Open Problem suggested by J.P. Demailly
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The well-known Hodge conjecture asserts that every rational cohomology class of bidegree (p, p) on a projective n-fold X is the cohomology class of an algebraic cycle of codimension p with rational coefficients. One of the main difficulties for a proof of the conjecture is the arithmetic nature of the rationality assumption. Even for abelian varieties, the conjecture remains unsettled. Questions (1), (2) below are an attempt to split the difficulties in several steps by taking apart the arithmetic considerations.

Let $\mathcal{F}$ be a foliation on X with complex analytic leaves of codimension p. If $\mathcal{F}$ has a transversal smooth volume form, then this form can be seen as a closed positive (p, p)-current α with $T\mathcal{F}$ in its kernel. Conversely, every smooth closed positive (p, p)-form α which is decomposable at each point is associated to a foliation $\mathcal{F}$ as above and will be called a *foliated (p, p)-current* (according to D. Sullivan). When X is a complex torus $\mathbf{C}^n/\Gamma$, it is clear that $H^{p,p}(X)$ is generated by the cohomology classes $\{\alpha\}$ of constant decomposable (p, p)-forms, which are associated to linear foliations of the torus. Moreover, it is easy to check in this case that $\{\alpha\} \in H^{2p}(X, \mathbf{R})$ is rational if and only if the foliation has compact leaves. When X is an arbitrary projective n-fold, we are thus led to the following questions. A positive answer to these would imply the Hodge conjecture for X.

(1) *Is $H^{p,p}(X)$ generated by cohomology classes of currents $\pi_\star\alpha$ which are direct images of a foliated (q, q)-current α on Y by some morphism $\pi : Y \to X$, where Y is projective and $q - p = \dim Y - \dim X$? Is $H^{p,p}(X) \cap H^{2p}(X, \mathbf{Q})$ generated by those classes $\{\pi_\star\alpha\}$ for which $\{\alpha\}$ is rational ?*

(2) *If α is a foliated (p, p)-current with cohomology class in $H^{2p}(X, \mathbf{Q})$, can one say that the leaves of the foliation $\mathcal{F}$ associated to α are compact ?*

Note that taking direct images in (1) cannot be avoided in general (except perhaps if highly singular foliations were admitted). For instance $\mathbf{CP}^n$ has no foliated $(1, 1)$-current: if α is decomposable, we have $\alpha^n = 0$, hence $\{\alpha\} = 0$. In that case, however, we can take Y to be the flag manifold of pairs (x, S), $x \in S$, where S is a linear subspace of codimension p in $\mathbf{CP}^n$ and take $\pi : Y \longrightarrow \mathbf{CP}^n$ to be the obvious projection. Then $H^{p,p}(\mathbf{CP}^n, \mathbf{Q})$ is generated by the foliated current given by the fibers of the projection of Y onto the Grassmannian G of codimension p subspaces, with the unitary invariant volume form of G as transversal volume form.

The answer to (2) is negative in general, e.g. for some Hilbert modular surfaces, but statement (2) can be weakened by requiring only that the foliation $\mathcal{F}$ associated to α can be deformed (within the same cohomology class $\{\alpha\}$) into a foliation with compact leaves.