists h_\varepsilon,\ \ \Theta_{L,h_\varepsilon}\geq-\varepsilon\omega,\ \ \text{such that}$$
$$\int_{X\smallsetminus Z_\varepsilon}(\mathrm{i}\,\Theta_{L,h_\varepsilon}+\varepsilon\omega)^p\wedge\omega^{n-p}\geq c\Big\},$$

where h_ε runs over all metrics with analytic singularities on L. It may happen in general that $\mathrm{nd}(L,h_{\min})<\mathrm{nd}(L)$, even when L is nef; in that case the h_ε can be taken to be smooth in the definition of $\mathrm{nd}(L)$, and therefore $\mathrm{nd}(L)$ is the largest integer p such that $c_1(L)^p\neq 0$. In fact, for the line bundle L already mentioned in Remark 2.2.4 , it is shown in [DPS94] that there is unique positive current $T\in c_1(L)$, namely the current of integration $T=[C]$ on the negative curve $C\subset X$, hence $\mathrm{nd}(L,h_{\min})=\mathrm{nd}([C])=0$, although we have $\mathrm{nd}(L)=1$ here.

4. Proof of Junyan Cao's vanishing theorem

This section is a brief account and a simplified exposition of Junyan Cao's proof, as detailed in his PhD thesis [JC13]. The key curvature and singularity estimates are contained in the following technical statement, which depends in a crucial way on Bergman regularization and on Yau's theorem [Yau78] for solutions of Monge-Ampère equations.

4.1. Proposition. *Let (L,h) be a pseudoeffective line bundle on a compact Kähler manifold (X,ω). Let us write $T=\frac{\mathrm{i}}{2\pi}\Theta_{L,h}=\alpha+dd^c\varphi$ where α is smooth and φ is a quasi-psh potential. Let $p=\mathrm{nd}(L,h)$ be the numerical dimension of (L,h). Then, for every $\gamma\in\,]0,1]$ and $\delta\in\,]0,1]$, there exists a quasi-psh potential $\Phi_{\gamma,\delta}$ on X satisfying the following properties:*

(a) *$\Phi_{\gamma,\delta}$ is smooth in the complement $X\smallsetminus Z_\delta$ of an analytic set $Z_\delta\subset X$.*

(b) *$\alpha+\delta\omega+dd^c\Phi_{\gamma,\delta}\geq\frac{\delta}{2}(1-\gamma)\omega$ on X.*

(c) *$(\alpha+\delta\omega+dd^c\Phi_{\gamma,\delta})^n\geq a\,\gamma^n\delta^{n-p}\omega^n$ on $X\smallsetminus Z_\delta$.*

(d) *$\Phi_{\gamma,\delta}\leq(1+b\delta)\psi_{B,k}+C_{\gamma,\delta}$ where $\psi_{B,k}\geq\varphi$ is a Bergman approximation of φ of sufficiently high index $k=k_0(\delta)$.*

(e) *$\sup_X\Phi_{1,\delta}=0$, and for all $\gamma\in\,]0,1]$ there are estimates $\Phi_{\gamma,\delta}\leq A$ and*

$$\exp\big(-\Phi_{\gamma,\delta}\big)\leq e^{-(1+b\delta)\varphi}\exp\big(A-\gamma\Phi_{1,\delta}\big)$$

(f) *For γ_0, $\delta_0>0$ small, $\gamma\in\,]0,\gamma_0]$, $\delta\in\,]0,\delta_0]$ and $k=k_0(\delta)$ large enough, we have*

$$\mathcal{I}(\Phi_{\gamma,\delta})=\mathcal{I}_+(\varphi).$$

Here a, b, A, γ_0, δ_0, $C_{\gamma,\delta}>0$ are suitable constants ($C_{\gamma,\delta}$ being the only one that depends on γ, δ).

Before starting the proof, notice that the family of multiplier ideals $\lambda\mapsto\mathcal{I}(\lambda\varphi)$ is nonincreasing in λ. By the Noetherian property of ideal sheaves, they can jump only for a locally finite set of values λ in $[0,+\infty[$, and in particular, there exists a real value $\lambda_0>1$ such that

$$\mathcal{I}_+(\varphi):=\lim_{\lambda\to1+0}\mathcal{I}(\lambda\varphi)=\mathcal{I}(\lambda\varphi),\quad\forall\lambda\in\,]1,\lambda_0]. \tag{4.2}$$

Proof. Denote by $\psi_{B,k}$ the nonincreasing sequence of Bergman approximations of φ (obtained with denominators $m_k = 2^k$, say). We have $\psi_{B,k} \geq \varphi$ for all k, the $\psi_{B,k}$ have analytic singularities and $\alpha + dd^c\psi_{B,k} \geq -\varepsilon_k\omega$ with $\varepsilon_k \downarrow 0$. Then $\varepsilon_k \leq \frac{\delta}{4}$ for $k \geq k_0(\delta)$ large enough, and so

$$\begin{aligned}\alpha + \delta\omega + dd^c\big((1+b\delta)\psi_{B,k}\big) &\geq \alpha + \delta\omega - (1+b\delta)(\alpha+\varepsilon_k\omega)\\ &\geq \delta\omega - (1+b\delta)\varepsilon_k\omega - b\delta\alpha \geq \tfrac{\delta}{2}\omega\end{aligned}$$

for $b > 0$ small enough (independent of δ and k). Let $\mu : \widehat{X} \to X$ be a log-resolution of $\psi_{B,k}$, so that

$$\mu^*\big(\alpha + \delta\omega + dd^c((1+b\delta)\psi_{B,k})\big) = c_k[D_k] + \beta_k$$

where $\beta_k \geq \frac{\delta}{2}\mu^*\omega \geq 0$ is a smooth closed $(1,1)$-form on $\widehat{X}$ that is > 0 in the complement $\widehat{X} \smallsetminus E$ of the exceptional divisor, $c_k = \frac{1+b\delta}{m_k} > 0$, and D_k is a divisor that includes all components E_ℓ of E. The map μ can be obtained by Hironaka [Hir64] as a composition of a sequence of blow-ups with smooth centers, and we can even achieve that D_k and E are normal crossing divisors. In this circumstance, it is well known that there exist arbitrary small numbers $\eta_\ell > 0$ such that $\beta_k - \sum \eta_\ell[E_\ell]$ is a Kähler class on $\widehat{X}$. Hence we can find a quasi-psh potential $\widehat{\theta}_k$ on $\widehat{X}$ such that $\widehat{\beta}_k := \beta_k - \sum \eta_\ell[E_\ell] + dd^c\widehat{\theta}_k$ is a Kähler metric on $\widehat{X}$, and by taking the η_ℓ small enough, we may assume that $\int_{\widehat{X}}(\widehat{\beta}_k)^n \geq \frac{1}{2}\int_{\widehat{X}}\beta_k^n$. Now, we write

$$\begin{aligned}\alpha + \delta\omega + dd^c\big((1+b\delta)\psi_{B,k}\big) &\geq \alpha + \varepsilon_k\omega + dd^c\psi_{B,k} + (\delta-\varepsilon_k)\omega - b\delta(\alpha+\varepsilon_k\omega)\\ &\geq (\alpha + \varepsilon_k\omega + dd^c\psi_{B,k}) + \tfrac{\delta}{2}\omega\end{aligned}$$

for $k \geq k_0(\delta)$ and $b > 0$ small (independent of δ and k). The assumption on the numerical dimension of $\frac{\mathrm{i}}{2\pi}\Theta_{L,h} = \alpha + dd^c\varphi$ implies the existence of a constant $c > 0$ such that, with $Z = \mu(E) \subset X$, we have

$$\begin{aligned}\int_{\widehat{X}}\beta_k^n &= \int_X \mathbf{1}_{X\smallsetminus Z}\big(\alpha + \delta\omega + dd^c((1+b\delta)\psi_{B,k})\big)^n\\ &\geq \binom{n}{p}\Big(\frac{\delta}{2}\Big)^{n-p}\int_{X\smallsetminus Z}\big(\alpha + \varepsilon_k\omega + dd^c\psi_{B,k}\big)^p \wedge \omega^{n-p} \geq c\,\delta^{n-p}\int_X \omega^n\end{aligned}$$

for all $k \geq k_0(\delta)$. Therefore, we may assume

$$\int_{\widehat{X}}(\widehat{\beta}_k)^n \geq \frac{c}{2}\,\delta^{n-p}\int_X \omega^n.$$

By Yau's theorem [Yau78], there exists a quasi-psh potential $\widehat{\tau}_k$ on $\widehat{X}$ such that $\widehat{\beta}_k + dd^c\widehat{\tau}_k$ is a Kähler metric on $\widehat{X}$ with a prescribed volume form $\widehat{f} > 0$ such that $\int_{\widehat{X}} f = \int_{\widehat{X}}\beta_k^n$. By the above discussion, we can take here $\widehat{f} > \frac{c}{3}\delta^{n-p}\mu^*\omega^n$ everywhere on $\widehat{X}$.

Now, we consider $\theta_k = \mu_*\widehat{\theta}_k$ and $\tau_k = \mu_*\widehat{\tau}_k \in L^1_{\mathrm{loc}}(X)$. Since $\widehat{\theta}_k$ was defined in such a way that $dd^c\widehat{\theta}_k = \widehat{\beta}_k - \beta_k + \sum_\ell \eta_\ell[E_\ell]$, we get

$$\begin{aligned}&\mu^*\big(\alpha + \delta\omega + dd^c((1+b\delta)\psi_{B,k} + \gamma(\theta_k + \tau_k))\big)\\ &\qquad = c_k[D_k] + (1-\gamma)\beta_k + \gamma\Big(\sum_\ell \eta_\ell[E_\ell] + \widehat{\beta}_k + dd^c\widehat{\tau}_k\Big) \geq 0.\end{aligned}$$

This implies in particular that $\Phi_{\gamma,\delta} := (1+b\delta)\psi_{B,k} + \gamma(\theta_k + \tau_k)$ is a quasi-psh potential on X and that

$$\mu^*\big(\alpha + \delta\omega + dd^c\Phi_{\gamma,\delta}\big) \geq (1-\gamma)\beta_k \geq \frac{\delta}{2}(1-\gamma)\,\mu^*\omega,$$

thus condition (b) is satisfied. Putting $Z_\delta = \mu(|D_k|) \supset \mu(E) = Z$, we also have

$$\mu^*\mathbf{1}_{X\smallsetminus Z_\delta}\big(\alpha + \delta\omega + dd^c\Phi_{\gamma,\delta}\big)^n \geq \gamma^n\,\widehat{\beta}_k^n \geq \frac{c}{3}\,\gamma^n\delta^{n-p}\mu^*\omega^n,$$

therefore condition (c) is satisfied as well with $a = c/3$. Property (a) is clear, and (d) holds since the quasi-psh function $\widehat{\theta}_k + \widehat{\tau}_k$ must be bounded from above on $\widehat{X}$. We will actually adjust constants in $\widehat{\theta}_k + \widehat{\tau}_k$ (as we may), so that $\sup_X \Phi_{1,\delta} = 0$. Since $\varphi \leq \psi_{B,k} \leq \psi_{B,0} \leq A_0 := \sup_X \psi_{B,0}$ and

$$\Phi_{\gamma,\delta} = (1+b\delta)\psi_{B,k} + \gamma\big(\Phi_{1,\delta} - \psi_{B,k}\big) = (1-\gamma+b\delta)\psi_{B,k} + \gamma\Phi_{1,\delta},$$

we have

$$(1+b\delta)\varphi - \gamma(A_0 - \psi_{B,k}) \leq \Phi_{\gamma,\delta} \leq (1-\gamma+b\delta)A_0$$

and the estimates in (e) follow with $A = (1+b)A_0$. The only remaining property to be proved is (f). Condition (d) actually implies $\mathcal{I}(\Phi_{\gamma,\delta}) \subset \mathcal{I}((1+b\delta)\psi_{B,k})$, and Cor. 1.12 also gives $\mathcal{I}((1+b\delta)\psi_{B,k}) \subset \mathcal{I}((1+b\delta/2)\varphi)$ if we take $k \geq k_0(\delta)$ large enough, hence $\mathcal{I}(\Phi_{\gamma,\delta}) \subset \mathcal{I}_+(\varphi)$ for $\delta \leq \delta_0$ small. In the opposite direction, we observe that $\Phi_{1,\gamma}$ satisfies $\alpha + \omega + dd^c\Phi_{1,\delta} \geq 0$ and $\sup_X \Phi_{1,\delta} = 0$, hence $\Phi_{1,\delta}$ belongs to a compact family of quasi-psh functions. A standard result of potential theory then shows the existence of a uniform small constant $c_0 > 0$ such that $\int_X \exp(-c_0\Phi_{1,\delta})dV_\omega < +\infty$ for all $\delta \in\,]0,1]$. If $f \in \mathcal{O}_{X,x}$ is a germ of holomorphic function and U a small neighborhood of x, the Hölder inequality combined with estimate (e) implies

$$\int_U |f|^2\exp(-\Phi_{\gamma,\delta})dV_\omega \leq e^A\Big(\int_U |f|^2e^{-p(1+b\delta)\varphi}dV_\omega\Big)^{\frac{1}{p}}\Big(\int_U |f|^2e^{-q\gamma\Phi_{1,\delta}}dV_\omega\Big)^{\frac{1}{q}}.$$

We fix $\lambda_0 > 1$ so that $\mathcal{I}(\lambda_0\varphi) = \mathcal{I}_+(\varphi)$, $p \in\,]1,\lambda_0[$ (say $p = 1+\lambda_0)/2$), and take

$$\gamma \leq \gamma_0 := \frac{c_0}{q} = c_0\frac{\lambda_0-1}{\lambda_0+1} \quad\text{ and }\quad \delta \leq \delta_0 \in\,]0,1] \text{ so small that } p(1+b\delta_0) \leq \lambda_0.$$

Then clearly $f \in \mathcal{I}(\lambda_0\varphi)$ implies $f \in \mathcal{I}(\Phi_{\gamma,\delta})$, and (f) is proved. □

The rest of the arguments proceeds along the lines of [Dem82], [Mou95] and [DP02]. Let (L,h) be a pseuffective line bundle and $p = \mathrm{nd}(L,h) = \mathrm{nd}(\,\mathrm{i}\,\Theta_{L,h})$. We equip L be the hermitian metric h_δ defined by the quasi-psh weight $\Phi_\delta = \Phi_{\gamma_0,\delta}$ obtained in Prop. 4.1 , with $\delta \in\,]0,\delta_0]$. Since Φ_δ is smooth on $X \smallsetminus Z_\delta$, the well-known Bochner-Kodaira identity shows that for every smooth (n,q)-form u with values in $K_X \otimes L$ that is compactly supported on $X \smallsetminus Z_\delta$, one has

$$\|\overline{\partial}u\|_\delta^2 + \|\overline{\partial}^*u\|_\delta^2 \geq 2\pi\int_X(\lambda_{1,\delta} + \ldots + \lambda_{q,\delta} - q\delta)|u|^2e^{-\Phi_\delta}dV_\omega,$$

where $\|u\|_\delta^2 := \int_X |u|^2_{\omega,h_\delta}dV_\omega = \int_X |u|^2e^{-\Phi_\delta}dV_\omega$ and

$$0 < \lambda_{1,\delta}(x) \leq \ldots \leq \lambda_{n,\delta}(x)$$

are, at each point $x \in X$, the eigenvalues of $\alpha + \delta\omega + dd^c\Phi_\delta$ with respect to the base Kähler metric ω. Notice that the $\lambda_{j,\delta}(x) - \delta$ are the actual eigenvalues of $\frac{\mathrm{i}}{2\pi}\Theta_{L,h_\delta} = \alpha + dd^c\Phi_\delta$ with respect to ω and that the inequality $\lambda_{j,\delta}(x) \geq \frac{\delta}{2}(1-\gamma) > 0$ is guaranted by Prop. 4.1 (b). After dividing by $2\pi q$ (and neglecting that constant in the left hand side), we get

$$\|\overline{\partial}u\|_\delta^2 + \|\overline{\partial}^*u\|_\delta^2 + \delta\|u\|_\delta^2 \geq \int_X(\lambda_{1,\delta} + \ldots + \lambda_{q,\delta})|u|^2e^{-\Phi_\delta}dV_\omega. \tag{4.3}$$

A standard Hahn-Banach argument in the L^2-theory of the $\overline{\partial}$-operator then yields the following conclusion.

4.5. Proposition. *For every L^2 section of $\Lambda^{n,q}T_X^*\otimes L$ such that $\|f\|_\delta<+\infty$ and $\overline{\partial}f=0$ in the sense of distributions, there exists a L^2 section $v=v_\delta$ of $\Lambda^{n,q-1}T_X^*\otimes L$ and a L^2 section $w=w_\delta$ of $\Lambda^{n,q}T_X^*\otimes L$ such that $f=\overline{\partial}v+w$ with*

$$\|v\|_\delta^2+\frac{1}{\delta}\|w\|_\delta^2\le\int_X\frac{1}{\lambda_{1,\delta}+\ldots+\lambda_{q,\delta}}|f|^2e^{-\Phi_\delta}dV_\omega.$$

Because of the singularities of the weight on Z_δ, one should in fact argue first on $X\smallsetminus Z_\delta$ and approximate the base Kähler metric ω by a metric $\widehat{\omega}_{\delta,\varepsilon}=\omega+\varepsilon\widehat{\omega}_\delta$ that is complete on $X\smallsetminus Z_\delta$, exactly as explained in [Dem82]; we omit the (by now standard) details here. A consequence of Prop. 4.5 is that the "error term" w satisfies the L^2 bound

$$\int_X|w|^2e^{-\Phi_\delta}dV_\omega\le\int_X\frac{\delta}{\lambda_{1,\delta}+\ldots+\lambda_{q,\delta}}|f|^2e^{-\Phi_\delta}dV_\omega. \tag{4.6}$$

The idea for the next estimate is taken from Mourougane's PhD thesis [Mou95].

4.7. Lemma. *The ratio $\rho_\delta(x):=\delta/(\lambda_{1,\delta}(x)+\ldots+\lambda_{q,\delta}(x))$ is uniformly bounded on X (independently of δ), and, as soon as $q\ge n-\mathrm{nd}(L,h)+1$, there exists a subsequence $(\rho_{\delta\ell})$, $\delta_\ell\to 0$, that tends almost everywhere to 0 on X.*

Proof. By estimates (b,c) in Prop. 4.1 , we have $\lambda_{j,\delta}(x)\ge\frac{\delta}{2}(1-\gamma_0)$ and

$$\lambda_{1,\delta}(x)\ldots\lambda_{n,\delta}(x)\ge a\gamma_0^n\delta^{n-p}\quad\text{where}\quad p=\mathrm{nd}(L,h). \tag{4.8}$$

Therefore we already find $\rho_\delta(x)\le 2/q(1-\gamma_0)$. Now, we have

$$\int_{X\smallsetminus Z_\delta}\lambda_{n,\delta}(x)dV_\omega\le\int_X(\alpha+\delta\omega+dd^c\Phi_\delta)\wedge\omega^{n-1}=\int_X(\alpha+\delta\omega)\wedge\omega^{n-1}\le\text{Const},$$

therefore the "bad set" $S_\varepsilon\subset X\smallsetminus Z_\delta$ of points x where $\lambda_{n,\delta}(x)>\delta^{-\varepsilon}$ has a volume $\mathrm{Vol}(S_\varepsilon)\le C\delta^\varepsilon$ converging to 0 as $\delta\to 0$ (with a slightly more elaborate argument we could similarly control any elementary symmetric function in the $\lambda_{j,\delta}$'s, but this is not needed here). Outside of S_ε, the inequality (4.8) yields

$$\lambda_{q,\delta}(x)^q\delta^{-\varepsilon(n-q)}\ge\lambda_{q,\delta}(x)^q\lambda_{n,\delta}(x)^{n-q}\ge a\gamma_0^n\delta^{n-p}$$

hence

$$\lambda_{q,\delta}(x)\ge c\delta^{\frac{n-p+(n-q)\varepsilon}{q}}\qquad\text{and}\qquad\rho_\delta(x)\le C\delta^{1-\frac{n-p+(n-q)\varepsilon}{q}}.$$

If we take $q\ge n-p+1$ and $\varepsilon>0$ small enough, the exponent of δ in the final estimate is positive, and Lemma 4.7 follows. □

Proof of Junyan Cao's Theorem, Th. 0.10 . Let $\{f\}$ be a cohomology class in the group $H^q(X,K_X\otimes L\otimes\mathcal{I}_+(h))$, $q\ge n-\mathrm{nd}(L,h)+1$. Consider a finite Stein open covering $\mathcal{U}=(U_\alpha)_{\alpha=1,\ldots,N}$ by coordinate balls U_α. There is an isomorphism between Čech cohomology $\check{H}^q(\mathcal{U},\mathcal{F})$ with values in the sheaf $\mathcal{F}=\mathcal{O}(K_X\otimes L)\otimes\mathcal{I}_+(h)$ and the cohomology of the complex $(K_\delta^\bullet,\overline{\partial})$ of (n,q)-forms u such that both u and $\overline{\partial}u$ are L^2 with respect to the weight Φ_δ, i.e. $\int_X|u|^2\exp(-\Phi_\delta)dV_\omega<+\infty$ and $\int_X|\overline{\partial}u|^2\exp(-\Phi_\delta)dV_\omega<+\infty$. The isomorphism comes from

Leray's theorem and from the fact that the sheafified complex $(\mathcal{K}^\bullet_\delta, \overline\partial)$ is a complex of $\mathcal{C}^\infty$-modules that provides a resolution of the sheaf $\mathcal{F}$: the main point here is that $\mathcal{I}(\Phi_\delta) = \mathcal{I}_+(\varphi) = \mathcal{I}_+(h)$, as asserted by Prop. 4.1 (f), and that we can locally solve $\overline\partial$-equations by means of Hörmander's estimates [Hör66].

Let (ψ_α) be a partition of unity subordinate to $\mathcal{U}$. The explicit isomorphism between Čech cohomology and L^2 cohomology yields a smooth L^2 representative $f = \sum_{|I|=q} f_I(z)dz_1 \wedge ... \wedge dz_n \wedge d\overline{z}_I$ which is a combination

$$f = \sum_{\alpha_0} \psi_{\alpha_0} c_{\alpha_0\alpha_1\ldots\alpha_q} \overline\partial\omega_{\alpha_1} \wedge \ldots \wedge \overline\partial\psi_{\alpha_q}$$

of the components of the corresponding Čech cocycle

$$c_{\alpha_0\alpha_1\ldots\alpha_q} \in \Gamma\big(U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_q}, \mathcal{O}(\mathcal{F})\big).$$

Estimate (e) in Prop. 4.1 implies the Hölder inequality

$$\int_X \rho_\delta |f|^2 \exp(-\Phi_\delta) dV_\omega \le e^A \Big(\int_X \rho_\delta^p |f|^2 e^{-p(1+b\delta)\varphi} dV_\omega\Big)^{\frac{1}{p}} \Big(\int_X |f|^2 e^{-q\gamma_0\Phi_{1,\delta}} dV_\omega\Big)^{\frac{1}{q}}.$$

Our choice of $\delta \le \delta_0$, γ_0 and p, q shows that the integrals in the right hand side are convergent, and especially $\int_X |f|^2 e^{-p(1+b\delta)\varphi} dV_\omega < +\infty$. Lebesgue's dominated convergence theorem combined with Lemma 4.7 implies that the L^p-part goes to 0 as $\delta = \delta_\ell \to 0$, hence the "error term" w converges to 0 in L^2 norm by estimate (4.6). If we express the corresponding class $\{w\}$ in Čech cohomology and use Hörmander's estimates on the intersections $U_\alpha = \bigcap U_{\alpha_j}$, we see that $\{w\}$ will be given by a Čech cocycle $(\widetilde{w}_\alpha)$ such that $\int_{U_\alpha} |\widetilde{w}_\alpha|^2 e^{-\Phi_\delta} dV_\omega \to 0$ as $\delta = \delta_\ell \to 0$ (we may lose here some fixed constants since Φ_δ is just quasi-psh on our balls, but this is irrelevant thanks to the uniform lower bounds for the Hessian). The inequality $\Phi_\delta \le A$ in Prop. 4.1 (e) shows that we have as well an unweighted L^2 estimate $\int_{U_\alpha} |\widetilde{w}_\alpha|^2 dV \to 0$. However it is well-known that when one takes unweighted L^2 norms on spaces of Čech cocyles (or uniform convergence on compact subsets, for that purpose), the resulting topology on the finite dimensional space $\check{H}^q(\mathcal{U}, \mathcal{F})$ is Hausdorff, so the subspace of coboundaries is closed in the space of cocycles. Hence we conclude from the above that f is a coboundary, as desired. □

4.9. Remark. In this proof, it is remarkable that we can control the error term w, but a priori completely lose control on the element v such that $\overline\partial v \approx f$ when $\delta \to 0$!

5. Compact Kähler threefolds without nontrivial subvarieties

The bimeromorphic classification of compact Kähler manifolds leads to considering those, termed as "simple", that have as little internal structure as possible, and are somehow the elementary bricks needed to reconstruct all others through meromorphic fibrations (cf. [Cam80], [Cam85]).

5.1. Definition. *A compact Kähler manifold X is said to be simple if there does not exist any irreducible analytic subvariety Z with $0 < \dim Z < \dim X$ through a very generic point $x \in X$, namely a point x in the complement $X \smallsetminus \bigcup S_j$ of a countable union of analytic sets $S_j \subsetneq X$.*

Of course, every one dimensional manifold X is simple, but in higher dimensions $n > 1$, one can show that a very generic torus $X = \mathbb{C}^n/\Lambda$ has no nontrivial analytic subvariety Z at all (i.e.

none beyond finite sets and X itself), in any dimension n. In even dimension, a very generic Hyperkähler manifold can be shown to be simple as well. It has been known since Kodaira that there are no other simple Kähler surfaces (namely only very generic 2-dimensional tori and K3 surfaces). Theefore, the next dimension to be investigated is dimension 3. A partial answer has been recently given for "strongly simple" Kähler threefolds in [CDV13]; we give here a short account of these results and refer to the latter paper for further details.

The simplicity assumption implies that the algebraic dimension is $a(X) = 0$, in particular X cannot be projective, and cannot either be uniruled (i.e. covered by rational curves). By the Kodaira embedding theorem, we also infer that $H^0(X, \Omega_X^2) \neq 0$, otherwise X would be projective. One of the most crucial arguments is the following strong and difficult theorem of Brunella [Bru10].

5.2. Theorem. ([Bru10]) *Let X be a compact Kähler manifold with a 1-dimensional holomorphic foliation F given by a nonzero morphism of vector bundle $L \to T_X$, where L is a line bundle on X, and T_X is its holomorphic tangent bundle. If L^{-1} is not pseudoeffective, the closures of the leaves of F are rational curves, and X is thus uniruled.*

We use this result in the form of the following corollary, which has been observed in [HPR11], Proposition 4.2.

5.3. Corollary. *If X is a non uniruled n-dimensional compact Kähler manifold with $H^0(X, \Omega_X^{n-1}) \neq 0$, then K_X is pseudoeffective.*

Proof. Ω_X^{n-1} is canonically isomorphic to $K_X \otimes T_X$. Any nonzero section of Ω_X^{n-1} thus provides a nonzero map $K_X^{-1} \to T_X$, and an associated foliation. □

It follows from the above that the canonical line bundle K_X of our simple threefold X must be pseudoeffective. We then use the following simple observation.

5.4. Proposition. *Assume that X is a strongly simple compact complex manifold. Then every pseudoeffective line bundle (L, h) is nef, and all multiplier sheaves $\mathcal{I}(h^m)$ are trivial, i.e. $\mathcal{I}(h^m) = \mathcal{O}_X$. Moreover, we have $c_1(L)^n = 0$.*

Proof. Since there are not positive dimensional analytic subvarieties, the zero varieties of the ideal sheaves $\mathcal{I}(h^m)$ must be finite sets of points, hence, by Skoda [Sko72a], the Lelong numbers $\nu(\mathrm{i}\,\Theta_{L,h}, x)$ are zero except on a countable set $S \subset X$. By [Dem92], this implies that L is nef and $c_1(L)^n \geq \sum_{x \in S} \nu(\mathrm{i}\,\Theta_{L,h}, x)^n$. However, by the Grauert-Riemenschneider conjecture solved in [Siu84], [Siu85] and [Dem85b], the positivity of $c_1(L)^n$ would imply that $a(X) = n$ (i.e. X Moishezon, a contradiction). Therefore $c_1(L)^n = 0$ and $S = \emptyset$. □

5.5. Proposition. *Let X be a compact Kähler manifold of dimension $n > 1$ without any non-trivial subvariety, and with K_X pseudoeffective. Then*

$$h^j(X, K_X^{\otimes m}) \leq h^0(X, \Omega_X^j \otimes K_X^{\otimes m}) \leq \binom{n}{j} \quad \text{for every } j \geq 0,$$

and the Hilbert polynomial $P(m) := \chi(X, K_X^{\otimes m})$ is constant, equal to $\chi(X, \mathcal{O}_X)$.

Proof. The inequality $h^j(X, K_X^{\otimes m}) \leq h^0(X, \Omega_X^j \otimes K_X^{\otimes m})$ follows from the Hard Lefschetz Theorem 0.8 applied with $L = K_X$ and the corresponding trivial multiplier ideal sheaf. Also, for any holomorphic vector bundle E on X, we have $h^0(X, E) \leq \mathrm{rank}(E)$, otherwise, some ratios of determinants of sections would produce a nonconstant meromorphic function, and thus $a(X) > 0$, contradiction; here we take $E = \Omega_X^j \otimes K_X^{\otimes m}$ and get $\operatorname{rank} E = \binom{n}{j}$. The final claim is clear because a polynomial function $P(m)$ which remains bounded as $m \to +\infty$ is necessarily constant. □

5.6. Corollary. *Let X be a strongly simple Kähler threefold. Let $h^{i,j} = \dim H^{i,j}(X, \mathbb{C})$ be the Hodge numbers. We have*

$$c_1(X)^3 = c_1(X) \cdot c_2(X) = 0, \quad \chi(X, \mathcal{O}_X) = 0 \quad \textit{and} \quad q := h^{1,0} > 0.$$

Proof. The intersection number $K_X^3 = -c_1(X)^3$ vanishes because it is the leading term of $P(m)$, up to the factor 3!. The Riemann-Roch formula then gives

$$P(m) = \frac{(1 - 12\,m)}{24}\, c_1(X) \cdot c_2(X).$$

The boundedness of $P(m)$ implies $\chi(X, \mathcal{O}_X) = \frac{1}{24} c_1(X) \cdot c_2(X) = 0$. Now, we write

$$0 = \chi(X, \mathcal{O}_X) = 1 - h^{1,0} + h^{2,0} - h^{3,0}.$$

By Kodaira's theorem, $h^{2,0} > 0$ since X is not projective, and $h^{3,0} \le 1$ since $a(X) = 0$. Thus $0 = 1 - h^{1,0} + h^{2,0} - h^{3,0} \ge 1 - q + 1 - 1 = 1 - q$, and $q > 0$. □

Everything is now in place for the final conclusion.

5.7. Theorem. *For any strongly simple Kähler threefold X, the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is a biholomorphism of 3-dimensional tori.*

Proof. Since $q = h^{1,0} > 0$, the Albanese map α is non constant. By simplicity, X cannot possess any fibration with positive dimensional fibers, so we must have $\dim \alpha(X) = \dim X = 3$, and as $q = h^{1,0} = h^0(X, \Omega_X^1) \le 3$ (Prop. 5.5 with $j = 1$, $m = 0$) the Albanese map α must be surjective. The function $\det(d\alpha)$ cannot vanish, otherwise we would get a non trivial divisor, so α is étale. Therefore X is a 3-dimensional torus, as a finite étale cover of the 3-dimensional torus $\mathrm{Alb}(X)$, and α must be an isomorphism. □

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