POTENTIAL THEORY IN SEVERAL COMPLEX VARIABLES

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1. Monge-Ampère operators.

Let X be a complex manifold of dimension n . We denote as usual $d = d' + d''$ the exterior derivative and we set

$$d^c = \frac{1}{2i\pi}(d' - d'') ,$$

so that $dd^c = \frac{i}{\pi} d'd''$. In this context, we have the following integration by parts formula.

FORMULA 1.1. — *Let $\Omega \subset\subset X$ be a smooth open subset of X and f, g forms of class C^2 on $\overline{\Omega}$ of pure bidegrees (p,p) and (q,q) with $p + q = n - 1$. Then*

$$\int_{\Omega} f \wedge dd^c g - dd^c f \wedge g = \int_{\partial\Omega} f \wedge d^c g - d^c f \wedge g .$$

Proof. — By Stokes' theorem the right hand side is the integral over Ω of

$$d(f \wedge d^c g - d^c f \wedge g) = f \wedge dd^c g - dd^c f \wedge g + (df \wedge d^c g + d^c f \wedge dg) .$$

As all forms of total degree $2n$ and bidegree $\neq (n,n)$ are zero, we get

$$df \wedge d^c g = \frac{i}{2\pi}(d'f \wedge d''g - d''f \wedge d'g) = -d^c f \wedge dg . \ \square$$

Let u be a psh function on X and T a closed positive current of bidimension (p,p), *i.e.* of bidegree $(n-p, n-p)$. Our desire is to define the wedge product $dd^c u \wedge T$ even when neither u nor T are smooth. A priori, this product makes no sense since $dd^c u$ and T have measure coefficients in general. Assume that u is a *locally bounded* psh function. Then the current uT is well defined since u is a locally bounded Borel function and T has measure coefficients. According to Bedford-Taylor [B-T2] one defines

$$dd^c u \wedge T = dd^c(uT)$$

where $dd^c(\ \ )$ is taken in the sense of distribution (or current) theory.

PROPOSITION 1.2. — *The wedge product $dd^c u \wedge T$ is again a closed positive current.*

Proof. — The result is local. We may assume that X is an open set $\Omega \subset \mathbb{C}^n$, and after shrinking Ω , that $|u| \leqslant M$ on Ω . Let (ρ_ε) be a family of regularizing kernels with $\operatorname{Supp} \rho_\varepsilon \subset B(0,\varepsilon)$ and $\int \rho_\varepsilon = 1$. The sequence of convolutions $u_k = u \star \rho_{1/k}$ is decreasing, bounded by M , and converges pointwise to u as $k \to +\infty$. By Lebesgue's dominated convergence theorem $u_k T$ converges weakly to uT , thus $dd^c(u_k T)$ converges weakly to $dd^c(uT)$. However, since u_k is smooth, $dd^c(u_k T)$ coincides with the product $dd^c u_k \wedge T$ in its usual sense. As $T \geqslant 0$ and as $dd^c u_k$ is a positive $(1,1)$–form, we have $dd^c u_k \wedge T \geqslant 0$, hence the weak limit $dd^c u \wedge T$ is $\geqslant 0$ (and obviously closed). $\square$

Given locally bounded psh functions $u_1, \ldots, u_q$, one defines inductively

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T = dd^c(u_1 dd^c u_2 \ldots \wedge dd^c u_q \wedge T)$$

and the result is a closed positive current. In particular, when u is a locally bounded psh function, there is a well defined positive measure $(dd^c u)^n$. If u is of class C^2, a computation in local coordinates gives

$$(dd^c u)^n = \det\Big(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\Big) \cdot \frac{n!}{\pi^n}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n \ .$$

The expression "Monge-Ampère operator" refers generally to the non-linear partial differential operator $u \longmapsto \det(\partial^2 u/\partial z_j \partial \overline{z}_k)$.

Now, let Θ be a current of order 0 . If K is a compact subset contained in a coordinate patch of X , we define the *mass* of $\Theta = \sum \Theta_{I,J}\, dz_I \wedge d\overline{z}_J$ on K by

$$||\Theta||_K = \int_K \sum_{I,J} |\Theta_{I,J}|$$

where $|\Theta_{I,J}|$ is the absolute value of the measure $\Theta_{I,J}$. When $\Theta \geqslant 0$ we have $|\Theta_{I,J}| \leqslant C.\Theta \wedge \beta^p$ with $\beta = dd^c|z|^2$; up to constants, the mass $||\Theta||_K$ is then equivalent to the integral $\int_K \Theta \wedge \beta^p$. When $K \subset\subset X$ is arbitrary, we take a partition $K = \bigcup K_j$ where each $\overline{K}_j$ is contained in a coordinate patch and write

$$||\Theta||_K = \sum ||\Theta||_{K_j} \ .$$

Up to constants, the semi-norm $||\Theta||_K$ does not depend on the choice of the coordinate systems involved.

Chern-Levine-Nirenberg inequalities 1.3 ([C-L-N]). — *For all compact sets K, L of X with $L \subset K^\circ$, there exists a constant $C_{K,L} \geqslant 0$ such that*

$$||dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T||_L \leqslant C_{K,L}\ ||u_1||_{L^\infty(K)} \ldots ||u_q||_{L^\infty(K)}\, ||T||_K \ .$$

Proof. — By induction, it is sufficient to prove the result for $q = 1$ and $u_1 = u$. There is a covering of L by a family of balls $B'_j \subset\subset B_j \subset K$ contained in coordinate patches of X . Let $\chi \in \mathcal{D}(B_j)$ be equal to 1 on $\overline{B}'_j$. Then

$$||dd^c u \wedge T||_{L\cap \overline{B}'_j} \leqslant C \int_{\overline{B}'_j} dd^c u \wedge T \wedge \beta^{p-1} \leqslant C \int_{B_j} \chi\, dd^c u \wedge T \wedge \beta^{p-1} \ .$$

As T and β are closed, an integration by parts yields

$$||dd^c u \wedge T||_{L\cap \overline{B}'_j} \leqslant C \int_{B_j} u\, T \wedge dd^c \chi \wedge \beta^{p-1} \leqslant C' ||u||_{L^\infty(K)} ||T||_K$$

where C' is equal to C multiplied by a bound for the coefficients of $dd^c\chi \wedge \beta^{p-1}$. □

Exercise 1.4. — *Denote by $L^1(K)$ the space of integrable functions with respect to some smooth positive density on K . For any V psh on X show*

(a) $||dd^c V||_L \leqslant C_{K,L}\, ||V||_{L^1(K)}$.

(b) $\sup_L V_+ \leqslant C_{K,L}\, ||V||_{L^1(K)}$.

Now, we prove a rather important continuity theorem due to [B-T2].

THEOREM 1.5. — *Let $u_1, \ldots, u_q$ be locally bounded psh functions and let $u_1^k, \ldots, u_q^k$ be decreasing sequences of psh functions converging pointwise to $u_1, \ldots, u_q$. Then*

(a) $u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \longrightarrow u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$ *weakly.*

(b) $dd^c u_1^k \wedge \ldots \wedge dd^c u_q^k \wedge T \longrightarrow dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ *weakly.*

Proof. — As the sequence (u_j^k) is non increasing and as u_j is locally bounded, the family $(u_j^k)_{k\in\mathbb{N}}$ is locally uniformly bounded. The result is local, so we can work on a strongly pseudoconvex open set $\Omega \subset\subset X$. Let ψ be a strongly psh function of class C^∞ near $\overline{\Omega}$ with $\psi < 0$ on Ω , $\psi = 0$ and $d\psi \neq 0$ on $\partial\Omega$. After addition of a constant we can assume that $-M \leqslant u_j^k \leqslant -1$ near $\overline{\Omega}$. Let us denote by $(u_j^{k,\varepsilon})$, $\varepsilon \in]0, \varepsilon_0]$, an increasing family of regularizations converging to u_j^k as $\varepsilon \to 0$ and such that $-M \leqslant u_j^{k,\varepsilon} \leqslant -1$ on $\overline{\Omega}$. Set $A = M/\delta$ with $\delta > 0$ small and replace u_j^k by $v_j^k = \max\{A\psi, u_j^k\}$, $u_j^{k,\varepsilon}$ by $v_j^{k,\varepsilon} = \max_\varepsilon\{A\psi, u_j^{k,\varepsilon}\}$ where $\max_\varepsilon = \max \star \rho_\varepsilon$ is a regularized max function.

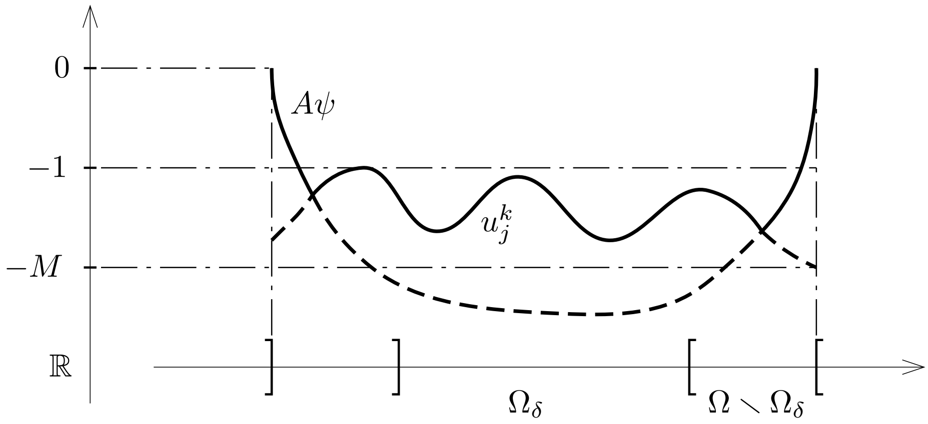

Fig. 1 Construction of v_j^k

Then v_j^k coincides with u_j^k on $\Omega_\delta = \{\psi < -\delta\}$ since $A\psi < -A\delta = -M$ on Ω_δ , and v_j^k is equal to $A\psi$ on the corona $\Omega \setminus \Omega_{\delta/M}$. Without loss of generality, we can therefore assume that all u_j^k (and similarly all $u_j^{k,\varepsilon}$) coincide with $A\psi$ on a fixed neighborhood of $\partial\Omega$.

Now, we argue by induction on q and observe that (b) is an immediate consequence of (a). When $q = 1$, (a) follows directly from the bounded convergence theorem. We need a lemma.

LEMMA 1.6. — *Let f_k be a non-increasing sequence of upper semi-continuous functions converging to f on some separable locally compact space X and μ_k a sequence of positive measures converging weakly to μ on X . Then every weak limit ν of $f_k\mu_k$ satisfies $\nu \leqslant f\mu$.*

Indeed if (g_p) is a decreasing sequence of continuous functions converging to f_{k_0} for some k_0 , then $f_k\mu_k \leqslant f_{k_0}\mu_k \leqslant g_p\mu_k$ for $k \geqslant k_0$, thus $\nu \leqslant g_p\mu$ as $k \to +\infty$. The monotone convergence theorem then gives $\nu \leqslant f_{k_0}\mu$ as $p \to +\infty$ and $\nu \leqslant f\mu$ as $k_0 \to +\infty$. $\square$

End of proof of theorem 1.5. — Assume that (a) has been proved for $q-1$. Then

$$S^k = dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \longrightarrow S = dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \ .$$

By 1.3 the sequence $(u_1^k S^k)$ has locally bounded mass, hence is relatively compact for the weak topology. In order to prove (a), we only have to show that every weak limit Θ of $u_1^k S^k$ is equal to $u_1 S$. Let (m,m) be the bidimension of S and let γ be an arbitrary smooth and strongly positive form of bidegree (m,m) . Then the positive measures $S^k \wedge \gamma$ converge weakly to $S\wedge\gamma$ and lemma 1.6 shows that $\Theta\wedge\gamma \leqslant u_1 S\wedge\gamma$, hence $\Theta \leqslant u_1 S$. To get the equality, we set $\beta = dd^c\psi > 0$ and show that $\int_\Omega u_1 S\wedge\beta^m \leqslant \int_\Omega \Theta\wedge\beta^m$, *i.e.*

$$\int_\Omega u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \leqslant \liminf \int_\Omega u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \wedge \beta^m \ .$$

As $u_1 \leqslant u_1^k \leqslant u_1^{k,\varepsilon_1}$ for every $\varepsilon_1 > 0$ we get

$$\begin{aligned} &\int_\Omega u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \\ &\qquad \leqslant \int_\Omega u_1^{k,\varepsilon_1} dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \\ &\qquad = \int_\Omega dd^c u_1^{k,\varepsilon_1} \wedge u_2 dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \end{aligned}$$

after an integration by parts (there is no boundary term because u_1^{k,ε_1} and u_2 vanish on $\partial\Omega$). Repeating this argument with $u_2,\ldots,u_q$, we obtain

$$\begin{aligned} &\int_\Omega u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \\ &\qquad \leqslant \int_\Omega dd^c u_1^{k,\varepsilon_1} \wedge \ldots \wedge dd^c u_{q-1}^{k,\varepsilon_{q-1}} \wedge u_q T \wedge \beta^m \\ &\qquad \leqslant \int_\Omega u_1^{k,\varepsilon_1} dd^c u_2^{k,\varepsilon_2} \wedge \ldots \wedge dd^c u_q^{k,\varepsilon_q} \wedge T \wedge \beta^m \ . \end{aligned}$$

Now let $\varepsilon_q \to 0, \ldots, \varepsilon_1 \to 0$ in this order. We have weak convergence at each step and $u_1^{k,\varepsilon_1} = 0$ on the boundary; therefore the last integral converges and we get the desired inequality

$$\int_\Omega u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \leqslant \int_\Omega u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \wedge \beta^m \ . \ \square$$

COROLLARY 1.7. — *The product $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ is symmetric with respect to $u_1,\ldots,u_q$.*

Observe that the definition was unsymmetric. The result is true when $u_1, \ldots, u_q$ are smooth and follows in general from theorem 1.5 applied to $u_1^k = u_1 \star \rho_{1/k}$. □

THEOREM 1.8. — *Let K, L be compact subsets of X such that $L \subset K^\circ$. For any psh functions $V, u_1, \ldots, u_q$ on X such that $u_1, \ldots, u_q$ are locally bounded, there is an inequality*

$$||V dd^c u_1 \wedge \ldots \wedge dd^c u_q||_L \leqslant C_{K,L}\, ||V||_{L^1(K)} ||u_1||_{L^\infty(K)} \cdots ||u_q||_{L^\infty(K)} \ .$$

Proof. — First, we may assume that L is contained in a strictly pseudoconvex open set $\Omega = \{\psi < 0\} \subset K$ (otherwise cover L by small balls contained in K). A suitable normalization gives $-2 \leqslant u_j \leqslant -1$ on K; then we can modify u_j on $\Omega \setminus L$ so that $u_j = A\psi$ on $\Omega \setminus \Omega_\delta$ with a fixed constant A and $\delta > 0$ such that $L \subset \Omega_\delta$. Let $\chi \geqslant 0$ be a smooth function equal to $-\psi$ on Ω_δ with compact support in Ω . If we take $||V||_{L^1(K)} = 1$, we see that V_+ is uniformly bounded on Ω_δ by 1.4 (b); after subtraction of a fixed constant we get $V \leqslant 0$ on Ω_δ . As $u_j = A\psi$ on $\Omega \setminus \Omega_\delta$, we find for $q \leqslant n-1$:

$$\begin{aligned}
&\int_{\Omega_\delta} - V\, dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q} \\
&\qquad = \int_\Omega V\, dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q-1} \wedge dd^c\chi - A^q \int_{\Omega\setminus\Omega_\delta} V\, \beta^{n-1} \wedge dd^c \chi \\
&\qquad = \int_\Omega \chi\, dd^c V \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q-1} - A^q \int_{\Omega\setminus\Omega_\delta} V\, \beta^{n-1} \wedge dd^c \chi \ .
\end{aligned}$$

The first integral of the last line is uniformly bounded thanks to 1.3 and 1.4 (a), and the second one is bounded by $||V||_{L^1(\Omega)} \leqslant$ constant. Inequality 1.8 follows if $q \leqslant n-1$. If $q = n$, we can work instead on $X \times \mathbb{C}$ and consider $V, u_1, \ldots, u_q$ as functions on $X \times \mathbb{C}$ independent of the extra factor $\mathbb{C}$. □

Now, we would like to define $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ also in some cases when $u_1, \ldots, u_q$ are not bounded below everywhere. Consider first the case $q = 1$ and let u be a plurisubharmonic function on X . The polar set of u is by definition $u^{-1}(-\infty)$.

ASSUMPTIONS 1.9. — *We make two additional assumptions :*

(a) *T has non zero bidimension (p,p) (i.e. degree of $T < 2n$) .*

(b) *X is covered by a family of strongly pseudoconvex open sets $\Omega = \{\psi < 0\}$, $\Omega \subset\subset X$, with the following property : there is an open set ω_T containing* $\mathrm{Supp}\, T \cap \Omega$ *and an open set ω_u containing $u^{-1}(-\infty) \cap \Omega$ such that $\overline{\omega}_T \cap \overline{\omega}_u$ is compact in Ω and u is bounded on $\omega_T \setminus \omega_u$.*

Example. — For any T , hypothesis 1.9 (b) is clearly satisfied if u has a discrete set P of poles; an interesting example is $u = \log|F|$ where $F = (F_1, \ldots, F_N)$ are holomorphic functions having a discrete set of common zeroes. □

Let us replace u by the everywhere finite function

$$u_{\geqslant s}(z) = \max\{u(z), s\} \ .$$

We shall let hereafter s tend to $-\infty$. Let $\beta = dd^c\psi$ and let s_0 be a lower bound for u on a neighborhood of $\partial\Omega\cap\operatorname{Supp} T$. For $s < s_0$, the integral $\int_\Omega dd^c u_{\geqslant s}\wedge T\wedge\beta^{p-1}$ does not depend on s; in fact, Stokes' theorem shows that

$$\int_\Omega (dd^c u_{\geqslant r} - dd^c u_{\geqslant s})\wedge T\wedge(dd^c\psi)^{p-1} = \int_\Omega dd^c\big[(u_{\geqslant r} - u_{\geqslant s})\, T\wedge(dd^c\psi)^{p-1}\big] = 0$$

because $u_{\geqslant r}$ and $u_{\geqslant s}$ both coincide with u near $\partial\Omega\cap\operatorname{Supp} T$, hence the current $[\ldots]$ has compact support in Ω . This shows that the mass of $dd^c u_{\geqslant s}\wedge T$ is uniformly bounded on Ω . Now let χ be a function with compact support in Ω equal to ψ on a neighborhood Ω' of $\overline{\omega}_T\cap\overline{\omega}_u$. As u is bounded on $(\Omega\setminus\Omega')\cap\operatorname{Supp} T$, we have

$$\int_\Omega \chi dd^c u_{\geqslant s}\wedge T\wedge(dd^c\psi)^{p-1} = \int_\Omega u_{\geqslant s} T\wedge(dd^c\psi)^{p-1}\wedge dd^c\chi \leqslant C + \int_{\Omega'} u_{\geqslant s} T\wedge(dd^c\psi)^p.$$

The first integral remains bounded as $s\to-\infty$. Hence the last integral cannot decrease to $-\infty$ and we see that uT has bounded mass on Ω' . We can therefore define $dd^c u\wedge T = dd^c(uT)$ as before.

Remark 1.10. — The current uT has not necessarily a finite mass when T has degree $2n$ (*i.e.* T is a measure); example : $T=\delta_0$ and $u(z)=\log|z|$ in $\mathbb{C}^n$. □

Assume now that $u_1,\ldots,u_q$ are psh functions on X that are bounded on $\omega_T\setminus\omega_u$, where ω_u is an open set containing all polar sets $u_j^{-1}(-\infty)$ such that $\overline{\omega}_T\cap\overline{\omega}_u\subset\subset\Omega$. One can again use induction to define

$$dd^c u_1\wedge dd^c u_2\wedge\ldots\wedge dd^c u_q\wedge T = dd^c(u_1 dd^c u_2\ldots\wedge dd^c u_q\wedge T)\ . \tag{1.11}$$

THEOREM 1.12. — *If $u_1^k,\ldots,u_q^k$ are non-increasing sequences converging pointwise to $u_1,\ldots,u_q$, then*

$$u_1^k dd^c u_2^k\wedge\ldots\wedge dd^c u_q^k\wedge T \longrightarrow u_1 dd^c u_2\wedge\ldots\wedge dd^c u_q\wedge T \quad \textit{weakly},$$
$$dd^c u_1^k\wedge\ldots\wedge dd^c u_q^k\wedge T \longrightarrow dd^c u_1\wedge\ldots\wedge dd^c u_q\wedge T \quad \textit{weakly}.$$

Proof. — Same proof as in theorem 1.5, with the following minor modification : the max procedure $\max\{u_j^k, A\psi\}$ is applied only on $\omega_T\setminus\Omega_\delta$ and u_j^k is left unchanged on $\omega_T\cap\Omega_\delta$, assuming that $\Omega_\delta\supset\overline{\omega}_T\cap\overline{\omega}_u$; observe that the functions u_j^k and $u_j^{k,\varepsilon}$ are needed only on ω_T .

THEOREM 1.13. — *Let P be a compact subset of a strongly pseudoconvex open set $\Omega\subset X$. If V is a psh function on X and $u_1,\ldots,u_q$, $1\leqslant q\leqslant n-1$, are psh functions that are locally bounded on $\overline{\Omega}\setminus P$, then $V dd^c u_1\wedge\ldots\wedge dd^c u_q$ has finite mass on Ω .*

Proof. — Same proof as 1.8, taking $P\subset\Omega_\delta$. □

2. Generalized Lelong numbers.

Assume from now on that X is a Stein manifold, *i.e.* that X has a strictly psh exhaustion function. Let $\varphi : X \longrightarrow [-\infty, +\infty[$ be a continuous psh function. The sets

$$S(r) = \{x \in X \,;\, \varphi(x) = r\} \,, \tag{2.1}$$

$$B(r) = \{x \in X \,;\, \varphi(x) < r\} \,, \tag{2.1'}$$

$$\overline{B}(r) = \{x \in X \,;\, \varphi(x) \leqslant r\} \tag{2.1''}$$

will be called *pseudo-spheres* and *pseudo-balls* associated to φ . It may happen in some cases that $\overline{B}(r)$ is distinct from the closure of $B(r)$. For simplicity, we sometimes denote

$$\alpha = dd^c\varphi \,, \qquad \beta = \frac{1}{2}dd^c(e^{2\varphi}) \,. \tag{2.2}$$

The most simple example we have in mind is $\varphi(z) = \log|z-a|$ on an open subset $X \subset \mathbb{C}^n$; in this case $B(r)$ is the euclidean ball of center a and radius e^r , and β is the usual hermitian metric $\frac{i}{2\pi}d'd''|z|^2$ of $\mathbb{C}^n$. When $a = 0$, α is the pull back on $\mathbb{C}^n$ of the standard Fubini-Study metric on $\mathbb{P}^{n-1}$.

DEFINITION 2.3. — *We say that φ is semi-exhaustive if there exists a real number R such that $B(R) \subset\subset X$. Similarly, φ is said to be semi-exhaustive on a closed subset $A \subset X$ if there exists R such that $A \cap B(R) \subset\subset X$.*

We are interested especially in the set of poles $S(-\infty) = \{\varphi = -\infty\}$ and in the behaviour of φ near $S(-\infty)$. Let T be a closed positive current of bidimension (p,p) on X . Assume that φ is semi-exhaustive on $\operatorname{Supp} T$ and that $B(R) \cap \operatorname{Supp} T \subset\subset X$. Then $P = S(-\infty) \cap \mathrm{SuppT}$ is compact and the results of §1 show that the measure $T \wedge (dd^c\varphi)^p$ is well defined.

DEFINITION 2.4. — *For $r \in]-\infty, R[$, we set*

$$\nu(T,\varphi,r) = \int_{B(r)} T \wedge (dd^c\varphi)^p \,,$$

$$\nu(T,\varphi) = \int_{S(-\infty)} T \wedge (dd^c\varphi)^p = \lim_{r \to -\infty} \nu(T,\varphi,r) \,.$$

The number $\nu(T,\varphi)$ will be called the (generalized) Lelong number of T with respect to the weight φ .

It is clear that $r \longmapsto \nu(T,\varphi,r)$ is an increasing function of r . Before giving an example, we need a formula.

FORMULA 2.5. — *For any convex increasing function $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ one has*

$$\int_{B(r)} T \wedge (dd^c\chi\circ\varphi)^p = \chi'(r-0)^p\,\nu(T,\varphi,r)$$

where $\chi'(r-0)$ denotes the left derivative of χ at r .

Proof. — Let χ_ε be the convex function equal to χ on $[r-\varepsilon,+\infty[$ and to a linear function of slope $\chi'(r-\varepsilon-0)$ on $]-\infty,r-\varepsilon]$. We get $dd^c(\chi_\varepsilon\circ\varphi)=\chi'(r-\varepsilon-0)dd^c\varphi$ on $B(r-\varepsilon)$ and Stokes' theorem implies

$$\begin{aligned}\int_{B(r)} T\wedge(dd^c\chi\circ\varphi)^p &= \int_{B(r)} T\wedge(dd^c\chi_\varepsilon\circ\varphi)^p\\ &\geqslant \int_{B(r-\varepsilon)} T\wedge(dd^c\chi_\varepsilon\circ\varphi)^p=\chi'(r-\varepsilon-0)^p\nu(T,\varphi,r-\varepsilon)\ .\end{aligned}$$

Similarly, taking $\widetilde\chi_\varepsilon$ equal to χ on $]-\infty,r-\varepsilon]$ and linear on $[r-\varepsilon,r]$, we obtain

$$\int_{B(r-\varepsilon)} T\wedge(dd^c\chi\circ\varphi)^p\leqslant\int_{B(r)} T\wedge(dd^c\widetilde\chi_\varepsilon\circ\varphi)^p=\chi'(r-\varepsilon-0)^p\nu(T,\varphi,r)\ .$$

The expected formula follows when ε tends to 0 . □

We get in particular $\int_{B(r)}T\wedge(dd^ce^{2\varphi})^p=(2e^{2r})^p\nu(T,\varphi,r)$, whence the formula

$$\nu(T,\varphi,r)=e^{-2pr}\int_{B(r)} T\wedge\beta^p\ . \tag{2.6}$$

Now, assume that X is an open subset of $\mathbb{C}^n$ and that $\varphi(z)=\log|z-a|$ for some $a\in X$. Formula (2.6) gives

$$\nu(T,\varphi,\log r)=r^{-2p}\int_{|z-a|<r} T\wedge\Big(\frac{i}{2\pi}d'd''|z|^2\Big)^p\ .$$

The positive measure $\sigma_T=\frac{1}{p!}T\wedge(\frac{i}{2}d'd''|z|^2)^p=2^{-p}\sum T_{I,I}\,.\,i^n dz_1\wedge d\overline z_1\ldots d\overline z_n$ is called the *trace measure* of T . We get

$$\nu(T,\varphi,\log r)=\frac{\sigma_T\big(B(a,r)\big)}{\pi^pr^{2p}/p!} \tag{2.7}$$

and $\nu(T,\varphi)$ is the limit of this ratio as $r\to0$. This limit is called the Lelong number of T at point a and denoted $\nu(T,a)$. This was precisely the original definition of Lelong (*cf.* [Le3]). Let us mention an important consequence.

Consequence 2.8. — *The ratio $\sigma_T\big(B(a,r)\big)/r^{2p}$ is an increasing function of the radius r . In particular, we have*

$$\sigma_T\big(B(a,r)\big)\leqslant Cr^{2p}$$

for $r<r_0$ small enough.

All these results are particularly interesting when $T=[A]$ is the current of integration over an analytic subset $A\subset X$ of pure dimension p . Then $\sigma_T\big(B(a,r)\big)$ is the euclidean area of $A\cap B(a,r)$, and $\nu(T,\varphi,\log r)$ is the ratio of this area to the area of a ball of radius r in $\mathbb{C}^p$.

EXERCISE 2.9. — *When A is a smooth submanifold of X , show that*

$$\nu([A],x)=\begin{cases}1 & \text{for } x\in A\\ 0 & \text{for } x\notin A\ .\end{cases}$$

Remark 2.10. — When $X=\mathbb{C}^n$, $\varphi(z)=\log|z-a|$ and $A=X$ (*i.e.* $T=1$), we obtain in particular $\int_{B(a,r)}(dd^c\log|z-a|)^n=1$ for all r . This implies

$$(dd^c\log|z-a|)^n=\delta_a\ .$$

This fundamental formula can be viewed as a higher dimensional analogue of the usual formula $\Delta\log|z-a|=2\pi\delta_a$ in $\mathbb{C}$.

3. The Lelong-Jensen formula.

Assume in this paragraph that φ is *semi-exhaustive* on X and that $B(R)\subset\subset X$. For every $r\in]-\infty,R[$, the measures $dd^c(\varphi_{\geqslant r})^n$ are well defined. The map $r\longmapsto(dd^c\varphi_{\geqslant r})^n$ is continuous on $]-\infty,R[$ with respect to the weak topology : right continuity follows immediately from theorem 1.5, while left continuity is obtained similarly from the equality $(dd^c\varphi_{\geqslant r})^n=(dd^c\max\{\varphi-r,0\})^n$. As $(dd^c\varphi_{\geqslant r})^n=(dd^c\varphi)^n$ on $X\setminus\overline{B}(r)$ and $\varphi_{\geqslant r}\equiv r$, $(dd^c\varphi_{\geqslant r})^n=0$ on $B(r)$, the left continuity implies $(dd^c\varphi_{\geqslant r})^n\geqslant\mathbf{1}_{X\setminus B(r)}(dd^c\varphi)^n$. Here $\mathbf{1}_A$ denotes the characteristic function of any subset $A\subset X$. According to the definition introduced in [De2], the collection of *Monge-Ampère measures* associated to φ is the family of positive measures μ_r such that

$$\mu_r=(dd^c\varphi_{\geqslant r})^n-\mathbf{1}_{X\setminus B(r)}(dd^c\varphi)^n\ ,\qquad r\in]-\infty,R[\ . \tag{3.1}$$

The measure μ_r is supported on $S(r)$ and $r\longmapsto\mu_r$ is weakly continuous on the left by the bounded convergence theorem. Stokes' formula shows that $\int_{B(s)}(dd^c\varphi_{\geqslant r})^n-(dd^c\varphi)^n=0$ for $s>r$, hence the total mass $\mu_r\big(S(r)\big)=\mu_r\big(B(s)\big)$ is equal to the difference between the masses of $(dd^c\varphi)^n$ and $\mathbf{1}_{X\setminus B(r)}(dd^c\varphi)^n$ over $B(s)$, *i.e.*

$$\mu_r\big(S(r)\big)=\int_{B(r)}(dd^c\varphi)^n\ . \tag{3.2}$$

Example 3.3. — When $(dd^c\varphi)^n=0$ on $X\setminus\varphi^{-1}(-\infty)$, formula (3.1) simplifies into $\mu_r=(dd^c\varphi_{\geqslant r})^n$. This is so for $\varphi(z)=\log|z|$. In this case, the invariance of φ under unitary transformations implies that μ_r is also invariant. As the total mass of μ_r is equal to 1 by 2.10 and (3.2), we see that μ_r is the invariant measure of mass 1 on the euclidean sphere of radius e^r .

THEOREM 3.4. — *Assume that φ is smooth near $S(r)$ and that $d\varphi\neq 0$ on $S(r)$, i.e. r is a non critical value. Then $S(r)=\partial B(r)$ is a smooth oriented real hypersurface and μ_r is given by the $(2n-1)$–volume form $(dd^c\varphi)^{n-1}\wedge d^c\varphi_{\restriction S(r)}$.*

Proof. — Write $\max\{t,r\} = \lim_{k\to+\infty}\chi_k(t)$ where χ is a non-increasing sequence of smooth convex functions with $\chi_k(t)=r$ for $t\leqslant r-1/k$, $\chi_k(t)=t$ for $t\geqslant r+1/k$. Theorem 1.5 shows that $(dd^c\chi_k\circ\varphi)^n$ converges weakly to $(dd^c\varphi_{\geqslant r})^n$. Let h be a smooth function h with compact support near $S(r)$. Let us apply Stokes' theorem with $S(r)$ considered as the boundary of $X\setminus B(r)$:

$$\begin{aligned}
\int_X h(dd^c\varphi_{\geqslant r})^n &= \lim_{k\to+\infty}\int_X h(dd^c\chi_k\circ\varphi)^n \\
&= \lim_{k\to+\infty}\int_X -dh\wedge(dd^c\chi_k\circ\varphi)^{n-1}\wedge d^c(\chi_k\circ\varphi) \\
&= \lim_{k\to+\infty}\int_X -\chi_k'(t)^n\,dh\wedge(dd^c\varphi)^{n-1}\wedge d^c\varphi \\
&= \int_{X\setminus B(r)} -dh\wedge(dd^c\varphi)^{n-1}\wedge d^c\varphi \\
&= \int_{S(r)} h\,(dd^c\varphi)^{n-1}\wedge d^c\varphi + \int_{X\setminus B(r)} h\,(dd^c\varphi)^{n-1}\wedge d^c\varphi\ .
\end{aligned}$$

Near $S(r)$ we thus have an equality of measures

$$(dd^c\varphi_{\geqslant r})^n = (dd^c\varphi)^{n-1}\wedge d^c\varphi_{\restriction S(r)} + \mathbf{1}_{X\setminus B(r)}(dd^c\varphi)^n\ .\ \square$$

Lelong-Jensen formula 3.5. — *Let V be any psh function on X . Then V is μ_r–integrable for every $r\in]-\infty,R[$ and*

$$\mu_r(V) - \int_{B(r)} V(dd^c\varphi)^n = \int_{-\infty}^r \nu(dd^cV,\varphi,t)\,dt\ .$$

Proof. — Theorem 1.8 shows that V is integrable with respect to $(dd^c\varphi_{\geqslant r})^n$, hence V is μ_r–integrable. By definition $\nu(dd^cV,\varphi,t) = \int_{\varphi(z)<t} dd^cV\wedge(dd^c\varphi)^{n-1}$ and Fubini's theorem gives

$$\begin{aligned}
\int_{-\infty}^r \nu(dd^cV,\varphi,t)\,dt &= \iint_{\varphi(z)<t<r} dd^cV(z)\wedge(dd^c\varphi(z))^{n-1}\,dt \\
&= \int_{B(r)} (r-\varphi)dd^cV\wedge(dd^c\varphi)^{n-1}\ . \qquad (3.6)
\end{aligned}$$

We first show that formula 3.5 is true when φ and V are smooth. As both members of the formula are left continuous with respect to r and as almost all values of φ are non critical by Sard's theorem, we may assume r non critical. Formula 1.1 applied with $f=(r-\varphi)(dd^c\varphi)^{n-1}$ and $g=V$ shows that integral (3.6) is equal to

$$\int_{S(r)} V(dd^c\varphi)^{n-1}\wedge d^c\varphi - \int_{B(r)} V\,(dd^c\varphi)^n = \mu_r(V) - \int_{B(r)} V\,(dd^c\varphi)^n\ .$$

Formula 3.5 is thus proved when φ and V are smooth. If V is smooth and φ merely continuous and finite, one can write $\varphi=\lim\varphi_k$ where φ_k is a non-increasing sequence of smooth plurisubharmonic functions (because X is Stein).

Then $dd^cV\wedge(dd^c\varphi_k)^{n-1}$ converges weakly to $dd^cV\wedge(dd^c\varphi)^{n-1}$ and (3.6) converges, since $\mathbf{1}_{B(r)}(r-\varphi)$ is continuous with compact support on X . The left hand side of formula 3.5 also converges because the definition of μ_r implies

$$\mu_{k,r}(V)-\int_{\varphi_k<r}V(dd^c\varphi_k)^n=\int_X V\big((dd^c\varphi_{k,\geqslant r})^n-(dd^c\varphi_k)^n\big)$$

and we can apply again weak convergence on a neighborhood of $\overline{B}(r)$. If φ takes $-\infty$ values, replace φ by $\varphi_{\geqslant -k}$ where $k\to+\infty$. Then $\mu_r(V)$ is unchanged, $\int_{B(r)}V(dd^c\varphi_{\geqslant -k})^n$ converges to $\int_{B(r)}V(dd^c\varphi)^n$ and the right hand side of formula 3.5 is replaced by $\int_{-k}^r\nu(dd^cV,\varphi,t)\,dt$. Finally, for V arbitrary, write $V=\lim\downarrow V_k$ with a sequence of smooth functions V_k . Then $dd^cV_k\wedge(dd^c\varphi)^{n-1}$ converges weakly to $dd^cV\wedge(dd^c\varphi)^{n-1}$ by theorem 1.13, thus integral (3.6) converges to the expected limit, and the same is true for the left hand side of 3.5 by the monotone convergence theorem. □

For $r<r_0<R$, the Lelong-Jensen formula implies

$$\mu_r(V)-\mu_{r_0}(V)+\int_{B(r_0)\setminus B(r)}V(dd^c\varphi)^n=\int_{r_0}^r\nu(dd^cV,\varphi,t)\,dt\ . \tag{3.7}$$

Corollary 3.8. — *Assume that $(dd^c\varphi)^n=0$ on $X\setminus S(-\infty)$. Then $r\longmapsto\mu_r(V)$ is a convex increasing function of r and the lelong number $\nu(dd^cV,\varphi)$ is given by*

$$\nu(dd^cV,\varphi)=\lim_{r\to-\infty}\frac{\mu_r(V)}{r}\ .$$

Proof. — By (3.7) we have

$$\mu_r(V)=\mu_{r_0}(V)+\int_{r_0}^r\nu(dd^cV,\varphi,t)\,dt\ .$$

As $\nu(dd^cV,\varphi,t)$ is increasing and non-negative, it follows that $r\longmapsto\mu_r(V)$ is convex and increasing. The formula for $\nu(dd^cV,\varphi)=\lim_{t\to-\infty}\nu(dd^cV,\varphi,t)$ is then obvious. □

Example 3.9. — Take $\varphi(z)=\log|z-a|$ on an open subset of $\mathbb{C}^n$ containing the point a . The Lelong-Jensen formula becomes

$$\mu_r(V)=V(a)+\int_{-\infty}^r\nu(dd^cV,\varphi,t)\,dt\ .$$

As μ_r is the mean value measure on the sphere $S(a,e^r)$, we make the change of variables $r\mapsto\log r$, $t\mapsto\log t$ and obtain the more familiar formula

$$\mu(V,S(a,r))=V(a)+\int_0^r\nu(dd^cV,t)\,\frac{dt}{t}$$

where $\nu(dd^cV,t)=\nu(dd^cV,\varphi,\log t)$ is given by (2.7) :

$$\nu(dd^cV,t)=\frac{1}{\pi^{n-1}t^{2n-2}/(n-1)!}\int_{B(a,t)}\frac{1}{2\pi}\Delta V\ .$$

In particular, take $V = \log|f|$ where f is a holomorphic function on X . The Poincaré-Lelong formula shows that $dd^c\log|f|$ is equal to the zero divisor $[Z_f]=\sum m_j[H_j]$, where H_j are the irreducible components of $f^{-1}(0)$ and m_j the multiplicity of f on H_j . The trace $\frac{1}{2\pi}\Delta f$ is then the euclidean area measure of Z_f (with corresponding multiplicities m_j). In dimension $n=1$, we have $\frac{1}{2\pi}\Delta f=\sum m_j\delta_{a_j}$. Then we get the usual Jensen formula

$$\mu\big(\log|f|,S(0,r)\big)-\log|f(0)|=\int_0^r \nu(t)\frac{dt}{t}=\sum m_j\log\frac{r}{|a_j|}$$

where $\nu(t)$ is the number of zeroes a_j in the disk $D(0,t)$, counted with multiplicities m_j .

Example 3.10. — Take $\varphi(z)=\log\max|z_j|^{\lambda_j}$. where $\lambda_j>0$. Then $B(r)$ is the polydisk of radii $(e^{r/\lambda_1},\ldots,e^{r/\lambda_n})$. If some coordinate z_j is non zero, say z_1 , we can write $\varphi(z)$ as $\lambda_1\log|z_1|$ plus some function depending only on the $(n-1)$ variables $z_j/z_1^{\lambda_1/\lambda_j}$. Hence $(dd^c\varphi)^n=0$ on $\mathbb{C}^n\setminus\{0\}$. It will be shown later that

$$(dd^c\varphi)^n=\lambda_1\ldots\lambda_n\,\delta_0\ . \tag{3.11}$$

We now determine the measures μ_r . At any point z where not all terms $|z_j|^{\lambda_j}$ are equal, the smallest one can be omitted without changing φ in a neighborhood of z . Thus φ depends only on $(n-1)$–variables and $(dd^c\varphi_{\geqslant r})^n=0$, $\mu_r=0$ near z . It follows that μ_r is supported by the distinguished boundary $|z_j|=e^{r/\lambda_j}$ of the polydisk $B(r)$. As φ is invariant by all rotations $z_j\longmapsto e^{i\theta_j}z_j$, the measure μ_r is also invariant and we see that μ_r is a constant multiple of $d\theta_1\ldots d\theta_n$. By formula (3.2) and (3.11) we get

$$\mu_r=\lambda_1\ldots\lambda_n\,(2\pi)^{-n}d\theta_1\ldots d\theta_n\ .$$

In particular, the Lelong number $\nu(dd^cV,\varphi)$ is given by

$$\nu(dd^cV,\varphi)=\lambda_1\ldots\lambda_n\lim_{r\to-\infty}\int_{\theta_j\in[0,2\pi]}V(e^{r/\lambda_1+i\theta_1},\ldots,e^{r/\lambda_n+i\theta_n})\,\frac{d\theta_1\ldots d\theta_n}{(2\pi)^n}\ .$$

These numbers have been introduced and studied by Kiselman [Ki3]; we shall denote them $\nu(T,x,\lambda)$, where $\lambda=(\lambda_1,\ldots,\lambda_n)$.

4. Comparison theorem for Lelong numbers.

We show here that the Lelong numbers $\nu(T,\varphi)$ only depend on the asymptotic behaviour of φ near the polar set $S(-\infty)$. In a precise way :

Theorem 4.1. — *Let $\varphi,\psi:X\longrightarrow[-\infty,+\infty[$ be continuous psh functions. We assume that φ,ψ are semi-exhaustive on* $\operatorname{Supp}T$ *and that*

$$l:=\limsup\frac{\psi(x)}{\varphi(x)}<+\infty\quad\text{ as }\ x\in\operatorname{Supp}T\ \text{ and }\ \varphi(x)\to-\infty\ .$$

Then $\nu(T,\psi)\leqslant l^p\nu(T,\varphi)$, and the equality holds if $l=\lim\psi/\varphi$.

Proof. — Definition 2.4 gives immediately

$$\nu(T, \lambda\varphi) = \lambda^p \nu(T, \varphi)$$

for every scalar $\lambda > 0$. It is thus sufficient to verify the inequality $\nu(T, \psi) \leqslant \nu(T, \varphi)$ under tha hypothesis $\limsup \psi/\varphi < 1$. For all $c > 0$, consider the psh function

$$u_c = \max(\psi - c, \varphi) \ .$$

Let R_φ and R_ψ be such that $B_\varphi(R_\varphi) \cap \operatorname{Supp} T$ and $B_\psi(R_\psi) \cap \operatorname{Supp} T$ be relatively compact in X . Let $r < R_\varphi$ be fixed and $a < r$. For $c > 0$ large enough, we have $u_c = \varphi$ on $\varphi^{-1}([a, r])$ and Stokes' formula gives

$$\nu(T, \varphi, r) = \nu(T, u_c, r) \geqslant \nu(T, u_c) \ .$$

The hypothesis $\limsup \psi/\varphi < 1$ implies on the other hand that there exists $t_0 < 0$ such that $u_c = \psi - c$ on $\{u_c < t_0\} \cap \operatorname{Supp} T$. We infer

$$\nu(T, u_c) = \nu(T, \psi - c) = \nu(T, \psi) \ ,$$

hence $\nu(T, \psi) \leqslant \nu(T, \varphi)$. The equality case is obtained by reversing the roles of φ and ψ and observing that $\lim \varphi/\psi = 1/l$. □

Assume in particular that $z^k = (z_1^k, \ldots, z_n^k)$, $k = 1, 2$, are coordinate systems centered at a point $x \in X$ and let

$$\varphi_k(z) = \log |z^k| = \log\big(|z_1^k|^2 + \ldots + |z_n^k|^2\big)^{1/2}.$$

We have $\lim_{z \to x} \varphi_2(z)/\varphi_1(z) = 1$, hence $\nu(T, \varphi_1) = \nu(T, \varphi_2)$ by theorem 4.1 .

Corollary 4.2. — *The usual Lelong numbers $\nu(T, x)$ are independent of the choice of local coordinates.* □

This result had been originally proved by [Siu] with a much more delicate proof. Another interesting consequence is :

Corollary 4.3. — *On an open subset of $\mathbb{C}^n$, the Lelong numbers and Kiselman numbers are related by*

$$\nu(T, x) = \nu\big(T, x, (1, \ldots, 1)\big) \ .$$

Proof. — By definition, the number $\nu(T, x)$ is associated to the weight $\varphi(z) = \log |z - x|$ and $\nu\big(T, x, (1, \ldots, 1)\big)$ to the weight $\psi(z) = \log \max |z_j - x_j|$. It is clear that $\lim_{z \to x} \psi(z)/\varphi(z) = 1$, whence the conclusion. □

Another consequence of theorem 4.1 is that $\nu(T, x, \lambda)$ is an increasing function of each variable λ_j . Moreover, if $\lambda_1 \leqslant \ldots \leqslant \lambda_n$, we get the inequalities

$$\lambda_1^p \nu(T, x) \leqslant \nu(T, x, \lambda) \leqslant \lambda_n^p \nu(T, x) \ .$$

It can be shown ([De1]) that the stronger inequalities

$$\lambda_1 \ldots \lambda_p \nu(T, x) \leqslant \nu(T, x, \lambda) \leqslant \lambda_{n-p+1} \ldots \lambda_n \nu(T, x)$$

hold for every current T of bidimension (p,p) . By formula (3.11), this is easily checked for any p–plane of coordinates $T = [\mathbb{C}e_{i_1} \oplus \ldots \oplus \mathbb{C}e_{i_p}]$. The general case can be deduced from this special case.

Now, we assume that $T = [A]$ is the current of integration over an analytic set $A \subset X$ of pure dimension p (*cf.* P. Lelong[Le1]). The above comparison theorem will enable us to give a simple proof of P. Thie's main result [Th] : the Lelong number $\nu([A], x)$ can be interpreted as the multiplicity of the analytic set A at point x .

Let $x \in A$ be a given point and $\mathcal{I}_{A,x}$ the ideal of germs of holomorphic functions at x vanishing on A . Then, one can find local coordinates $z = (z_1, \ldots, z_n)$ on X centered at x such that there exist distinguished Weierstrass polynomials $P_j \in \mathcal{I}_{A,x}$ in the variable z_j , $p < j \leqslant n$, of the type

$$P_j(z) = z_j^{d_j} + \sum_{k=1}^{d_j} a_{j,k}(z_1, \ldots, z_{j-1}) z_j^{d_j - k} \ , \ a_{j,k} \in \mathcal{M}^k_{\mathbb{C}^{j-1},0} \tag{4.4}$$

where $\mathcal{M}_{X,x}$ is the maximal ideal of X at x .

Indeed, let us prove this property by induction on $\operatorname{codim} X = n - p$. We fix a coordinate system $(w_1, \ldots, w_n)$ by which we identify the germ (X, x) to $(\mathbb{C}^n, 0)$.

If $n - p \geqslant 1$, there exists a non zero element $f \in \mathcal{I}_{A,x}$. Let d be the smallest integer such that $f \in \mathcal{M}^d_{\mathbb{C}^n,0}$ and let $e_n \in \mathbb{C}^n$ be a non zero vector such that $\lim_{t\to 0} f(te_n)/t^d \neq 0$. Complete e_n into a basis $(\tilde{e}_1, \ldots, \tilde{e}_{n-1}, e_n)$ of $\mathbb{C}^n$ and denote by $(\tilde{z}_1, \ldots, \tilde{z}_{n-1}, z_n)$ the corresponding coordinates. The Weierstrass preparation theorem gives a factorization $f = gP$ where P is a distinguished polynomial of type (4.4) in the variable z_n and where g is an invertible holomorphic function at point x . If $n - p = 1$, the polynomial $P_n = P$ satisfies the requirements.

If $n - p \geqslant 2$, $\mathcal{O}_{A,x} = \mathcal{O}_{X,x}/\mathcal{I}_{A,x}$ is a $\mathcal{O}_{\mathbb{C}^{n-1},0} = \mathbb{C}\{\tilde{z}_1, \ldots, \tilde{z}_{n-1}\}$-module of finite type, *i.e.* the projection $\mathrm{pr} : (X, x) \approx (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ is a finite morphism of (A, x) onto a germ $(Z, 0) \subset (\mathbb{C}^{n-1}, 0)$ of dimension p . The induction hypothesis applied to $\mathcal{I}_{Z,0} = \mathcal{O}_{\mathbb{C}^{n-1},0} \cap \mathcal{I}_{A,x}$ implies the existence of a new basis $(e_1, \ldots, e_{n-1})$ of $\mathbb{C}^{n-1}$ and polynomials $P_{p+1}, \ldots, P_{n-1} \in \mathcal{I}_{Z,0}$ of type (4.4) in the coordinates $(z_1, \ldots, z_{n-1})$ associated to the basis $(e_1, \ldots, e_{n-1})$. If we choose $P_n = P$, the expected property is proved in codimension $n - p$.

For any polynomial $Q(w) = w^d + a_1 w^{d-1} + \ldots + a_d \in \mathbb{C}[w]$, the roots w of Q satisfy

$$|w| \leqslant 2 \max_{1 \leqslant k \leqslant d} |a_k|^{1/k}, \tag{4.5}$$

otherwise $Q(w)w^{-d} = 1 + a_1 w^{-1} + \ldots + a_d w^{-d}$ would have a modulus larger than $1 - (2^{-1} + \ldots + 2^{-d}) = 2^{-d}$, a contradiction. Let us denote $z = (z', z'')$ with $z' = (z_1, \ldots, z_p)$ and $z'' = (z_{p+1}, \ldots, z_n)$. As $a_{j,k} \in \mathcal{M}^k_{\mathbb{C}^{j-1},0}$, we get

$$|a_{j,k}(z_1, \ldots, z_{j-1})| = \mathrm{O}\big((|z_1| + \ldots + |z_{j-1}|)^k\big) \quad \text{if} \quad j > p \ ,$$

and we deduce from (4.4), (4.5) that $|z_j| = O(|z_1| + \ldots + |z_{j-1}|)$ on (A, x) . Therefore, the germ (A, x) is contained in a cone $|z''| \leqslant C|z'|$.

We shall use this property in order to compute the Lelong number of $[A]$ at point x . When $z \in A$ tends to x , the functions

$$\varphi(z) = \log|z| = \log(|z'|^2 + |z''|^2)^{1/2} \ , \qquad \psi(z) = \log|z'| \ .$$

are equivalent. As φ, ψ are semi-exhaustive on A , theorem 4.1 implies

$$\nu([A], x) = \nu([A], \varphi) = \nu([A], \psi).$$

Let $B' \subset \mathbb{C}^p$ the ball of center 0 and radius r' , $B'' \subset \mathbb{C}^{n-p}$ the ball of center 0 and radius $r'' = Cr'$. The inclusion of germ (A, x) in the cone $|z''| \leqslant C|z'|$ shows that for r' small enough the projection

$$\mathrm{pr} : A \cap (B' \times B'') \longrightarrow B'$$

is proper. The fibers are finite by (4.4). Hence this projection is a ramified covering with a finite sheet number m .

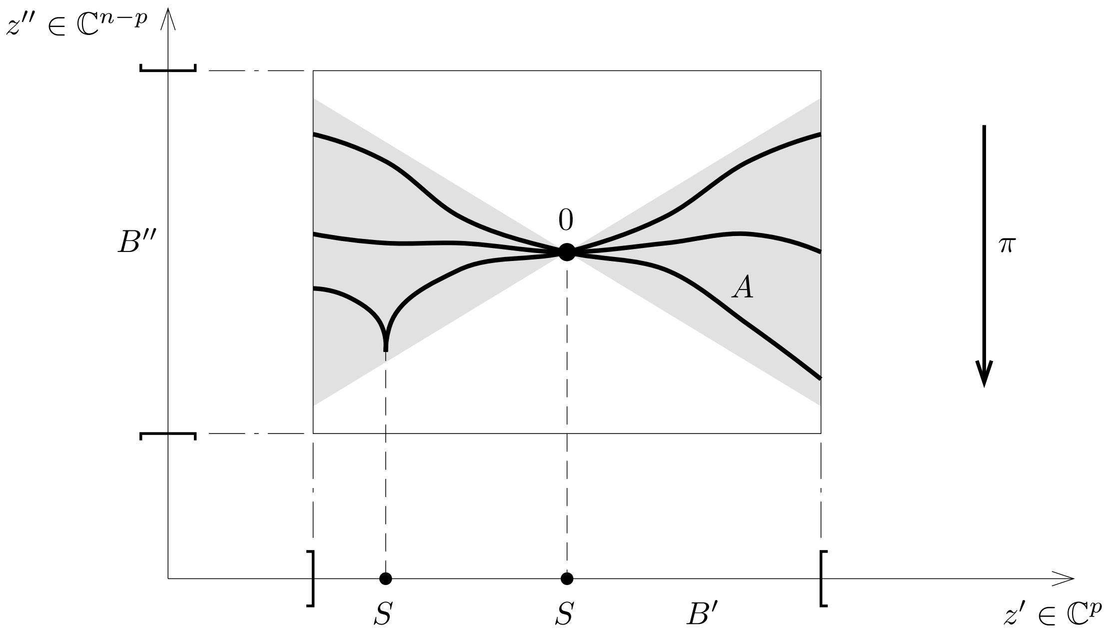

Fig. 2 Ramified covering from A to $\Delta' \subset \mathbb{C}^p$.

Let us apply formula (2.6) to ψ : for every $t < r'$ we get

$$\begin{aligned}\nu([A], \psi, \log t) &= t^{-2p} \int_{\{\psi < \log t\}} [A] \wedge \Big(\frac{1}{2} dd^c e^{2\psi}\Big)^p \\ &= t^{-2p} \int_{A \cap \{|z'| < t\}} \Big(\frac{1}{2} \mathrm{pr}^\star dd^c |z'|^2\Big)^p \\ &= m\, t^{-2p} \int_{\mathbb{C}^p \cap \{|z'| < t\}} \Big(\frac{1}{2} dd^c |z'|^2\Big)^p = m \ ,\end{aligned}$$

hence $\nu(T,\psi) = m$. Here, we used the fact that pr is actually a covering with m sheets over the complement of the ramification locus $S \subset B'$, which is of zero Lebesgue measure. We thus obtain :

THEOREM 4.6 (P. Thie [Th]). — *Let A be an analytic set of dimension p in a complex manifold of dimension p . For every point $x \in A$, there exist local coordinates*

$$z = (z', z'') , \quad z' = (z_1, \ldots, z_p) , \quad z'' = (z_{p+1}, \ldots, z_n)$$

centered at x and balls $B' \subset \mathbb{C}^p$, $B'' \subset \mathbb{C}^{n-p}$ in of radii r', r'' in these coordinates, such that $A \cap (B' \times B'')$ is contained in the cone $|z''| \leqslant (r''/r')|z'|$. The multiplicity of A at x is defined as the number m of sheets of the ramified covering map $A \cap (B' \times B'') \longrightarrow B'$. Then $\nu([A], x) = m$.

EXERCISE 4.7. — *Show that the measures*

$$\big(dd^c \log \max |z_j|^{\lambda_j}\big)^n , \qquad \Big(dd^c \log \sum |z_j|^{\lambda_j}\Big)^n$$

are equal to $c(\lambda)\delta_0$ with the same coefficient $c(\lambda)$ in both cases. When all λ_j are even integers, compute explicitly $c(\lambda)$ by means of formula (2.6). Extend the result to arbitrary rational (resp. real) numbers $\lambda_j > 0$.

5. Siu's semi-continuity theorem.

Let X, Y be complex manifolds of dimension n, n' such that X is Stein. Let $\varphi : X \times Y \longrightarrow [-\infty, +\infty[$ be a continuous psh function. We assume that φ is *semi-exhaustive* with respect to $\operatorname{Supp} T$, *i.e.* that for every compact subset $L \subset Y$ there exists $R = R(L) < 0$ such that

$$\{(x,y) \in \operatorname{Supp} T \times L \,;\, \varphi(x,y) \leqslant R\} \subset\subset X \times Y. \tag{5.1}$$

Let T be a closed positive current of bidimension (p,p) on X . For every point $y \in Y$, the function $\varphi_y(x) := \varphi(x,y)$ is semi-exhaustive on $\operatorname{Supp} T$; one can therefore associate to y a generalized Lelong number $\nu(T, \varphi_y)$.

LEMMA 5.2. — (a) *The measure $T \wedge (dd^c \varphi_{y,\geqslant t})^p$ depends continuously on y.*

(b) *For all $r_1 < r_2 < R(L)$ and $y, y_0 \in L$ one has*

$$\limsup_{y \to y_0} \ \nu(T, \varphi_y, r_1) \leqslant \nu(T, \varphi_{y_0}, r_2) .$$

(c) *The map $y \mapsto \nu(T, \varphi_y)$ is upper semi-continuous.*

Proof. — (a) We prove by induction on q that $T \wedge (dd^c \varphi_{y,\geqslant t})^q$ is weakly continuous in y . Let h be a smooth form on X . Then

$$\int_X h \wedge T \wedge (dd^c \varphi_{y,\geqslant t})^q = \int_X dd^c h \wedge T \wedge \varphi_{y,\geqslant t} (dd^c \varphi_{y,\geqslant t})^{q-1} .$$

Taking the difference for two points y, y_0 , we get

$$\int_X h \wedge T \wedge \big((dd^c\varphi_{y,\geqslant t})^q - (dd^c\varphi_{y_0,\geqslant t})^q\big)$$

$$= \int_X dd^c h \wedge T \wedge (\varphi_{y,\geqslant t} - \varphi_{y_0,\geqslant t})(dd^c\varphi_{y,\geqslant t})^{q-1}\big)$$

$$+ \int_X \varphi_{y_0,\geqslant t} dd^c h \wedge T \wedge \big((dd^c\varphi_{y,\geqslant t})^{q-1} - (dd^c\varphi_{y_0,\geqslant t})^{q-1}\big) \ .$$

The last integral tends to 0 thanks to the induction hypothesis, and the previous one thanks to the Chern-Levine Nirenberg inequalities and the uniform convergence of $\varphi_{y,\geqslant t}$ to $\varphi_{y_0,\geqslant t}$.

(b) As $\nu(T, \varphi_y, r) = \int_{B(r)} T \wedge (dd^c\varphi_{y,\geqslant t})^p$ when $t < r$, (b) follows from (a).

(c) Letting $r_1, r_2 \to -\infty$, we get easily $\limsup_{y\to y_0} \ \nu(T, \varphi_y) \leqslant \nu(T, \varphi_{y_0})$. □

As a consequence of 5.2 (c), we see that the sublevel sets

$$E_c = \{y \in Y; \nu(T, \varphi_y) \geqslant c\} \ , \ c > 0 \tag{5.3}$$

are closed. Under mild additional hypotheses, we are going to show that the sets E_c are in fact analytic subsets of Y .

DEFINITION 5.3. — *We say that a function $f(x, y)$ is locally Hölder continuous with respect to y on $X \times Y$ if every point of $X \times Y$ has a neighborhood Ω on which*

$$|f(x, y_1) - f(x, y_2)| \leqslant M|y_1 - y_2|^\gamma$$

for all $(x, y_1) \in \Omega$, $(x, y_2) \in \Omega$, for some constants $M > 0$, $\gamma \in]0, 1]$, and for suitable coordinates on Y .

THEOREM 5.4. — *Let T be a closed positive current on X and*

$$\varphi : X \times Y \longrightarrow [-\infty, +\infty[$$

a continuous psh function. Assume that φ is semi-exhaustive on $\mathrm{Supp}\, T$ *and that $e^{\varphi(x,y)}$ is locally Hölder continuous with respect to y on $X \times Y$. Then the sublevel sets*

$$E_c = \{y \in Y; \nu(T, \varphi_y) \geqslant c\}$$

are analytic subsets of Y .

This theorem, proved in [De3], can be rephrased by saying that $y \longmapsto \nu(T, \varphi_y)$ is upper semi-continuous with respect to the analytic Zariski topology. As a special case, we get the following important result of [Siu] :

COROLLARY 5.5. — *If T is a closed positive current on a manifold X , the sublevel sets $E_c = \{x \in X \, ; \, \nu(T, x) \geqslant c\}$ of the usual Lelong numbers are analytic.*

Proof. — The result is local, so we may assume that $X \subset \mathbb{C}^n$ is an open subset. Then apply theorem 5.4 with $Y = X$ and $\varphi(x, y) = \log|x - y|$. □

If X is an open subset of $\mathbb{C}^n$, the sublevel sets for Kiselman's numbers $\nu(T,x,\lambda)$ are also analytic in X . However, this result is not intrinsically significant on a manifold, because Kiselman's numbers depend on the choice of coordinates. To give another application, set $u_\lambda(z) = \log\max|z_j|^{\lambda_j}$ and set $\varphi(x,y,g) = u_\lambda\big(g(x-y)\big)$ where $x,y \in \mathbb{C}^n$ and $g \in \mathrm{Gl}(\mathbb{C}^n)$. Then $\nu(T,\varphi_{y,g})$ is the Kiselman number of T at y when the coordinates have been rotated by g . It is clear that φ is psh in (x,y,g) and semi-exhaustive with respect to x , and that e^φ is Hölder continuous with exponent $\gamma = \min\{1,\lambda_j\}$. Thus the sublevel sets

$$E_c = \{(y,g) \in X \times \mathrm{Gl}(\mathbb{C}^n)\,;\, \nu(T,\varphi_{y,g}) \geqslant c\}$$

are analytic in $X \times \mathrm{Gl}(\mathbb{C}^n)$. Theorem 5.4 can be applied more generally to weight functions of the type

$$\varphi = \max_j\ \log\Big(\sum_k |F_{j,k}|^{\lambda_{j,k}}\Big)$$

where $F_{j,k}$ are holomorphic functions on $X \times Y$ and $\gamma_{j,k}$ positive real constants; in this case e^φ is Hölder continuous of exponent $\gamma = \min\{\lambda_{j,k},1\}$.

Now let us prove theorem 5.4. As the result is local on Y , we may assume without loss of generality that Y is a ball in $\mathbb{C}^{n'}$. After addition of a constant to φ , we may also assume that there exists a compact subset $K \subset X$ such that

$$\{(x,y) \in X \times Y; \varphi(x,y) \leqslant 0\} \subset K \times Y.$$

By theorem 4.1, the Lelong numbers depend only on the asymptotic behaviour of φ near the (compact) polar set $\varphi^{-1}(-\infty) \cap (\mathrm{SuppT} \times Y)$. We can add a smooth strictly plurisubharmonic function on $X \times Y$ to make φ strictly plurisuharmonic. Then Richberg's approximation theorem for continuous psh functions shows that there exists a smooth psh function $\widetilde\varphi$ such that $\varphi \leqslant \widetilde\varphi \leqslant \varphi + 1$. We may therefore assume that φ is smooth on $(X \times Y) \setminus \varphi^{-1}(-\infty)$.

• **First step** : *construction of a local psh potential.*

Our goal is to generalize the usual construction of psh potentials associated to a closed positive current (*cf.* P. Lelong[Le2] and H. Skoda[Sk]). We replace here the usual kernel $|z-\zeta|^{-2p}$ arising from the hermitian metric of $\mathbb{C}^n$ by a kernel depending on the weight φ . Let $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ be an increasing function such that $\chi(t) = t$ for $t \leqslant -1$ and $\chi(t) = 0$ for $t \geqslant 0$. We consider the half-plane $H = \{z \in \mathbb{C}\,;\, \mathrm{Re} z < -1\}$ and associate to T the potential function V on $Y \times H$ defined by

$$(5.6)\qquad V(y,z) = -\int_{\mathrm{Re} z}^0 \nu(T,\varphi_y,t)\chi'(t)\,dt\ .$$

For every $t > \mathrm{Re}\, z$, Stokes' formula gives

$$\nu(T,\varphi_y,t) = \int_{\varphi(x,y)<t} T(x) \wedge (dd^c_x\widetilde\varphi(x,y,z))^p$$

with $\widetilde{\varphi}(x,y,z) := \max\{\varphi(x,y), \mathrm{Re} z\}$. Fubini's theorem applied to (5.6) gives

$$V(y,z) = -\int_{\substack{x\in X, \varphi(x,y)<t \\ \mathrm{Re}\, z<t<0}} T(x) \wedge (dd^c_x \widetilde{\varphi}(x,y,z))^p \ \chi'(t)dt$$
$$= \int_{x\in X} T(x) \wedge \chi(\widetilde{\varphi}(x,y,z))\, (dd^c_x \widetilde{\varphi}(x,y,z))^p \ .$$

For all $(n-1,n-1)$–form h of class C^∞ with compact support in $Y\times H$, we get

$$\langle dd^c V, h\rangle = \langle V, dd^c h\rangle = \int_{X\times Y\times H} T(x)\wedge \chi(\widetilde{\varphi}(x,y,z))(dd^c\widetilde{\varphi}(x,y,z))^p \wedge dd^c h(y,z).$$

Observe that the replacement of dd^c_x by the total differentiation $dd^c = dd^c_{x,y,z}$ does not modify the integrand, because the terms in dx , $d\overline{x}$ must have total bidegree (n,n) . The current $T(x)\wedge\chi(\widetilde{\varphi}(x,y,z))h(y,z)$ has compact support in $X\times Y\times H$. An integration by parts can thus be performed to obtain

$$\langle dd^c V, h\rangle = (2\pi)^{-p} \int_{X\times Y\times H} T(x) \wedge dd^c(\chi\circ\widetilde{\varphi}(x,y,z)) \wedge (dd^c\widetilde{\varphi}(x,y,z))^p . h(y,z).$$

On the corona $\{-1 \leqslant \varphi(x,y) \leqslant 0\}$ we have $\widetilde{\varphi}(x,y,z) = \varphi(x,y)$, whereas for $\varphi(x,y) < -1$ we get $\widetilde{\varphi} < 1$ and $\chi\circ\widetilde{\varphi} = \widetilde{\varphi}$. As $\widetilde{\varphi}$ is psh, we see that $dd^c V(y,z)$ is the sum of the positive $(1,1)$–form

$$(y,z) \longmapsto \int_{\{x\in X; \varphi(x,y)<-1\}} T(x)\wedge (dd^c_{x,y,z}\widetilde{\varphi}(x,y,z))^{p+1}$$

and of the $(1,1)$–form independent of z

$$y \longmapsto \int_{\{x\in X; -1\leqslant\varphi(x,y)\leqslant 0\}} T\wedge dd^c_{x,y}(\chi\circ\varphi)\wedge (dd^c_{x,y}\varphi)^p \ ;$$

as φ is smooth outside $\varphi^{-1}(-\infty)$, this last form has locally bounded coefficients. We obtain therefore the following result.

Theorem 5.7. — *There exists a positive psh function $\rho\in C^\infty(Y)$ such that $\rho(y)+V(y,z)$ is psh on $Y\times H$.*

To be quite complete, we must observe in addition that V is continuous on $Y\times H$ because $T\wedge (dd^c\widetilde{\varphi}_{y,z})^p$ is weakly continuous in the variables (y,z) by lemma 5.1 (a).

If we let $\mathrm{Re}\, z$ tend to $-\infty$, we see that the function

$$U_0(y) = \rho(y) + V(y,-\infty) = \rho(y) - \int_{-\infty}^0 \nu(T,\varphi_y,t)\chi'(t)dt$$

is locally psh or $\equiv -\infty$ on Y . Moreover, it is clear that $U_0(y) = -\infty$ at every point y such that $\nu(T,\varphi_y) > 0$. If Y is connected and $U_0 \not\equiv -\infty$, we already conclude that the density set $\bigcup_{c>0} E_c$ is pluripolar in Y .

• **Second step** : *application of Kiselman's minimum principle.*

KISELMAN'S MINIMUM PRINCIPLE 5.8 ([Ki1]). — *Let M be a complex manifold, $\omega \subset \mathbb{R}^n$ a convex open subset and Ω the "tube domain" $\Omega = \omega + i\mathbb{R}^n$. For every plurisubharmonic function $v(\zeta, z)$ on $M \times \Omega$ that does not depend on* $\mathrm{Im}\, z$ *, the function*

$$u(\zeta) = \inf_{z\in\Omega} v(\zeta, z)$$

is plurisubharmonic or locally $\equiv -\infty$ on M .

Proof. — Without loss of generality, we may assume that M is a ball and that $0 \in M \times \Omega$. The hypothesis implies that $v(\zeta, z)$ is convex in $x = \mathrm{Re}\, z$. Let $\tau \geqslant 0$ be a smooth strictly convex exhaustion function on ω with $\tau(0) = 0$. We approximate v by the sequence of smooth functions defined on $(1-\varepsilon)M \times \Omega$ by

$$v_\varepsilon = v \star \rho_{\varepsilon^2} + \varepsilon\tau\big((1-\varepsilon)^{-1}x\big)$$

Then $u_\varepsilon(\zeta) = \inf_{z\in\Omega} v_\varepsilon(\zeta, z)$ is increasing in ε and converges to u . We may therefore assume that v has all properties of v_ε , *i.e.* v is smooth, plurisubharmonic in (ζ, z) , strictly convex in x and $\lim_{|x|\to+\infty} v(\zeta, x) = +\infty$ for every $\zeta \in M$. Then $x \longmapsto v(\zeta, x)$ has a unique minimum point $x = g(\zeta)$, solution of the equations $\partial v/\partial x_j(x, \zeta) = 0$. As the matrix $(\partial^2 v/\partial x_j \partial x_k)$ is positive definite, the implicit function theorem shows that g is smooth. Now, if $w \longmapsto \zeta_0 + wa$, $a \in \mathbb{C}^n$, $|w| \leqslant 1$ is a complex disk Δ contained in M , there exists a holomorphic function f on the unit disk and smooth up to the boundary, whose real part solves the Dirichlet problem

$$\mathrm{Re}\, f(e^{i\theta}) = g(\zeta_0 + e^{i\theta}a) .$$

Since $v(\zeta_0 + wa, f(w))$ is subharmonic in w , we get the mean value inequality

$$v(\zeta_0, f(0)) \leqslant \frac{1}{2\pi}\int_0^{2\pi} v\big(\zeta_0 + e^{i\theta}a, f(e^{i\theta})\big)d\theta = \frac{1}{2\pi}\int_{\partial\Delta} v(\zeta, g(\zeta))d\theta .$$

The last equality holds because $\mathrm{Re}\, f = g$ on $\partial\Delta$ and $v(\zeta, z) = v(\zeta, \mathrm{Re}\, z)$ by hypothesis. As $u(\zeta_0) \leqslant v(\zeta_0, f(0))$ and $u(\zeta) = v(\zeta, g(\zeta))$, we see that u satisfies the mean value inequality. □

Let $a \geqslant 0$ be arbitrary. The function

$$Y \times H \ni (y, z) \longmapsto \rho(y) + V(y, z) - a\mathrm{Re} z$$

is psh and independent of $\mathrm{Im}\, z$. By 5.8, the Legendre transform

$$U_a(y) = \inf_{r<-1} [\rho(y) + V(y, r) - ar]$$

is locally psh or $\equiv -\infty$ on Y .

LEMMA 5.9. — *Let $y_0 \in Y$ be a given point.*

(a) *If $a > \nu(T, \varphi_{y_0})$, then U_a is bounded below on a neighborhood of y_0 .*

(b) *If $a < \nu(T, \varphi_{y_0})$, then $U_a(y_0) = -\infty$.*

Proof. — By definition of V (*cf.* (5.6)) we have

$$V(y,r) \leqslant -\nu(T,\varphi_y,r)\int_r^0 \chi'(t)dt = r\nu(T,\varphi_y,r) \leqslant r\nu(T,\varphi_y)\ . \tag{5.10}$$

Then clearly $U_a(y_0) = -\infty$ if $a < \nu(T,\varphi_{y_0})$. On the other hand, if $\nu(T,\varphi_{y_0}) < a$, there exists $t_0 < 0$ such that $\nu(T,\varphi_{y_0},t_0) < a$. Fix $r_0 < t_0$. The semi-continuity property 5.2 (b) shows that there exists a neighborhood ω of y_0 such that $\sup_{y\in\omega}\ \nu(T,\varphi_y,r_0) < a$. For all $y\in\omega$, we get

$$V(y,r) \geqslant -C - a\int_r^{r_0} \chi'(t)dt = -C + a(r-r_0)\ ,$$

and this implies $U_a(y) \geqslant -C - ar_0$. □

THEOREM 5.11. — *If Y is connected and if $E_c \neq Y$, then E_c is a closed complete pluripolar subset of Y , i.e. there exists a continuous psh function $w : Y \longrightarrow [-\infty,+\infty[$ such that $E_c = w^{-1}(-\infty)$.*

Proof. — We first observe that the family (U_a) is increasing in a , that $U_a = -\infty$ on E_c for all $a < c$ and that $\sup_{a<c} U_a(y) > -\infty$ if $y \in Y\setminus E_c$ (apply lemma 5.9). For any integer $k \geqslant 1$, let $w_k \in C^\infty(Y)$ be a psh regularization of $U_{c-1/k}$ such that $w_k \geqslant U_{c-1/k}$ on Y and $w_k \leqslant -2^k$ on $E_c\cap Y_k$. Then lemma 5.10 (a) shows that the family (w_k) is uniformly bounded below on every compact subset of $Y\setminus E_c$. We can also choose w_k uniformly bounded above on every compact subset of Y because $U_{c-1/k} \leqslant U_c$. The function

$$w = \sum_{k=1}^{+\infty} 2^{-k} w_k$$

satifies our requirements. □

• **Third step** : *estimation of the singularities of the potentials U_a.*

LEMMA 5.12. — *Let $y_0 \in Y$ be a given point, L a compact neighborhood of y_0 , $K \subset X$ a compact subset and r_0 a real number < -1 such that*

$$\{(x,y)\in X\times L; \varphi(x,y) \leqslant r_0\} \subset K\times L\ .$$

Assume that $e^\varphi(x,y)$ is locally Hölder continuous in y and that

$$|f(x,y_1) - f(x,y_2)| \leqslant M|y_1-y_2|^\gamma$$

for all $(x,y_1,y_2)\in K\times L\times L$. Then, for all $\varepsilon\in]0,1[$, there exists a real number $\eta(\varepsilon) > 0$ such that all $y \in Y$ with $|y-y_0| < \eta(\varepsilon)$ satisfy

$$U_a(y) \leqslant \rho(y) + \big((1-\varepsilon)^p\nu(T,\varphi_{y_0}) - a\big)\Big(\gamma\log|y-y_0| + \log\frac{2eM}{\varepsilon}\Big)\ .$$

Proof. — First, we try to estimate $\nu(T,\varphi_y,r)$ when $y\in L$ is near y_0 . Set

$$\begin{cases} \psi(x) = (1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 & \text{if} \quad \varphi_{y_0}(x) \leqslant r-1 \\ \psi(x) = \max\big(\varphi_y(x), (1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2\big) & \text{if} \quad r-1 \leqslant \varphi_{y_0}(x) \leqslant r \\ \psi(x) = \varphi_y(x) & \text{if} \quad r \leqslant \varphi_{y_0}(x) \leqslant r_0 \end{cases}$$

and verify that this definition is coherent when $|y-y_0|$ is small enough. By hypothesis

$$|e^{\varphi_y(x)} - e^{\varphi_{y_0}(x)}| \leqslant M|y-y_0|^\gamma.$$

This inequality implies

$$\varphi_y(x) \leqslant \varphi_{y_0}(x) + \log\big(1 + M|y-y_0|^\gamma e^{-\varphi_{y_0}(x)}\big)$$
$$\varphi_y(x) \geqslant \varphi_{y_0}(x) + \log\big(1 - M|y-y_0|^\gamma e^{-\varphi_{y_0}(x)}\big).$$

In particular, for $\varphi_{y_0}(x) = r$, we have $(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 = r - \varepsilon/2$, thus

$$\varphi_y(x) \geqslant r + \log(1 - M|y-y_0|^\gamma e^{-r}) \ .$$

Similarly, for $\varphi_{y_0}(x) = r-1$, we have $(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 = r - 1 + \varepsilon/2$, thus

$$\varphi_y(x) \leqslant r - 1 + \log(1 + M|y-y_0|^\gamma e^{1-r}).$$

The definition of ψ is thus coherent as soon as $M|y-y_0|^\gamma e^{1-r} \leqslant \varepsilon/2$, *i.e.*

$$\gamma \log|y-y_0| + \log\frac{2eM}{\varepsilon} \leqslant r \ .$$

In this case ψ coincides with φ_y on a neighborhood of $\{\psi = r\}$, and with

$$(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2$$

on a neighborhood of the polar set $\psi^{-1}(-\infty)$. By Stokes' formula applied to $\nu(T,\psi,r)$, we infer

$$\nu(T,\varphi_y,r) = \nu(T,\psi,r) \geqslant \nu(T,\psi) = (1-\varepsilon)^p \nu(T,\varphi_{y_0}).$$

From (5.10) we get $V(y,r) \leqslant r\nu(T,\varphi_y,r)$, hence

$$U_a(y) \leqslant \rho(y) + V(y,r) - ar \leqslant \rho(y) + r\big(\nu(T,\varphi_y,r) - a\big) \ ,$$
$$U_a(y) \leqslant \rho(y) + r\big((1-\varepsilon)^p \nu(T,\varphi_{y_0}) - a\big) \ . \tag{5.13}$$

Suppose $\gamma\log|y-y_0| + \log(2eM/\varepsilon) \leqslant r_0$, *i.e.* $|y-y_0| \leqslant (\varepsilon/2eM)^{1/\gamma} e^{r_0/\gamma}$; one can then choose $r = \gamma\log|y-y_0| + \log(2eM/\varepsilon)$, and by (5.13) this yields the inequality asserted in theorem 5.12 . □

• **Fourth step** : *application of Hörmander's L^2 estimates.*

The end of the proof rests upon the following crucial result, known as the Hörmander-Bombieri-Skoda theorem (*cf.* [Hö] , [Bo] and [Sk]).

THEOREM 5.14. — *Let u be a psh function on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every point $z_0 \in \Omega$ such that e^{-u} is integrable in a neighborhood of z_0 , there exists a holomorphic function F on Ω such that $F(z_0) = 1$ and*

$$\int_\Omega \frac{|F(z)|^2 e^{-u(z)}}{(1+|z|^2)^{n+\varepsilon}} d\lambda(z) < +\infty \ .$$

COROLLARY 5.15. — *Let u be a psh function on a complex manifold Y. The set of points in a neighborhood of which e^{-u} is not integrable is an analytic subset of Y.*

Proof. — The result is local, so we may assume that Y is a ball in $\mathbb{C}^n$. Then the set of non integrability points of e^{-u} is the intersection of all hypersurfaces $F^{-1}(0)$ defined by the holomorphic functions F such that $\int_Y |F|^2 e^{-u} d\lambda < +\infty$. Indeed F must vanish at any non integrability point, and on the other hand theorem 5.14 shows that one can choose $F(z_0) = 1$ at any integrability point z_0. $\square$

The main idea in what follows is due to Kiselman [Ki2]. For all real numbers $a, b > 0$, we let $Z_{a,b}$ be the set of points in a neighborhood of which $\exp(-U_a/b)$ is not integrable. Then $Z_{a,b}$ is analytic, and as the family (U_a) is increasing in a, we have $Z_{a',b'} \supset Z_{a'',b''}$ if $a' \leqslant a''$, $b' \leqslant b''$.

Let $y_0 \in Y$ be a given point. If $y_0 \notin E_c$, then $\nu(T, \varphi_{y_0}) < c$ by definition of E_c. Choose a such that $\nu(T, \varphi_{y_0}) < a < c$. Lemma 5.9 (a) implies that U_a is bounded below in a neighborhood of y_0, thus $\exp(-U_a/b)$ is integrable and $y_0 \notin Z_{a,b}$ for all $b > 0$.

On the other hand, if $y_0 \in E_c$ and if $a < c$, then lemma 5.12 implies for all $\varepsilon > 0$ that

$$U_a(y) \leqslant (1-\varepsilon)(c-a)\gamma \log|y - y_0| + C(\varepsilon)$$

on a neighborhood of y_0. Hence $\exp(-U_a/b)$ is non integrable at y_0 as soon as $b < (c-a)\gamma/2n'$, where $n' = \dim Y$. We obtain therefore

$$E_c = \bigcap_{\substack{a<c \\ b<(c-a)\gamma/2n'}} Z_{a,b}.$$

This proves that E_c is an analytic subset of Y. $\square$

EXERCISE 5.16. — *Combine El Mir's extension theorem and Siu's theorem in order to get the following result : let P be a complete pluripolar set in a complex manifold X and A an analytic subset of $X \setminus P$. If A has finite mass near every point of P, then $\overline{A}$ is analytic in X.*

This result has been first obtained by E. Bishop when P is an analytic subset of X. The general case is due to Siu.

6. Monge-Ampère capacities and quasi-continuity.

Let Ω be a bounded open subset of $\mathbb{C}^n$. We denote by $P(\Omega)$ the set of all psh functions that are $\not\equiv -\infty$ on each connected component of Ω. The following fundamental definition has been introduced in [B–T2].

DÉFINITION 6.1. — *For every Borel subset $E \subset \Omega$, we set*

$$c(E, \Omega) = \sup\Big\{ \int_E (dd^c u)^n \ ;\ u \in P(\Omega),\ 0 \leqslant u \leqslant 1 \Big\}.$$

The Chern-Levine-Nirenberg inequalities show that $c(E, \Omega) < +\infty$ as soon as $E \subset\subset \Omega$. If $\Omega \subset B(z_0, R)$, we can choose $u(z) = R^{-2}|z - z_0|^2$ and we obtain therefore

$$c(E,\Omega) \geqslant \frac{2^n n!}{\pi^n R^{2n}} \lambda(E)$$

where λ is the Lebesgue measure. From the standard properties of measures (countable additivity, monotone convergence theorem), we immediately deduce :

PROPERTIES 6.2. — *Let $\Omega \subset\subset \mathbb{C}^n$ and $E, E_1, E_2, \ldots$ Borel subsets of Ω. Denote $c(E) = c(E,\Omega)$ for simplicity.*

(a) *If $E_1 \subset E_2$ then $c(E_1) \leqslant c(E_2)$.*

(b) *$c(\bigcup_{j\geqslant 1} E_j) \leqslant \sum_{j\geqslant 1} c(E_j)$.*

(c) *If $E_1 \subset E_2 \subset \ldots$ then $c(\bigcup E_j) \leqslant \lim_{j\to+\infty} c(E_j)$.* □

A set function $c : E \mapsto c(E)$ defined on all Borel subsets $E \subset \Omega$ with values in $[0,+\infty]$ is called a capacity, resp. a subadditive capacity, if c satisfies the axioms (a,c), resp. (a,b,c) and $c(\emptyset) = 0$. The capacity is said to be inner regular if all Borel subsets satisfy

(d) $$c(E) = \sup_{K \text{ compact} \subset E} c(K).$$

Similarly, c is said to be outer regular if all Borel subsets E satisfy

(e) $$c(E) = \inf_{G \text{ open } \supset E} c(G)$$

Example 6.3. — If (μ_α) is a family of positive Radon measures on Ω, then $c(E) = \sup \mu_\alpha(E)$ is a subadditive capacity. In general, c does not satisfy the additivity property

$$E_1, E_2 \text{ disjoint } \Rightarrow c(E_1 \cup E_2) = c(E_1) + c(E_2) \ ;$$

for a specific example, consider the measures $\mu_1 = \delta_0$, $\mu_2 = d\lambda$ on $\mathbb{R}$ and the sets $E_1 = \{0\}$, $E_2 =]0,1]$; then

$$c(\{0\}) = 1 \ , \quad c(]0,1]) = 1 \ , \quad c([0,1]) = 1.$$

Moreover, the capacity $c = \sup \mu_\alpha$ is inner regular because all Radon measures on a separable locally compact space are inner regular. However, c need not be outer regular : for instance, take $d\mu_\alpha(x) = \alpha^{-1}\rho(x/\alpha)dx$ on $\mathbb{R}$, $\alpha > 0$, where $\rho \geqslant 0$ is a function with support in $[-1,1]$ and $\int_{\mathbb{R}} \rho(x)dx = 1$; then $c(\{0\}) = 0$ but every neighborhood of 0 has capacity 1. □

The capacity $c(\bullet,\Omega)$ defined in 6.1 is called the relative Monge-Ampère capacity on Ω. It is associated to the family of measures $\mu_u = (dd^c u)^n$, $u \in P(\Omega)$, $0 \leqslant u \leqslant 1$. In particular $c(\bullet,\Omega)$ is inner regular. It is also outer regular, but this fact is non trivial and will be proved only in § 9.

When c is a capacity and $E \subset \Omega$ an arbitrary subset, we define the inner capacity $c_\star(E)$ and outer capacity $c^\star(E)$ by

(6.4′) $$c_\star(E) = \sup_{K \text{ compact} \subset E} c(K),$$

$$(6.4'') \qquad c^\star(E) = \inf_{G \text{ open} \supset E} c(G),$$

A set $E \subset \Omega$ is said to be c-capacitable if $c_\star(E) = c^\star(E)$. By definition, c is thus regular if and only if all Borel subsets are c-capacitable.

Now we compare capacities associated to different open sets Ω.

THEOREM 6.5. — *Let $\Omega_1 \subset \Omega_2 \subset\subset \mathbb{C}^n$. Then*

(a) *$c(E, \Omega_1) \geqslant c(E, \Omega_2)$ for all Borel subsets $E \subset \Omega_1$.*

(b) *Let $\omega \subset\subset \Omega_1$. There exists a constant $A > 0$ such that $c(E, \Omega_1) \leqslant A\, c(E, \Omega_2)$ for all Borel subsets $E \subset \omega$.*

Proof. — Since every psh function $u \in P(\Omega_2)$ with $0 \leqslant u \leqslant 1$ induces a psh function in $P(\Omega_1)$ with the same property, (a) is clear.

(b) Use a finite covering of $\overline{\omega}$ by open balls contained in Ω_1 and cut E into pieces. The proof is then reduced to the case when $\omega \subset\subset \Omega_1$ are concentric balls, say $\Omega_1 = B(0, r)$ and $\omega = B(0, r - \varepsilon)$. For every $u \in P(\Omega_1)$ such that $0 \leqslant u \leqslant 1$, set

$$\tilde{u}(z) = \begin{cases} \max\{u(z), \lambda(|z|^2 - r^2) + 2\} & \text{on} \quad \Omega_1, \\ \lambda(|z|^2 - r^2) + 2 & \text{on} \quad \Omega_2 \setminus \Omega_1. \end{cases}$$

Choose λ so large that $\lambda((r - \varepsilon)^2 - r^2) \leqslant -2$. Then $\tilde{u} \in P(\Omega_2)$ and $\tilde{u} = u$ on ω. Moreover $0 \leqslant \tilde{u} \leqslant M$ for some constant $M > 0$, thus for $E \subset \omega$ we get

$$\int_E (dd^c u)^n = \int_E (dd^c \tilde{u})^n \leqslant M^n c(E, \Omega_2).$$

Therefore $c(E, \Omega_1) \leqslant M^n c(E, \Omega_2)$. $\square$

As a consequence of theorem 6.5, it is in general harmless to shrink the domain Ω when one wants to estimate capacities.

THEOREM 6.6. — *Let K be a compact subset of Ω and $\omega \subset\subset \Omega$ a neighborhood of K. There is a constant $A > 0$ such that for every $v \in P(\Omega)$*

$$c(K \cap \{v < -m\}, \Omega) \leqslant A \|v\|_{L^1(\overline{\omega})} \cdot \frac{1}{m}.$$

Proof. — For every $u \in P(\Omega)$, $0 \leqslant u \leqslant 1$, theorem 1.8 implies

$$\int_{K \cap \{v < -m\}} (dd^c u)^n \leqslant \frac{1}{m} \int_K |v| (dd^c u)^n \leqslant \frac{1}{m} C_{K, \overline{\omega}} \|v\|_{L^1(\overline{\omega})}. \ \square$$

DEFINITION 6.7. — *A set $P \subset \Omega$ is said to be pluripolar in Ω if there exists $v \in P(\Omega)$ such that $P \subset \{v = -\infty\}$.*

COROLLARY 6.8. — *If P is pluripolar in Ω, then*

$$c^\star(P, \Omega) = 0.$$

Proof. — Write $P \subset \{v = -\infty\}$ and $\Omega = \bigcup_{j \geqslant 1} \Omega_j$ with $\Omega_j \subset\subset \Omega$. Theorem 6.4 shows that there is an open set $G_j = \Omega_j \cap \{v < -m_j\}$ such that $c(G_j, \Omega) < \varepsilon 2^{-j}$. Then $\{v = -\infty\} \subset G = \bigcup G_j$ and $c(G, \Omega) < \varepsilon$. □

THEOREM 6.9. — *Let $v_k,\ v \in P(\Omega)$ be locally bounded psh functions such that (v_k) decreases to v. Then for every compact subset $K \subset \Omega$ and every $\delta > 0$*

$$\lim_{k \to +\infty} c(K \cap \{v_k > v + \delta\}, \Omega) = 0.$$

Proof. — It is sufficient to show that

$$\sup_{u \in P(\Omega), 0 \leqslant u \leqslant 1} \int_K (v_k - v)(dd^c u)^n$$

tends to 0, because this supremum is larger than $\delta.c(K \cap \{v_k > v + \delta\}, \Omega)$. By cutting K into pieces and modifying v, v_k, u with the max construction, we may assume that $K \subset \Omega = B(0, r)$ are concentric balls and that all functions v, v_k, u are equal to $\lambda(|z|^2 - r^2) + 2)$ on the corona $\Omega \setminus \omega$, $\omega = B(0, r - \varepsilon)$. An integration by parts yields

$$\int_\Omega (v_k - v)(dd^c u)^n = -\int_\Omega d(v_k - v) \wedge d^c u \wedge (dd^c u)^{n-1}.$$

The Cauchy-Schwarz inequality implies that this integral is bounded by

$$A\Big(\int_\Omega d(v_k - v) \wedge d^c(v_k - v) \wedge (dd^c u)^{n-1}\Big)^{1/2}$$

where

$$A^2 = \int_\Omega du \wedge d^c u \wedge (dd^c u)^{n-1} \leqslant \int_\Omega dd^c(u^2) \wedge (dd^c u)^{n-1}$$

and this last integral depends only on the constants λ, r. Another integration by parts yields

$$\begin{aligned}\int_\Omega d(v_k - v) \wedge d^c(v_k - v) \wedge (dd^c u)^{n-1} &= -\int_\Omega (v_k - v) dd^c(v_k - v) \wedge (dd^c u)^{n-1} \\ &\leqslant \int_\Omega (v_k - v) dd^c v \wedge (dd^c u)^{n-1}.\end{aligned}$$

We have thus replaced one factor $dd^c u$ by $dd^c v$ in the integral. Repeating the argument $(n-1)$ times we get

$$\int_\Omega (v_k - v)(dd^c u)^n \leqslant C\Big(\int_\Omega (v_k - v)(dd^c v)^n\Big)^{1/2^n}$$

and the last integral converges to 0 by the bounded convergence theorem. □

THEOREM 6.10 (Quasi-consinuity of psh functions). — *Let $\Omega \subset\subset \mathbb{C}^n$ and $v \in P(\Omega)$. Then for each $\varepsilon > 0$, there is an open subset G of Ω such that $c(G, \Omega) < \varepsilon$ and v is continuous on $\Omega \setminus G$.*

Proof. — Let $\omega \subset\subset \Omega$ be arbitrary. We first show that there exists $G \subset \omega$ such that $c(G, \Omega) < \varepsilon$ and v continuous on $\omega \setminus G$. For $m > 0$ large enough, the set $G_0 = \omega \cap \{v < -m\}$ has capacity $< \varepsilon/2$ by theorem 6.6. On $\omega \setminus G_0$ we have $v \geqslant -m$, thus $\tilde{v} = \max\{v, -m\}$ coincides with v there and $\tilde{v}$ is locally bounded on Ω. Let (v_k) be a sequence of smooth psh functions which decrease to $\tilde{v}$ in a neighborhood of $\overline{\omega}$. For each $\ell \geqslant 1$, theorem 6.9 shows that there is an index $k(\ell)$ and an open set

$$G_{k(\ell)} = \omega \cap \{v_{k(\ell)} > \tilde{v} + 1/\ell\}$$

such that $c(G_{k(\ell)}, \Omega) < \varepsilon 2^{-\ell-1}$. Then $G = G_0 \cup \bigcup G_{k(\ell)}$ has capacity $c(G, \Omega) < \varepsilon$ by 6.2 (b) and $(v_{k(\ell)})$ converges uniformly to $\tilde{v} = v$ on $\omega \setminus G$. Hence v is continuous on $\omega \setminus G$. Now, take an increasing sequence $\omega_1 \subset \omega_2 \subset \ldots$ with $\bigcup \omega_j = \Omega$ and $G_j \subset \omega_j$ such that $c(G_j, \Omega) < \varepsilon 2^{-j}$ and v continuous on $\omega_j \setminus G_j$. The set $G = \bigcup G_j$ satisfies all requirements. $\square$

As an example of application, we prove an interesting inequality for the Monge-Ampère operator.

Proposition 6.11. — *Let u, v be locally bounded psh functions on Ω. Then we have an inequality of measures*

$$(dd^c \max\{u, v\})^n \geqslant \mathbf{1}_{\{u \geqslant v\}} (dd^c u)^n + \mathbf{1}_{\{u < v\}} (dd^c v)^n.$$

Proof. — It is enough to check that

$$\int_K (dd^c \max\{u, v\})^n \geqslant \int_K (dd^c u)^n$$

for every compact set $K \subset \{u \geqslant v\}$; the other term is then obtained by reversing the roles of u and v. By shrinking Ω, adding and multiplying with constants, we may assume that $0 \leqslant u, v \leqslant 1$ and that u, v have regularizations $u_\varepsilon, v_\varepsilon$ with $0 \leqslant u_\varepsilon,\ v_\varepsilon \leqslant 1$ on Ω. Let $G \subset \Omega$ be an open set of small capacity such that u, v are continuous on $\Omega \setminus G$. By Dini's lemma, $u_\varepsilon, v_\varepsilon$ converge uniformly to u, v on $\Omega \setminus G$. Hence for any $\delta > 0$, we can find an arbitrarily small neighborhood L of K such that $u_\varepsilon > v_\varepsilon - \delta$ on $L \setminus G$ for ε small enough. As $(dd^c u_\varepsilon)^n$ converges weakly to $(dd^c u)^n$ on Ω, we get

$$\begin{aligned}
\int_K (dd^c u)^n &\leqslant \liminf_{\varepsilon \to 0} \int_L (dd^c u_\varepsilon)^n \\
&\leqslant \liminf_{\varepsilon \to 0} \Big(\int_G (dd^c u_\varepsilon)^n + \int_{L \setminus G} (dd^c u_\varepsilon)^n \Big) \\
&\leqslant c(G, \Omega) + \liminf_{\varepsilon \to 0} \int_{L \setminus G} (dd^c \max\{u_\varepsilon + \delta, v_\varepsilon\})^n.
\end{aligned}$$

Observe that $\max\{u_\varepsilon + \delta, v_\varepsilon\}$ coincides with $u_\varepsilon + \delta$ on a neighborhood of $L \setminus G$. By weak convergence again, we get

$$\int_K (dd^c u)^n \leqslant c(G, \Omega) + \int_{L \setminus G} (dd^c \max\{u + \delta, v\})^n.$$

By taking L very close to K and $c(G,\Omega)$ arbitrarily small, this implies

$$\int_K (dd^c u)^n \leqslant \int_K (dd^c \max\{u+\delta, v\})^n$$

and the desired conclusion follows by letting δ tend to 0. $\square$

7. Extremal functions and negligible sets.

Let (u_α) be a family of upper semi-continuous functions on Ω which is locally bounded from above. Then the upper envelope

$$u = \sup_\alpha u_\alpha(z)$$

need not be upper semi-continuous, so we consider its "upper semi-continuous regularization"

$$u^\star(z) = \lim_{\varepsilon\to 0} \sup_{B(z,\varepsilon)} u \geqslant u(z).$$

It is easy to check that $u^\star$ is upper semi-continuous and that $u^\star$ is the smallest upper semi-continuous function $\geqslant u$.

Let $B(z_j,\varepsilon_j)$ be a denumerable basis of the topology of Ω. For each j, let (z_{jk}) be a sequence in $B(z_j,\varepsilon_j)$ such that

$$\sup_k u(z_{jk}) = \sup_{B(z_j,\varepsilon_j)} u,$$

and for each (j,k), let $\alpha(j,k,\ell)$ be a sequence of indices α such that $u(z_{jk}) = \sup_\ell u_{\alpha(j,k,\ell)}(z_{jk})$. Set

$$v = \sup_{j,k,\ell} u_{\alpha(j,k,\ell)}.$$

Then $v \leqslant u$ and $v^\star \leqslant u^\star$. On the other hand

$$\sup_{B(z_j,\varepsilon_j)} v \geqslant \sup_k v(z_{jk}) \geqslant \sup_{k,\ell} u_{\alpha(j,k,\ell)}(z_{jk}) = \sup_k u(z_{jk}) = \sup_{B(z_j,\varepsilon_j)} u.$$

As every ball $B(z,\varepsilon)$ is a union of balls $B(z_j,\varepsilon_j)$, we easily conclude that $v^\star \geqslant u^\star$, hence $v^\star = u^\star$. Therefore :

Choquet's lemma 7.1. — *Every family (u_α) has a denumerable subfamily $(u_{\alpha(j)})$ whose upper envelope v satisfies $v \leqslant u \leqslant u^\star = v^\star$.*

Proposition 7.2. — *If all u_α are psh, then $u^\star$ is psh and equal almost everywhere to u.*

Proof. — By Choquet's lemma one may assume that (u_α) is denumerable. Then $u = \sup u_\alpha$ is a Borel function. For every $(z_0,a) \in \Omega\times\mathbb{C}^n$, u_α satisfies the mean value inequality on circles, hence

$$u(z_0) = \sup u_\alpha(z_0) \leqslant \sup \int_0^{2\pi} u_\alpha(z_0+ae^{i\theta})\frac{d\theta}{2\pi} \leqslant \int_0^{2\pi} u(z_0+ae^{i\theta})\frac{d\theta}{2\pi}.$$

It follows easily that each convolution $u\star\rho_\varepsilon$ also satisfies the mean value inequality, thus $u \star \rho_\varepsilon$ is smooth and psh. Therefore $(u \star \rho_\varepsilon) \star \rho_\eta$ is increasing in η. Letting ε tends to 0, we see that $u \star \rho_\eta$ in increasing in η. Since $u \star \rho_\varepsilon$ is smooth and $u \star \rho_\varepsilon \geqslant u$ by the mean value inequality, we also have $u \star \rho_\varepsilon \geqslant u^\star$. By the upper semi-continuity we get $\lim_{\varepsilon\to 0} u \star \rho_\varepsilon = u^\star$, in particular $u^\star$ is psh and coincides almost everywhere with the L^1_{loc} limit u. $\square$

A set of the form

$$N = \{z \in \Omega\ ;\ u(z) < u^\star(z)\} \tag{7.3}$$

is called negligible. Every pluripolar set $P = \{v = -\infty\}$ is negligible : let $w \in P(\Omega) \cap C^\infty(\Omega)$ such that $w \geqslant v$ and $u_\alpha = (1-\alpha)v + \alpha w$, $\alpha \in]0,1[$. Then u_α is increasing in α and $u = \sup_\alpha u_\alpha$ satisfies

$$\begin{aligned} u &= -\infty \qquad && \text{on } \{v = -\infty\},\\ u &= w && \text{on } \{v > -\infty\}. \end{aligned}$$

Hence $u^\star = w$ and $\{u < u^\star\} = \{v = -\infty\}$.

To study further properties of the capacity, we consider the extremal function associated to a subset E of Ω :

$$u_E(z) = \sup\{v(z); v \in P(\Omega),\ v \leqslant -1 \text{ on } E,\ v \leqslant 0 \text{ on } \Omega\}. \tag{7.4}$$

Proposition 7.2 implies $u_E^\star \in P(\Omega)$ and $-1 \leqslant u_E^\star \leqslant 0$. In the sequel, we need the fundamental result of Bedford-Taylor [B–T1] on the solution of the Dirichlet problem for complex Monge-Ampère equations.

THEOREM 7.5. — *Let $\Omega \subset\subset \mathbb{C}^n$ be a smooth strongly pseudoconvex domain and let $f \in C^0(\partial\Omega)$ be a continuous function on the boundary. Then*

$$u(z) = \sup\{v(z)\ ;\ v \in P(\Omega) \cap C^0(\overline{\Omega}),\ v \leqslant f \text{ on } \partial\Omega\}$$

is continuous on $\overline{\Omega}$ and psh on Ω, and solves the Dirichlet problem

$$(dd^c u)^n = 0 \text{ on } \Omega\ , \quad u = f \text{ on } \partial\Omega.$$

Sketch of proof. — Let $g \in C^2(\overline{\Omega})$ be an approximate extension of f such that $|g - f| < \varepsilon$ on $\partial\Omega$ and let $\psi < 0$ be a smooth strongly psh exhaustion of Ω. Then $g - \varepsilon + A\psi$ is psh for $A > 0$ large enough and $g - \varepsilon + A\psi = g - \varepsilon \leqslant f$ on $\partial\Omega$, hence $g - \varepsilon + A\psi \leqslant u$ on $\overline{\Omega}$. Similarly, for all $v \in P(\Omega) \cap C^0(\overline{\Omega})$ with $v \leqslant f$ on $\partial\Omega$, the function $v - g - \varepsilon + A\psi$ equals $v - g - \varepsilon \leqslant 0$ on $\partial\Omega$ and is psh for A large, thus $v - g - \varepsilon + A\psi \leqslant 0$ on $\overline{\Omega}$ by the maximum principle. Therefore we get $u \leqslant g + \varepsilon - A\psi$; as ε tends to 0, we see that $u = f$ on $\partial\Omega$ and that u is continuous at every point of $\partial\Omega$. Since $g + \varepsilon + A\psi = g + \varepsilon > f$ on $\partial\Omega$, there exists $\delta > 0$ such that $u^\star < g + \varepsilon + A\psi$ on $\Omega \setminus \Omega_\delta$, where $\Omega_\delta = \{\psi < -\delta\}$. For $\eta > 0$ small enough, the regularizations of $u^\star$ satisfy $u^\star \star \rho_\eta < g + \varepsilon + A\psi$ on a neighborhood of $\partial\Omega_\delta$. Then we let

$$v_\varepsilon = \begin{cases} \max\{u^\star \star \rho_\eta - 2\varepsilon, g - \varepsilon + A\psi\} & \text{on } \ \Omega_\delta \\ g - \varepsilon + A\psi & \text{on } \ \overline{\Omega} \setminus \Omega_\delta. \end{cases}$$

It is clear that v_ε is psh and continuous on $\overline{\Omega}$ and that $v_\varepsilon = g-\varepsilon \leqslant f$ on $\partial\Omega$, hence $v_\varepsilon \leqslant u$ on $\overline{\Omega}$. We obtain therefore $u^\star \star \rho_\eta \leqslant u + 2\varepsilon$ on Ω_δ. As $u \leqslant u^\star \leqslant u^\star \star \rho_\eta$, we see that $u^\star \star \rho_\eta$ converges uniformly to u on every compact subset of Ω. Hence u is psh and continuous on $\overline{\Omega}$.

We shall complete the proof under the following additional assumptions : $f \in C^2(\partial\Omega)$ and $u \in C^2(\Omega)$; the general case is difficult and rather technical (*cf.* [B–T1]). The plurisubharmonicity of u implies $\det(\partial^2 u/\partial z_j \partial\overline{z}_k) \geqslant 0$. If we had a strict inequality at one point $z_0 \in \Omega$, say $z_0 = 0$ for simplicity, the Taylor expansion of u at z_0 would give

$$u(z) = \mathrm{Re}\, P(z) + \sum c_{jk} z_j \overline{z}_k + o(|z|^2)$$

where P is a holomorphic polynomial of degree 2 and (c_{jk}) a positive definite hermitian matrix. Hence we would have $u > \mathrm{Re}\, P + \varepsilon$ on a small sphere $S(0,r)$ with $\overline{B}(0,r) \subset \Omega$. The function

$$v = \begin{cases} \max\{u, \mathrm{Re}\, P + \varepsilon\} & \text{on } \ B(0,r) \\ u & \text{on } \ \overline{\Omega} \setminus B(0,r) \end{cases}$$

is then continuous on $\overline{\Omega}$ and psh, and satisfies

$$v = u \leqslant f \quad \text{on } \partial\Omega.$$

By the definition of u, we thus have $u \geqslant v$ on $\overline{\Omega}$. This is a contradiction, because

$$v(0) > \mathrm{Re}\, P(0) = u(0). \ \square$$

Corollary 7.6. — *Fix a ball $\overline{B}(z_0,r) \subset \Omega$ and let $f \in P(\Omega)$ be locally bounded. There exists a function $\tilde{f} \in P(\Omega)$ such that $\tilde{f} \geqslant f$ on Ω, $\tilde{f} = f$ on $\Omega \setminus B(0,r)$ and $(dd^c\tilde{f})^n = 0$ on $B(z_0,r)$. Moreover, for $f_1 \leqslant f_2$ we have $\tilde{f}_1 \leqslant \tilde{f}_2$.*

Proof. — Assume first that $f \in C^0(\Omega)$. By theorem 7.5 applied on $B(z_0,r)$, there exists u psh and continuous on $\overline{B}(z_0,r)$ with $u = f$ on $S(z_0,r)$ and $(dd^c u)^n = 0$ on $B(z_0,r)$. Set

$$\tilde{f} = \begin{cases} u & \text{on } \ B(z_0,r) \\ f & \text{on } \ \Omega \setminus B(z_0,r). \end{cases}$$

By definition of u, we have $\tilde{f} = u \geqslant f$ on $B(z_0,r)$. Moreover, $\tilde{f}$ is the decreasing limit of the psh functions

$$g_k = \begin{cases} \max\{u, f + \frac{1}{k}\} & \text{on } \ B(z_0,r) \\ f + \frac{1}{k} & \text{near } \ \Omega \setminus B(z_0,r) \end{cases}$$

hence $\tilde{f}$ is psh. Also clearly, for $f_1 \leqslant f_2$ we have $u_1 \leqslant u_2$, hence $\tilde{f}_1 \leqslant \tilde{f}_2$. For an arbitrary locally bounded function $f \in P(\Omega)$, write f as a decreasing limit of smooth psh functions $f_k = f \star \rho_{1/k}$ and set $\tilde{f} = \lim_{k\to+\infty} \downarrow \tilde{f}_k$. Then $\tilde{f}$ has all required properties. $\square$

Now we prove the following three fundamental results by a simultaneous induction on n.

Proposition 7.7. — *Let $u, u_j \in P(\Omega)$ be locally bounded functions such that u_j increases to u almost everywhere. Then the measure $(dd^c u_j)^n$ converges weakly to $(dd^c u)^n$ on Ω.*

Proposition 7.8. — *Let Ω be a strongly pseudoconvex smooth open subset of $\mathbb{C}^n$. If $K \subset \Omega$ is compact, then*

(a) $(dd^c u_K^\star)^n = 0$ *on* $\Omega \setminus K$.

(b) $c(K, \Omega) = \int_K (dd^c u_K^\star)^n = \int_\Omega (dd^c u_K^\star)^n$.

Proposition 7.9. — *If a Borel set $N \subset \Omega$ is negligible, then*

$$c(N, \Omega) = 0.$$

The inductive proof is made in three steps.

Step 1 : 7.7 in $\mathbb{C}^n \Rightarrow$ 7.8 in $\mathbb{C}^n$.
Step 2 : 7.8 in $\mathbb{C}^n \Rightarrow$ 7.9 in $\mathbb{C}^n$.
Step 3 : 7.7 and 7.9 in $\mathbb{C}^n \Rightarrow$ 7.7 in $\mathbb{C}^{n+1}$.

In the case $n = 1$, proposition 7.7 is a well-known fact of distribution theory : u_j converges to u in $L^1_{\mathrm{loc}}(\Omega)$, thus $dd^c u_j$ converges weakly to $dd^c u$. By the inductive argument, propositions 7.7, 7.8, 7.9 hold in all dimensions.

Proof of step 1. — By Choquet's lemma, there is a sequence of functions $v_j \in P(\Omega)$ such that $v_j \leqslant 0$ on Ω, $v_j \leqslant -1$ on K and $v^\star = u_K^\star$. If we replace v_j by $\max\{-1, v_1, \ldots, v_j\}$, we see that we may assume $v_j \geqslant -1$ for all j and v_j increasing. Then fix an arbitrary ball $B(z_0, r) \subset \Omega \setminus K$ and consider the increasing sequence $\tilde{v}_j$ given by corollary 7.6. We still have $\tilde{v}_j \leqslant 0$ on Ω and $\tilde{v}_j \leqslant -1$ on K, thus $v_j \leqslant \tilde{v}_j \leqslant u_K$ and $\tilde{v} = \lim \tilde{v}_j$ satisfies $v^\star = \tilde{v}^\star = u_K^\star$, in particular $\lim \tilde{v}_j = \lim v_j = u_K^\star$ almost everywhere. Since $(dd^c \tilde{v}_j)^n = 0$ on $B(z_0, r)$, we conclude by 7.7 that $(dd^c u_K^\star)^n = 0$ on $B(z_0, r)$ and 7.8 (a) is proved.

To prove 7.8 (b), observe first that $-1 \leqslant u_K^\star \leqslant 0$ on Ω, hence $c(K, \Omega) \geqslant \int_K (dd^c u_K^\star)^n$ by definition of the capacity. If $\psi < 0$ is a smooth strictly psh exhaustion function of Ω, we have $A\psi \leqslant -1$ on K for A large enough. We can clearly assume $v_j \geqslant A\psi$ on Ω ; otherwise replace v_j by $\max\{v_j, A\psi\}$. Now, let $w \in P(\Omega)$ be such that $0 \leqslant w \leqslant 1$ and set

$$w' = (1-\varepsilon)w - 1 + \varepsilon/2 \ , \qquad w_j = \max\{w', v_j\}.$$

Since $-1 + \varepsilon/2 \leqslant w' \leqslant -\varepsilon/2$ on Ω, we have $w_j = v_j$ as soon as $A\psi > -\varepsilon/2$, whereas $w_j = w' \geqslant -1 + \varepsilon/2 > v_j$ on a neighborhood of K. Hence for $\delta > 0$ small enough Stokes' theorem implies

$$\int_{\Omega_\delta} (dd^c v_j)^n = \int_{\Omega_\delta} (dd^c w_j)^n \geqslant \int_K (dd^c w_j)^n = (1-\varepsilon)^n \int_K (dd^c w)^n.$$

By 7.7 $(dd^c v_j)^n$ converges weakly to $(dd^c u_K^\star)^n$ and we get

$$\limsup_{j \to +\infty} \int_{\Omega_\delta} (dd^c v_j)^n \leqslant \int_{\overline{\Omega}_\delta} (dd^c u_K^\star)^n = \int_K (dd^c u_K^\star)^n.$$

Therefore $\int_K (dd^c w)^n \leqslant \int_K (dd^c u_K^\star)^n$ and $c(K,\Omega) \leqslant \int_K (dd^c u_K^\star)^n$. $\square$

Proof of step 2. — Let $N = \{v < v^\star\}$ with $v = \sup v_\alpha$. By Choquet's lemma, we may assume that v_α is an increasing sequence of psh functions. The theorem of quasi-continuity shows that there exists an open set $G \subset \Omega$ such that all functions v_α and $v^\star$ are continuous on $\Omega \setminus G$ and $c(G,\Omega) < \varepsilon$. Write

$$N \subset G \cup (N \cap (\Omega \setminus G)) = G \cup \bigcup_{\delta,\lambda,\mu \in \mathbb{Q}} K_{\delta\lambda_\mu}$$

where $\delta > 0$, $\lambda < \mu$ and

$$K_{\delta\lambda\mu} = \big\{ z \in \overline{\Omega}_\delta \setminus G\ ;\ v(z) \leqslant \lambda < \mu \leqslant v^\star(z) \big\}.$$

As $v^\star$ is continuous and v lower semi-continuous on $\Omega \setminus G$, we see that $K_{\delta\lambda\mu}$ is compact. We only have to prove that $c(K_{\delta\lambda\mu},\Omega) = 0$. Set $K = K_{\delta\lambda\mu}$ for simplicity and take an open set $\omega \subset\subset \Omega$. By subtracting a large constant, we may assume $v^\star \leqslant 0$ on $\overline{\omega}$.

Multiplying by another constant, we may set $\lambda = -1$. Then all v_α satisfy $v_\alpha \leqslant 0$ on ω and $v_\alpha \leqslant v \leqslant -1$ on K. We infer that the extremal function u_K on ω satisfies $u_K \geqslant v$, $u_K^\star \geqslant v^\star$, in particular $u_K^\star \geqslant \mu > -1$ on K. By proposition 6.11 we obtain

$$c(K,\omega) = \int_K (dd^c u_K^\star)^n \leqslant \int_K (dd^c \max\{u_K^\star, \mu\})^n \leqslant |\mu|^n c(K,\omega)$$

because $-1 \leqslant |\mu|^{-1} \max\{u_K^\star,\mu\} \leqslant 0$. As $|\mu| < 1$, we conclude that $c(K,\omega) = 0$, hence $c(K,\Omega) = 0$. $\square$

Proof of step 3. — We have to show that if $\Omega \subset \mathbb{C}^{n+1}$,

$$\lim_{j\to+\infty} \int_\Omega \chi (dd^c u_j)^{n+1} = \int_\Omega \chi (dd^c u)^{n+1}$$

for all test functions $\chi \in C_0^\infty(\Omega)$. That is,

$$\lim_{j\to+\infty} \int_\Omega u_j (dd^c u_j)^{n+1} \wedge \gamma = \int_\Omega u (dd^c u)^n \wedge \gamma \tag{7.10}$$

with $\gamma = dd^c\chi$. As all $(1,1)$-forms γ can be written as linear combinations of forms of the type $i\alpha \wedge \overline{\alpha}$, $\alpha \in \Lambda^{1,0}(\mathbb{C}^n)^\star$, it is sufficient, after a change of coordinates, to consider forms of the type $\gamma = \frac{i}{2}\chi(z) dz_{n+1} \wedge d\overline{z}_{n+1}$, $\chi \in C_0^\infty(\Omega)$. In this case, for any locally bounded psh function v on Ω, the Fubini theorem yields

$$\int_\Omega v (dd^c v)^n \wedge \gamma = \int_{\mathbb{C}} d\lambda(z_{n+1}) \int_{\Omega(z_{n+1})} \chi(\bullet, z_{n+1}) (dd^c v(\bullet, z_{n+1}))^n$$

where $\Omega(z_{n+1}) = \{z \in \mathbb{C}^n ; (z, z_{n+1}) \in \Omega\}$ and $f(\bullet, z_{n+1})$ denotes the function $z \mapsto f(z, z_{n+1})$ on $\Omega(z_{n+1})$. Indeed, the result is clearly true if v is smooth. The general case follows by taking smooth psh functions v_j decreasing to v. The convergence of both terms in the equality is guaranteed by theorem 1.5 (a), combined with 1.3 and the bounded convergence theorem for the right hand side.

In order to prove (7.10), we thus have to show

$$\lim_{j\to+\infty}\int_\omega \chi u_j(dd^cu_j)^n=\int_\omega \chi u(dd^cu)^n \tag{7.11}$$

for $\omega\subset\mathbb{C}^n$, $\chi\in C_0^\infty(\omega)$ and $u_j\in P(\omega)\cap L^\infty_{\mathrm{loc}}(\omega)$ increasing to $u\in P(\omega)$ almost everywhere. To prove (7.11), we can clearly assume $0\leqslant\chi\leqslant 1$ and $0\leqslant u_j\leqslant u\leqslant 1$ on Ω. By our inductive hypothesis 7.7, $(dd^cu_j)^n$ converges weakly to $(dd^cu)^n$. As $u_j\leqslant u\leqslant u_\varepsilon=u\star\rho_\varepsilon$, we get

$$\begin{aligned}\limsup_{j\to+\infty}\int_\omega \chi u_j(dd^cu_j)^n &\leqslant \lim_{\varepsilon\to 0}\lim_{j\to+\infty}\int_\omega \chi u_\varepsilon(dd^cu_j)^n\\ &=\lim_{\varepsilon\to 0}\int_\omega \chi u_\varepsilon(dd^cu)^n=\int_\omega \chi u(dd^cu)^n.\end{aligned}$$

To prove the other inequality, let $\varepsilon>0$ and choose an open set $G\subset\omega$ such that $c(G,\omega)<\varepsilon$ and u,u_j are all continuous on $\omega\setminus G$. Let $v=\sup u_j$. Then $v^\star=u$ because $v^\star$ and u are psh and coincide almost everywhere. Let $\tilde u_j$ be a continuous extension of $u_j|_{\omega\setminus G}$ to ω such that $0\leqslant\tilde u_j\leqslant 1$. For $j\geqslant k$ we have $u_j\geqslant u_k$, hence

$$\begin{aligned}\int_\omega \chi u_j(dd^cu_j)^n &\geqslant \int_{\omega\setminus G}\chi\tilde u_k(dd^cu_j)^n\\ &\geqslant \int_\omega \chi\tilde u_k(dd^cu_j)^n-\int_G(dd^cu_j)^n.\end{aligned}$$

The last integral on the right is $\leqslant c(G,\omega)<\varepsilon$. Taking the limit as j tends to $+\infty$, we obtain

$$\begin{aligned}\liminf_{j\to+\infty}\int_\omega \chi u_j(dd^cu_j)^n &\geqslant \int_\omega \chi\tilde u_k(dd^cu)^n-\varepsilon\\ &\geqslant \int_\omega \chi u_k(dd^cu)^n-2\varepsilon.\end{aligned}$$

The second term ε comes from $\int_G(dd^cu)^n\leqslant c(G,\omega)<\varepsilon$. Now let $k\to+\infty$ and $\varepsilon\to 0$ to get

$$\liminf_{j\to+\infty}\int_\omega \chi u_j(dd^cu_j)^n\geqslant\int_\omega \chi v(dd^cu)^n.$$

Moreover, the Borel set $N=\{v<u=v^\star\}$ is negligible and the inductive hypothesis 7.9 implies $c(N,\omega)=0$. Therefore

$$\int_\omega \chi(u-v)(dd^cu)^n\leqslant\int_N(dd^cu)^n=0$$

and the proof is complete. □

THEOREM 7.12. — *For each $j=1,\ldots,q$, let u_j^k be an increasing sequence of locally bounded psh functions such that u_j^k converges almost everywhere to $u_j\in P(\Omega)$. Then*

(a) $dd^cu_1^k\wedge\ldots\wedge dd^cu_q^k\to dd^cu_1\wedge\ldots\wedge dd^cu_q$ *weakly.*

(b) $u_1^k dd^cu_2^k\wedge\ldots\wedge dd^cu_q^k\to u_1dd^cu_2\wedge\ldots\wedge dd^cu_q$ *weakly.*

Proof. — (a) Without loss of generality, we may assume $q = n$, otherwise we complete with additional stationary sequences $u^k_{q+1} = u_{q+1}, \ldots, u^k_n = u_n$ where $u_{q+1}, \ldots, u_n$ are chosen arbitrarily in $P(\Omega) \cap C^\infty(\Omega)$. Now apply proposition 7.7 to $u_k = \lambda_1 u^k_1 + \cdots + \lambda_n u^k_n$, $\lambda_j > 0$, and consider the coefficient of $\lambda_1 \ldots \lambda_n$ in $(dd^c u_k)^n$.

(b) Same proof as for (7.11). □

Exercise 7.13. — *Use formula 7.8 (b) in order to compute the capacity of* $K = \overline{B}(0, r)$ *in a ball* $\Omega = B(0, R)$; *show that the extremal function* u_K *is*

$$u_K(z) = \left(\log \frac{R}{r}\right)^{-1} \max\left\{\log \frac{|z|}{R}, \log \frac{r}{R}\right\}.$$

8. Characterization of pluripolar sets.

First we quote a few elementary properties of the extremal functions $u^\star_E$.

Properties 8.1. —

(a) *if* $E_1 \subset E_2 \subset \Omega$, *then* $u^\star_{E_1} \geqslant u^\star_{E_2}$.

(b) *if* $E \subset \Omega_1 \subset \Omega_2$, *then* $u^\star_{E,\Omega_1} \geqslant u^\star_{E,\Omega_2}$.

(c) *if* $E \subset \Omega$, *then* $u^\star_E = u_E = -1$ *on* E^0 *and* $(dd^c u^\star_E)^n = 0$ *on* $\Omega \setminus \overline{E}$; *hence* $(dd^c u^\star_E)^n$ *is supported by* ∂E.

(d) $u^\star_E = 0$ *if and only if there exists* $v \in P(\Omega)$, $v \leqslant 0$ *such that* $E \subset \{v = -\infty\}$.

(e) *if* $E \subset\subset \Omega$ *and if* Ω *is strongly pseudoconvex with exhaustion* $\psi < 0$, *then* $u^\star_E \geqslant A\psi$ *for some* $A > 0$.

Proof. — (a), (b) are obvious from definition (7.4); (e) is true as soon as $A\psi \leqslant -1$ on E ; the equality $(dd^c u^\star_E)^n = 0$ on $\Omega \setminus \overline{E}$ in (c) is proved exactly in the same way as 7.8 (a) in step 1.

(d) If $E \subset \{v = -\infty\}$, $v \in P(\Omega)$, $v \leqslant 0$, then for every $\varepsilon > 0$ we have $\varepsilon v \leqslant u_E$, hence $u_E = 0$ on $\Omega \setminus \{v = -\infty\}$ and $u^\star_E = 0$.

Conversely, Choquet's lemma shows that there is an increasing sequence $v_j \in P(\Omega)$, $-1 \leqslant v_j \leqslant u_E$, converging almost everywhere to $u^\star_E$. If $u^\star_E = 0$, we can extract a subsequence in such a way that $\int_\Omega |v_j| d\lambda < 2^{-j}$. As $v_j \leqslant 0$ and $v_j \leqslant -1$ on E, the function $v = \sum v_j$ is psh $\leqslant 0$ and $v = -\infty$ on E. □

If $G \subset\subset \Omega$ is an open subset, $K_1 \subset K_2 \subset \ldots$ compact subsets of G with $K_j \subset K^0_{j+1}$ and $\bigcup K_j = G$, then $u^\star_{K_j} = -1$ on $K^0_j \supset K_{j-1}$, $\lim \downarrow u^\star_{K_j} = -1$ on G. Therefore $u^\star_G \leqslant \lim u^\star_{K_j} \leqslant u_G \leqslant u^\star_G$ and theorems 1.5, 7.8, 8.1 (c) show that

$$c(G, \Omega) = \int_{\overline{G}} (dd^c u^\star_G)^n = \int_\Omega (dd^c u^\star_G)^n. \tag{8.2}$$

PROPOSITION 8.3. — *Let $\Omega \subset\subset \mathbb{C}^n$ and $K_1 \supset K_2 \supset \ldots$, $K = \bigcap K_j$ compact subsets of Ω. Then*

(a) $\big(\lim_{j\to+\infty} \uparrow u^\star_{K_j}\big)^\star = u^\star_K$.

(b) $\lim c(K_j, \Omega) = c(K, \Omega)$.

(c) $c^\star(K, \Omega) = c(K, \Omega)$.

Proof. — We have $\lim \uparrow u^\star_{K_j} \leqslant u^\star_K$ by 8.1 (a). On the other hand, let $v \in P(\Omega)$ be such that $v \leqslant 0$ on Ω and $v \leqslant -1$ on K. For every $\varepsilon > 0$ the open set $\{v < -1+\varepsilon\}$ is a neighborhood of K, thus $K \subset \{v < -1+\varepsilon\}$ for j large. We obtain therefore $v-\varepsilon \leqslant u^\star_{K_j}$ and $u_K = \sup\{v\} \leqslant \lim u^\star_{K_j}$, whence equality (a). Property (b) follows now from theorems 7.7 and 7.8 (b), and (c) is a consequence of (b) when K_j are neighborhoods of K. $\square$

LEMMA 8.4. — *Let $\Omega \subset\subset \mathbb{C}^n$ and $u, v \in P(\Omega)$ locally bounded psh functions such that $u \leqslant v \leqslant 0$ and $\lim_{z\to\partial\Omega} u(z) = 0$. Then*

$$\int_\Omega (dd^c v)^n \leqslant \int_\Omega (dd^c u)^n.$$

Moreover $\int_\Omega (dd^c u)^n = 0$ if and only if $u = 0$.

Proof. — As $\max\{u+\varepsilon, v\} = u+\varepsilon$ near $\partial\Omega$, we get

$$\int_\Omega (dd^c u)^n = \int_\Omega (dd^c \max\{u+\varepsilon, v\})^n.$$

Let ε tend to 0, and observe that the integrand on the right hand side converges weakly to $(dd^c v)^n$ by theorem 1.5. The asserted inequality follows.

Now, assume that $u(z_0) < 0$ at some point. Then

$$v(z) = \max\{u(z), \varepsilon^2|z|^2 - \varepsilon\}$$

coincides with u near $\partial\Omega$ and with $\varepsilon^2|z|^2 - \varepsilon$ on a neighborhood ω of z_0. We get therefore

$$\int_\Omega (dd^c u)^n = \int_\Omega (dd^c v)^n \geqslant \int_\omega (dd^c v)^n > 0. \ \square$$

PROPOSITION 8.5. — *Let $\Omega \subset\subset \mathbb{C}^n$ be strongly pseudoconvex. If $E \subset\subset \Omega$ is an arbitrary subset, then*

$$c^\star(E, \Omega) = \int_\Omega (dd^c u^\star_E)^n.$$

Proof. — Let $\psi < 0$ be a strictly psh exhaustion function on Ω. For every open set $G \supset E$, we have $u^\star_E \geqslant u^\star_G \geqslant A\psi$ by 8.1, and lemma 8.4 implies

$$\int_\Omega (dd^c u^\star_E)^n \leqslant \int_\Omega (dd^c u^\star_G)^n = c(G, \Omega),$$

thus $\int_\Omega (dd^c u_E^\star)^n \leqslant c^\star(E,\Omega)$.

Conversely, Choquet's lemma shows that there exists an increasing sequence $v_j \in P(\Omega)$ with $-1 \leqslant v_j \leqslant 0$, $v_j \geqslant A\psi$ on Ω and $\lim v_j = u_E$ almost everywhere. If

$$G_j = \{z \in \Omega; (1+1/j)v_j(z) < -1\}$$

then $G_j \supset E$, G_j is decreasing and $(1+1/j)v_j \leqslant u_{G_j}$. Thus $\lim \uparrow u_{G_j}^\star = u_E^\star$ almost everywhere and theorem 7.7 gives

$$\lim_{j\to+\infty} \int_\Omega (dd^c u_{G_j}^\star)^n = \int_\Omega (dd^c u_E^\star)^n. \ \square$$

Corollary 8.6. — *Let $\Omega \subset\subset \mathbb{C}^n$ be strongly pseudoconvex. If $E \subset\subset \Omega$, then $c^\star(E,\Omega) = 0$ if and only if $u_E^\star = 0$.* $\square$

Now, we prove an important result due to Josefson [Jo]. A set P in $\mathbb{C}^n$ is said to be *locally pluripolar* if for each $z \in P$ there is an open neighborhood Ω of z and $v \in P(\Omega)$ such that $P \cap \Omega \subset \{v = -\infty\}$.

Theorem 8.7 (Josefson). — *If $P \subset \mathbb{C}^n$ is locally pluripolar, there exists $v \in P(\mathbb{C}^n)$ with $P \subset \{v = -\infty\}$, i.e. P is globally pluripolar in $\mathbb{C}^n$.*

Proof. — By the definition of locally pluripolar, one can find sets P_j, Ω_j with $P_j \subset\subset \Omega_j \subset\subset \mathbb{C}^n$, $\bigcup_{j\geqslant 1} P_j = P$, each Ω_j strongly pseudoconvex. By 8.1 (d) and 8.5, we have $c^\star(P_j,\Omega_j) = 0$.

Let B_k be the ball of center 0 and radius k in $\mathbb{C}^n$ and $j(k)$ a sequence of integers such that each integer is repeated infinitely many times and $\Omega_{j(k)} \subset B_k$. By the comparison theorem 6.5 we have $c^\star(P_{j(k)}, B_{k+2}) = 0$, hence the extremal function $u_{P_{j(k)}}^\star$ in B_{k+2} is zero and we can find $v_k \in P(B_{k+2})$ with $-1 \leqslant v_k \leqslant 0$, $v_k = -1$ on $P_{j(k)}$ and $\int_{B_k} |v_k| d\lambda < 2^{-k}$. Now set

$$\tilde v_k(z) = \begin{cases} v_k(z) & \text{on} \quad B_k \\ \max\{v_k(z), |z|^2 - (k+1)^2\} & \text{on} \quad B_{k+2} \setminus B_k \\ |z|^2 - (k+1)^2 & \text{on} \quad \mathbb{C}^n \setminus B_{k+2}. \end{cases}$$

As $\tilde v_k \leqslant 0$ on B_k and $\int_{B_k} |\tilde v_k| d\lambda < 2^{-k}$, the series $v = \sum \tilde v_k$ defines a global psh function on $\mathbb{C}^n$. Moreover $\tilde v_k = -1$ on $P_{j(k)}$ and each P_j is repeated infinitely many times, therefore

$$v = -\infty \quad \text{on} \quad \bigcup P_j = P. \ \square$$

Corollary 8.8. — *Let $\Omega \subset \mathbb{C}^n$ and $P \subset \Omega$. Then P is pluripolar in Ω if and only if $c^\star(P,\Omega) = 0$.*

Proof. — That P is pluripolar implies $c^\star(P,\Omega) = 0$ was proved in corollary 6.8. Conversely, if $c^\star(P,\Omega) = 0$ then $c^\star(P \cap \omega', \omega) = 0$ for all concentric balls

$\omega' \subset\subset \omega \subset\subset \Omega$ and corollary 8.6 combined with 8.1 (d) shows that $P \cap \omega'$ is pluripolar in ω. Josefson's theorem implies that P is globally pluripolar in $\mathbb{C}^n$. $\square$

COROLLARY 8.9. — *Negligible sets are pluripolar.*

Proof. — By Choquet's lemma every negligible set is contained in a Borel negligible set $N = \{v < v^\star\}$ with $v = \sup v_j$. However, in step 2 of § 7, we showed that $N \subset G \cup \bigcup K_{\delta\lambda\mu}$ with G open, $c(G, \Omega) < \varepsilon$ and $c(K_{\delta\lambda\mu}, \Omega) = 0$. By 8.3 (c), we have $c^\star(N, \Omega) < \varepsilon$ for all $\varepsilon > 0$. Therefore $c^\star(N, \Omega) = 0$ and N is pluripolar. $\square$

9. Capacitability and outer regularity.

First we introduce some definitions and prove a general capacitability result due to G. Choquet. All topological spaces occurring here are assumed to be Hausdorff.

DEFINITION 9.1. — *Let X be a topological space.*

- *A F_σ subset of X is a countable union of closed subsets of X ;*
- *A $F_{\sigma\delta}$ subset of X is a countable intersection of F_σ subsets of X.*
- *A space X is said to be a K_σ (resp. $K_{\sigma\delta}$) space if it is homeomorphic to some F_σ (resp. $F_{\sigma\delta}$) subset of a compact space W.*

PROPERTIES 9.2. —

(a) *Every closed subset F of a $K_{\sigma\delta}$ space X is a $K_{\sigma\delta}$ space.*

(b) *Every countable disjoint sum $\coprod X_j$ of $K_{\sigma\delta}$ spaces is a $K_{\sigma\delta}$ space.*

(c) *Every countable product $\prod X_j$ of $K_{\sigma\delta}$ spaces is a $K_{\sigma\delta}$ space.*

Proof. — (a) Write $X = \bigcap_{\ell \geqslant 1} G_\ell$ and $G_\ell = \bigcup_{m \geqslant 1} K_{\ell m}$ where $K_{\ell m}$ are closed subsets of a compact space W. If $\overline{F}$ is the closure of F in W, we have

$$F = X \cap \overline{F} = \bigcap_{\ell \geqslant 1} G_\ell \cap \overline{F}, \qquad G_\ell \cap \overline{F} = \bigcup_{m \geqslant 1} K_{\ell m} \cap \overline{F}.$$

(b) Let X_j, $j \geqslant 1$, be $K_{\sigma\delta}$ spaces and write for each j

$$X_j = \bigcap_{\ell \geqslant 1} G_\ell^j, \qquad G_\ell^j = \bigcup_{m \geqslant 1} K_{\ell m}^j$$

where $K_{\ell m}^j$ is a closed subspace of a compact space W_j. Then $\coprod W_j$ can be embedded in the compact space $W = \prod (W_j \amalg \{\star\})$ via the obvious map that sends $w \in W_j$ to $(\star, \ldots, \star, w, \star, \ldots)$ where w is in the j-th position. Now $X = \coprod X_j$ can be written

$$X = \bigcap_{\ell \geqslant 1} G_\ell, \qquad G_\ell = \bigcup_{m \geqslant 1} \coprod_{j \geqslant 1} K_{\ell m}^j.$$

As $K^j_{\ell m}$ is sent onto a closed set by the embedding $\coprod W_j \longrightarrow W$, we conclude that X is a $K_{\sigma\delta}$ space.

(c) With the notations of (b), write $X = \prod X_j$ as

$$X = \bigcap_{\ell \geqslant 1} G_\ell, \qquad G_\ell = G^1_\ell \times G^2_{\ell-1} \times \ldots \times G^\ell_1 \times W_{\ell+1} \times \ldots \times W_j \times \ldots,$$

$$G_\ell = \bigcup_{m_1,\ldots,m_\ell \geqslant 1} K^1_{\ell\, m_1} \times K^2_{\ell-1\, m_2} \times \ldots \times K^\ell_{1\, m_\ell} \times W_{\ell+1} \times \ldots \times W_j \times \ldots$$

where each term in the union is closed in $W = \prod W_j$. $\square$

Definition 9.3. — *A space E is said to be K-analytic if E is a continuous image of a $K_{\sigma\delta}$ space X.*

Theorem 9.4. — *Let Ω be a topological space and $E_1 \subset E_2 \subset \ldots$ K-analytic subsets of Ω. Then $\bigcup E_j$ and $\bigcap E_j$ are K-analytic.*

Proof. — Let $f_j : X_j \to E_j$ be a continuous map from a $K_{\sigma\delta}$ space onto E_j. Set $X = \coprod X_j$ and $f = \coprod f_j : X \to \Omega$. Then X is a $K_{\sigma\delta}$ space, f is continuous and $f(X) = \bigcup E_j$. Now set

$$X = \{x = (x_1, x_2, \ldots) \in \prod X_j \ ; \ f_1(x_1) = f_2(x_2) = \cdots\},$$

$$f : X \to \Omega \ , \qquad f(x) = f_1(x_1) = f_2(x_2) = \cdots \ .$$

Then X is closed in $\prod X_j$, so X is a $K_{\sigma\delta}$ space by 9.2 (a,c) and $f(X) = \bigcap E_j$. $\square$

Corollary 9.5. — *Let Ω be a separable locally compact space. Then all Borel subsets of Ω are K-analytic.*

Proof. — Any open or closed open set in Ω is a countable union of compact subsets, hence K-analytic. On the other hand, theorem 9.4 shows that

$$\mathcal{A} = \{E \subset \Omega \ ; \ E \text{ and } \Omega \setminus E \text{ are } K-\text{analytic}\}$$

is a σ-algebra. Since $\mathcal{A}$ contains all open sets in E, $\mathcal{A}$ must also contain all Borel subsets. $\square$

Before going further, we need a simple lemma.

Lemma 9.6. — *Let E be a relatively compact K-analytic subset of a topological space Ω. There exists a compact space T, a continuous map $g : T \to \Omega$ and a $F_{\sigma\delta}$ subset $Y \subset T$ such that $g(Y) = E$.*

Proof. — There is a compact space W, a $F_{\sigma\delta}$ subset $X \subset W$ and a continuous map $f : X \to E$ onto E. Let

$$Y = \{(x, f(x))\,;\, x \in X\} \subset X \times E$$

be the graph of f and $T = \overline{Y}$ the closure of Y in the compact space $\overline{X} \times \overline{E}$. As f is continuous, Y is closed in $X \times \overline{E}$, thus $Y = T \cap (X \times \overline{E})$. Now, X is a $F_{\sigma\delta}$ subset of $\overline{X}$, so $X \times \overline{E}$ is a $F_{\sigma\delta}$ subset of $\overline{X} \times \overline{E}$ and Y is a $F_{\sigma\delta}$ subset of T. Finally E is the image of Y by the second projection $g : T \to \overline{E}$. $\square$

Definition 9.7. — *Let Ω be a topological space. A generalized capacity is a set function $\overline{c}$ defined on all subsets $E \subset \Omega$ satisfying the following axioms :*

(a) *If $E_1 \subset E_2 \subset \Omega$, then $\overline{c}(E_1) \leqslant \overline{c}(E_2)$.*

(b) *If $E_1 \subset E_2 \subset \ldots \subset \Omega$, then $\overline{c}(\bigcup E_j) = \lim_{j\to+\infty} \overline{c}(E_j)$.*

(c) *If $K_1 \supset K_2 \supset \ldots$ are compact subsets of Ω, then $\overline{c}(\bigcap K_j) = \lim_{j\to+\infty} \overline{c}(K_j)$.*

Choquet's capacitability theorem 9.8. — *Let Ω be a K_σ space and let $\overline{c}$ be a generalized capacity on Ω. Then every K-analytic subset $E \subset \Omega$ satisfies*

$$\overline{c}(E) = \sup_{K \text{ compact} \subset E} \overline{c}(K).$$

Proof. — As Ω is an increasing union of compact sets L_j, we have $\overline{c}(E) = \lim_{j\to+\infty} \overline{c}(E \cap L_j)$ by axiom (b); we may therefore assume that E is relatively compact in Ω. Then lemma 9.6 shows that there is a $F_{\sigma\delta}$ subset Y in a compact space T and a continuous map $g : T \to \Omega$ such that $g(Y) = E$. It is immediate to check that $\overline{c} \circ g$ is a generalized capacity on T. Hence we are reduced to proving the theorem when Ω is a compact space and E is a $F_{\sigma\delta}$ subset of Ω. Then write

$$E = \bigcap_{\ell \geqslant 1} G_\ell, \qquad G_\ell = \bigcup_{m \geqslant 1} K_{\ell m}$$

where $K_{\ell m}$ is a closed subset of Ω. Without loss of generality, we can arrange that $K_{\ell m}$ is increasing in m. Fix $\lambda < \overline{c}(E)$. Then

$$E = G_1 \cap \bigcap_{\ell \geqslant 2} G_\ell = \bigcup_{m \geqslant 1} (K_{1m} \cap \bigcap_{\ell \geqslant 2} G_\ell)$$

and axiom (b) implies that exists a subset $E_1 = K_{1m_1} \cap \bigcap_{\ell \geqslant 2} G_\ell$ of E such that $\overline{c}(E_1) > \lambda$. By induction, there is a decreasing sequence $E \supset E_1 \supset \ldots \supset E_s$ with

$$E_s = K_{1m_1} \cap \ldots \cap K_{sm_s} \cap \bigcap_{\ell \geqslant s+1} G_\ell$$

and $\overline{c}(E_s) > \lambda$. Set $K = \bigcap K_{sm_s} = \bigcap E_s \subset E$. Axiom (c) implies

$$\overline{c}(K) = \lim_{s\to+\infty} \overline{c}\big(K_{1m_s} \cap \ldots \cap K_{sm_s}\big) \geqslant \lim_{s\to+\infty} \overline{c}(E_s) \geqslant \lambda$$

and the theorem is proved. $\square$

Now, we apply these general results to the outer Monge-Ampère capacity $c^\star$ introduced in § 6.

Theorem 9.9. — *Let $\Omega \subset\subset \mathbb{C}^n$ be strongly pseudoconvex. Then the outer capacity $c^\star(\bullet, \Omega)$ is a generalized capacity in the sense of 9.7.*

Proof. — Axiom 9.7 (a) is clear, and 9.7 (c) is a consequence of 8.3. To prove 9.7 (b), we only have to show that $c^\star(\bigcup E_j, \Omega) \leqslant \lim_{j\to+\infty} c^\star(E_j, \Omega)$. It is no loss of generality to assume that $E_j \subset\subset \Omega$. Let N_j be the negligible set $N_j = \{u_{E_j} < u^\star_{E_j}\}$ and G_0 an open subset of Ω with $G_0 \supset \bigcup N_j$ and $c(G_0, \Omega) < \varepsilon$. Consider the open sets $V_j = \{u^\star_{E_j} < -1 + \eta\}$ and $G_j = G_0 \cup V_j \supset E_j$. Then $(1-\eta)^{-1} u^\star_{E_j} \leqslant u^\star_{V_j} \leqslant 0$ and lemma 8.4 implies

$$
\begin{aligned}
c(G_j, \Omega) &\leqslant \varepsilon + c(V_j, \Omega) = \varepsilon + \int_\Omega (dd^c u^\star_{V_j})^n \\
&\leqslant \varepsilon + (1-\eta)^{-n} \int_\Omega (dd^c u^\star_{E_j})^n = \varepsilon + (1-\eta)^{-n} c^\star(E_j, \Omega),
\end{aligned}
$$

thanks to proposition 8.5. Further $E_j \subset G_j$ and $G_1 \subset G_2 \subset \ldots$ since $u^\star_{E_j}$ is decreasing. Thus $G = \bigcup G_j \supset E = \bigcup E_j$ and

$$
c(G, \Omega) = \lim_{j\to+\infty} c(G_j, \Omega) \leqslant \varepsilon + (1-\eta)^{-n} \lim_{j\to+\infty} c^\star(E_j, \Omega).
$$

Letting $\varepsilon, \eta \to 0$ we get the desired inequality

$$
c^\star(E, \Omega) \leqslant \lim_{j\to+\infty} c^\star(E_j, \Omega). \ \square
$$

Choquet's capacitability theorem combined with 8.3 (c) implies that every K-analytic subset $E \subset \Omega$ satisfies

$$
c^\star(E, \Omega) = \sup_{K \text{ compact} \subset E} c(K, \Omega) = c_\star(E, \Omega). \tag{9.10}
$$

When E is a Borel set, we thus have $c^\star(E, \Omega) = c(E, \Omega)$. These results can be restated as :

THEOREM 9.11. — *Every K-analytic (in particular every Borel) subset of Ω is c-capacitable. Consequently c is outer regular.* $\square$

10. Siciak's extremal function and Alexander's capacity.

We work here on the whole space $\mathbb{C}^n$ rather than on a bounded open subset Ω. In this case, the relevant class of psh functions to consider is the set $P_{\log}(\mathbb{C}^n)$ of psh functions v with logarithmic growth at infinity, *i.e.* such that

$$
v(z) \leqslant \log_+ |z| + C \tag{10.1}
$$

for some real constant C. Let E be a bounded subset of $\mathbb{C}^n$. We consider the global extremal function introduced by Siciak [Sic] :

$$
U_E(z) = \sup\{v(z) \ ; \ v \in P_{\log}(\mathbb{C}^n), \ v \leqslant 0 \text{ on } E\}. \tag{10.2}
$$

THEOREM 10.3. — *If $U^\star_E$ is not identically $+\infty$, then $U^\star_E \in P_{\log}(\mathbb{C}^n)$ and $U^\star_E$ satisfies an inequality*

$$
\log_+(|z|/R) \leqslant U^\star_E(z) \leqslant \log_+ |z| + C
$$

for suitable constants $C, R > 0$. Moreover $U_E^\star = 0$ on E^0,

$$(dd^c U_E^\star) = 0 \text{ on } \mathbb{C}^n \setminus \overline{E}, \qquad \int_{\overline{E}} (dd^c U_E^\star)^n = 1.$$

Proof. — Assume that $U_E^\star$ is not identically $+\infty$. Then $U_E^\star(z_0) < +\infty$ for some point $z_0 \in \mathbb{C}^n$, say $z_0 = 0$ for simplicity. By the upper semi-continuity of $U_E^\star$, this implies that all functions $v \in P_{\log}(\mathbb{C}^n)$ such that $v \leqslant 0$ on E are uniformly bounded above by some constant M_0 on a small ball $B(0, r_0)$. As $\sup_{B(0,r)} v = \chi(\log r)$ where χ is a convex increasing function, (10.1) implies $\chi' \leqslant 1$, thus

$$\chi(\log r) \leqslant \chi(\log r_0) + \log r/r_0 \quad \text{for } r \geqslant r_0.$$

Therefore all functions v under consideration satisfy $v(z) \leqslant \log_+(|z|/r_0) + M_0$. In particular these functions are uniformly bounded above everywhere and $U_E^\star = (\sup\{v\})^\star$ is psh of logarithmic growth on $\mathbb{C}^n$. On the other hand, E is contained in a ball $\overline{B}(0, R)$ so $\log_+ |z|/R \leqslant 0$ on E and we get $U_E(z) \geqslant \log_+ |z|/R$. The equality $(dd^c U_E^\star)^n = 0$ on $\mathbb{C}^n \setminus \overline{E}$ is verified exactly in the same way as 7.8 (a). The integral over $\overline{E}$ is obtained by the following lemma. $\square$

Lemma 10.4. — *Let $v \in P_{\log}(\mathbb{C}^n)$ be such that*

$$\log_+ |z| - C_1 \leqslant v(z) \leqslant \log_+ |z| + C_2$$

for some constants. Then $\int_{\mathbb{C}^n} (dd^c v)^n = 1$.

Proof. — It is sufficient to check that

$$\int_{\mathbb{C}^n} (dd^c v_1)^n \leqslant \int_{\mathbb{C}^n} (dd^c v_2)^n$$

when v_1, v_2 are two such functions. Indeed, we have

$$\int_{\mathbb{C}^n} (dd^c \log_+ |z|)^n = \int_{\mathbb{C}^n} (dd^c \log |z|)^n = 1$$

by Stokes' theorem and remark 2.10, and we only have to choose $v_1(z)$ or $v_2(z) = \log_+ |z|$ and the other function equal to v. To prove the inequality, fix $r, \varepsilon > 0$ and choose $C > 0$ large enough so that $(1-\varepsilon)v_1 > v_2 - C$ on $B(0, r)$. As the function $u = \max\{(1-\varepsilon)v_1, v_2 - C\}$ is equal to $v_2 - C$ for $|z| = R$ large, we get

$$(1-\varepsilon)^n \int_{B(0,r)} (dd^c v_1)^n = \int_{B(0,r)} (dd^c u)^n \leqslant \int_{B(0,R)} (dd^c u)^n = \int_{B(0,R)} (dd^c v_2)^n$$

and the expected inequality follows as $\varepsilon \to 0$ and $r \to +\infty$. $\square$

Theorem 10.5. — *Let $E, E_1, E_2, \ldots \subset B(0, R) \subset \mathbb{C}^n$.*

(a) *If $E_1 \subset E_2$, then $U_{E_1}^\star \geqslant U_{E_2}^\star$.*

(b) *If $E_1 \subset E_2 \subset \ldots$ and $E = \bigcup E_j$, then $U_E^\star = \lim \downarrow U_{E_j}^\star$.*

(c) *If $K_1 \supset K_2 \supset \ldots$ and $K = \bigcap K_j$, then $U_K^\star = (\lim \uparrow U_{K_j}^\star)^\star$.*

(d) *For every set E, there exists a decreasing sequence of open sets $G_j \supset E$ such that $U_E^\star = (\lim \uparrow U_{G_j}^\star)^\star$.*

Proof. — (a) is obvious and the proof of (c) is similar to that of 8.3 (a).

(d) By Choquet's lemma, there is an increasing sequence $v_j \in P_{\log}(\mathbb{C}^n)$ with $U_E^\star = (\lim v_j)^\star$ and $v_j(z) \geqslant \log_+ |z|/R$. Set $G_j = \{v_j < 1/j\}$ and observe that $U_{G_j}^\star \geqslant v_j - 1/j$.

(b) Set $v = \lim \downarrow U_{E_j}^\star$. Then $v \in P_{\log}(\mathbb{C}^n)$ and $v = 0$ on E, except on the negligible set $N = \bigcup\{U_{E_j} < U_{E_j}^\star\}$. By Josefson's theorem, there exists $w \in P(\mathbb{C}^n)$ such that $N \subset \{w = -\infty\}$. We set

$$v_j(z) = \begin{cases} \big(1-\frac{1}{j}\big)v(z) + \max\{\varepsilon_j w(z), \frac{1}{j}\log|z| - j\} & \text{on} \quad B(0, e^{j^3}) \\ \big(1-\frac{1}{j}\big)v(z) + \frac{1}{j}\log|z| - j & \text{on} \quad \mathbb{C}^n \setminus B(0, e^{j^3}) \end{cases}$$

where ε_j is chosen such that $\varepsilon_j w < j^2 - j$ on $S(0, e^{j^3})$. Then $v_j \in P_{\log}(\mathbb{C}^n)$ and $v_j \leqslant 0$ everywhere on E for j large. Therefore

$$U_E^\star \geqslant U_E \geqslant v_j \geqslant \big(1 - \frac{1}{j}\big)v + \varepsilon_j w \quad \text{on } B(0, e^{j^3})$$

and letting $j \to +\infty$ we obtain $U_E^\star \geqslant v$. The other inequality is clear. □

Now, we show that the extremal function of a compact set can be computed in terms of polynomials. We denote by $\mathcal{P}_d$ the space of polynomials of degree $\leqslant d$ in $\mathbb{C}[z_1, \ldots, z_n]$.

THEOREM 10.6. — *Let K be a compact subset of $\mathbb{C}^n$. Then*

$$U_K(z) = \sup\Big\{\frac{1}{d}\log|P(z)|\,;\, d \geqslant 1,\ P \in \mathcal{P}_d,\ \|P\|_{L^\infty(K)} \leqslant 1\Big\}.$$

Proof. — For any of the polynomials P involved in the above formula, we clearly have $\frac{1}{d}\log|P| \in P_{\log}(\mathbb{C}^n)$ and this function is $\leqslant 0$ on K. Hence

$$\frac{1}{d}\log|P| \leqslant U_K.$$

Conversely, fix a point $z_0 \in \mathbb{C}^n$ and a real number $a < U_K(z_0)$. Then there exists $v \in P_{\log}(\mathbb{C}^n)$ such that $v \leqslant 0$ on K and $v(z_0) > a$. Replacing v by $v \star \rho_\delta - \varepsilon$ with $\delta \ll \varepsilon \ll 1$, we may assume that $v \in P_{\log}(\mathbb{C}^n) \cap C^\infty(\mathbb{C}^n)$, $v < 0$ on K and $v(z_0) > a$. Choose a ball $B(z_0, r)$ on which $v > a$, a smooth function χ with compact support in $B(z_0, r)$ such that $\chi = 1$ on $B(z_0, r/2)$ and apply Hörmander's L^2 estimates to the closed $(0,1)$-form $d''\chi$ and to the weight

$$\varphi(z) = 2dv(z) + 2n\log|z - z_0| + \varepsilon\log(1 + |z|^2).$$

We find a solution f of $d''f = d''\chi$ such that

$$\int_{\mathbb{C}^n} |f|^2 e^{-2dv}|z - z_0|^{-2n}(1+|z|^2)^{-\varepsilon} d\lambda$$
$$\leqslant \int_{B(z_0,r)} |d''\chi|^2 e^{-2dv}|z - z_0|^{-2n}(1+|z|^2)^{2-\varepsilon} d\lambda \leqslant C_1 e^{-2da}.$$

We thus have $f(z_0) = 0$ and $F = \chi - f$ is a holomorphic function on $\mathbb{C}^n$ such that $F(z_0) = 1$. In addition we get

$$\int_{\mathbb{C}^n} |F|^2 e^{-2dv}(1+|z|^2)^{-2n-2\varepsilon} d\lambda \leqslant C_2 e^{-2da}$$

where $C_1, C_2 > 0$ are constants independent of d. As $v(z) \leqslant \log_+ |z| + C_3$, it follows that $F \in \mathcal{P}_d$. Moreover, since $v > 0$ on a neighborhood of K, the mean value inequality applied to the subharmonic function $|F|^2$ gives

$$\sup_K |F|^2 \leqslant C_4 e^{-2da}.$$

The polynomial $P = C_4^{-1/2} e^{da} F \in \mathcal{P}_d$ is such that $\|P\|_K = 1$ and we have $\log |P(z_0)| \geqslant da - C_5$, whence

$$\sup \Big\{ \frac{1}{d} \log |P(z_0)| \,;\, d \geqslant 1,\ P \in \mathcal{P}_d,\ \|P\|_{L^\infty(K)} \leqslant 1 \Big\} \geqslant a.$$

As a was an arbitrary number $< U_K(z_0)$, the proof is complete. □

Now, we introduce a few concepts related to extremal polynomials. Let B be the unit ball of $\mathbb{C}^n$ and K a compact subset of B. The Chebishev constants $M_d(K)$ are defined by

$$M_d(K) = \inf \big\{ \|P\|_{L^\infty(K)} \,;\, P \in \mathcal{P}_d,\ \|P\|_{L^\infty(B)} = 1 \big\}. \tag{10.7}$$

It is clear that $M_d(K) \leqslant 1$ and that $M_d(K)$ satisfies

$$M_{d+d'}(K) \leqslant M_d(K) M_{d'}(K).$$

The Alexander capacity is defined by

$$T(K) = \inf_{d \geqslant 1} M_d(K)^{1/d}. \tag{10.8}$$

It is easy to see that we have in fact $T(K) = \lim_{d \to +\infty} M_d(K)^{1/d}$: for any integer $\delta \geqslant 1$, write $\delta = qd + r$ with $0 \leqslant r < d$ and observe that

$$M_\delta(K)^{1/\delta} \leqslant M_{qd}(K)^{1/(qd+r)} \leqslant M_d(K)^{q/(qd+r)} \;;$$

letting $\delta \to +\infty$ with d fixed, we get

$$T(K) \leqslant \liminf_{\delta \to +\infty} M_\delta(K)^{1/\delta} \leqslant \limsup_{\delta \to +\infty} M_\delta(K)^{1/\delta} \leqslant M_d(K)^{1/d},$$

whence the equality. Now, for an arbitrary subset $E \subset B$, we set

$$T_\star(E) = \sup_{K \subset E} T(K), \qquad T^\star(E) = \inf_{G \text{ open} \supset E} T_\star(G). \tag{10.9}$$

Siciak's theorem 10.10. — *For every set $E \subset B$,*

$$T^\star(E) = \exp(-\sup_B U_E^\star).$$

Proof. — The main step is to show that the equality holds for compact subsets $K \subset B$, *i.e.* that

$$T(K) = \exp(-\sup_B U_K^\star). \tag{10.11}$$

Indeed, it is clear that $\sup_B U_K^\star = \sup_B U_K$ and theorem 10.6 gives

$$\begin{aligned}\sup_B U_K &= \sup\Big\{\frac{1}{d}\log\|P\|_{L^\infty(B)}\ ;\ d\geqslant 1,\ P\in\mathcal{P}_d,\ \|P\|_{L^\infty(K)}=1\Big\}\\ &= \sup\Big\{-\frac{1}{d}\log\|P\|_{L^\infty(K)}\ ;\ d\geqslant 1,\ P\in\mathcal{P}_d,\ \|P\|_{L^\infty(B)}=1\Big\}\end{aligned}$$

after an obvious rescaling argument $P\mapsto \alpha P$. Taking the exponential, we get

$$\begin{aligned}\exp(-\sup_B U_K^\star) &= \inf_{d\geqslant 1}\inf\Big\{\|P\|_{L^\infty(K)}^{1/d}\ ;\ P\in\mathcal{P}_d,\ \|P\|_{L^\infty(B)}=1\Big\}\\ &= \inf_{d\geqslant 1} M_d(K)^{1/d} = T(K).\end{aligned}$$

Next, let G be an open subset of B and K_j an increasing sequence of compact sets such that $G=\bigcup K_j$ and $T_\star(G)=\lim T(K_j)$. Then 10.5 (b) implies $U_G^\star = \lim\downarrow U_{K_j}^\star$, hence

$$\lim_{j\to+\infty}\sup_B U_{K_j}^\star = \sup_B U_G^\star = \sup_{\overline{B}} U_G^\star$$

by Dini's lemma. Taking the limit in (10.11), we get

$$T_\star(G) = \exp(-\sup_B U_G^\star).$$

Finally, 10.5 (d) shows that there exists a decreasing sequence of open sets $G_j\supset E$ such that $U_E^\star = (\lim\uparrow U_{G_j}^\star)^\star$. We may take G_j so small that $T^\star(E)=\lim T_\star(G_j)$. Theorem 10.9 follows. □

Corollary 10.12. — *The set function $T^\star$ is a generalized capacity in the sense of definition 9.7 and we have $T^\star(E)=T_\star(E)$ for every K-analytic set $E\subset B$.*

Proof. — Axioms 9.7 (a,b,c) are immediate consequences of properties 10.5 (a,b,c) respectively. In addition, formulas 10.10 and 10.11 show that $T^\star(K)=T(K)$ for every compact set $K\subset B$. The last statement is then a consequence of Choquet's capacitability theorem. □

To conclude this section, we show that $1/|\log T^\star|$ is not very far from being subadditive. We need a lemma.

Lemma 10.13. — *For every $P\in\mathcal{P}_d$, one has*

$$\log\|P\|_{L^\infty(B)} - c_n d \leqslant \int_{\partial B}\log|P(z)|d\sigma(z)\leqslant \log\|P\|_{L^\infty(B)}$$

where $d\sigma$ is the unit invariant measure on the sphere and c_n a constant such that $c_n\sim\log(2n)$ as $n\to+\infty$.

Proof. — Without loss of generality, we may assume that $\|P\|_{L^\infty(B)}=1$. Since $\frac{1}{d}\log|P|\in P_{\log}(\mathbb{C}^n)$, the logarithmic convexity property already used implies that

$$\sup_{B(0,r)}\frac{1}{d}\log|P|\geqslant \log r\quad\text{for } r<1.$$

The Harnack inequality for the Poisson kernel implies now

$$\sup_{B(0,r)} \log|P| \leqslant \frac{1-r^2}{(1+r)^{2n}} \int_{\partial B} \log|P| d\sigma,$$

$$\int_{\partial B} \log|P| d\sigma \geqslant \frac{1+r^{2n}}{1-r^2} \log r.d.$$

The lemma follows with

$$c_n = \inf_{r\in]0,1[} \frac{(1+r)^{2n}}{1-r^2} \log \frac{1}{r} \ ;$$

the infimum is attained approximately for $r = 1/(2n \log 2n)$. $\square$

Corollary 10.14. — *For $P_j \in \mathcal{P}_{d_j}$, $1 \leqslant j \leqslant N$,*

$$\|P_1 \ldots P_N\|_{L^\infty(B)} \geqslant e^{-c_n(d_1+\cdots+d_N)} \|P_1\|_{L^\infty(B)} \ldots \|P_N\|_{L^\infty(B)}.$$

Proof. — Apply lemma 10.13 to each P_j and observe that

$$\int_{\partial B} \log|P_1 \ldots P_N| d\sigma = \sum_{1\leqslant j\leqslant N} \int_{\partial B} \log|P_j| d\sigma. \ \square$$

Theorem 10.15. — *For any set $E = \bigcup_{j\geqslant 1} E_j$, one has*

$$\frac{1}{c_n - \log T^\star(E)} \leqslant \sum_{j\geqslant 1} \frac{1}{|\log T^\star(E_j)|}.$$

Proof. — It is sufficient to check the inequality for a finite union $K = \bigcup K_j$ of compacts sets $K_j \subset B$, $1 \leqslant j \leqslant N$. Select $P_j \in \mathcal{P}_{d_j}$ such that

$$\|P_j\|_{L^\infty(B)} = 1, \qquad \|P_j\|_{L^\infty(K_j)} = M_{d_j}(K_j),$$

and set $P = P_1 \ldots P_N$, $d = d_1 + \cdots + d_N$. Then corollary 10.14 shows that $\|P_j\|_{L^\infty(B)} \geqslant e^{-c_n d}$, thus

$$M_d(K) \leqslant e^{c_n d} \|P\|_{L^\infty(K)}.$$

If $z \in K$ is in K_j, then $|P(z)| \leqslant |P_j(z)| \leqslant \|P_j\|_{L^\infty(K_j)}$ because all other factors are $\leqslant 1$. Thus

$$M_d(K) \leqslant e^{c_n d} \max\{\|P_j\|_{L^\infty(K_j)}\},$$

$$T(K) \leqslant M_d(K)^{1/d} \leqslant e^{c_n} \max\{M_{d_j}(K)^{1/d_j \cdot d_j/d}\}.$$

Take $d_j = [k\alpha_j]$ with arbitrary $\alpha_j > 0$ and let $k \to +\infty$. It follows that

$$T(K) \leqslant e^{c_n} \max\{T(K_j)^{\alpha_j/\alpha}\}$$

where $\alpha = \sum \alpha_j$. The inequality asserted in theorem 10.15 is obtained for the special choice $\alpha_j = 1/|\log T(K_j)| > 0$ which makes all terms in $\max\{\ldots\}$ equal.

11. Comparison of capacities and El Mir's theorem.

We first prove a comparison theorem for the capacities $c(\bullet, \Omega)$ and T, due to Alexander and Taylor [A–T].

THEOREM 11.1. — *Let K be a compact subset of the unit ball $B \subset \mathbb{C}^n$. Then*

(a) $T(K) \leqslant \exp(-c(K,B)^{-1/n})$.

For each $r < 1$, there is a constant $A(r)$ such that

(b) $T(K) \geqslant \exp(-A(r)\, c(K,B)^{-1})$ *when $K \subset B(0,r)$.*

Remark 11.2. — Both the set functions $c^\star(\bullet, \Omega)$ and $T^\star$ are generalized capacities. Hence, the estimates of the theorem also hold for all K-analytic sets, in particular all Borel sets.

Remark 11.3. — The inequalities are sharp, at least as far as the exponents on $c(K,B)$ are concerned. For if $K = \overline{B}(0,\varepsilon)$, then it is easy to check that $T(K) = \varepsilon$ and exercise 7.13 gives $c(K,B) = (\log 1/\varepsilon)^{-n}$. Hence, equality holds in (11.2). On the other hand, if K is a small polydisc

$$K = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \ ; \ |z_1| \leqslant \delta,\ |z_j| \leqslant 1/n,\ 1 < j \leqslant n\}$$

and $\delta \leqslant 1/n$, then $T(K) \leqslant \delta$, while $c(K,B) \geqslant C(\log 1/\delta)^{-1}$. To check the last inequality, put

$$u(z) = \Big(\log\frac{1}{\delta}\Big)^{-1} \log_+ \frac{|z_1|}{\delta} + (\log n)^{-1} \sum_{j=2}^{n} \log_+(n|z_j|).$$

Then $u = 0$ and $u \leqslant n$ on B, hence

$$c(K,B) \geqslant \int_B \Big(\frac{1}{n} dd^c u\Big)^n = \frac{n!}{n^n} (\log n)^{-(n-1)} (\log 1/\delta)^{-1}$$

because all measures $dd^c \log_+(|z_j|/r)$ have total mass 1 in $\mathbb{C}$.

Proof of 11.1 (a). — Set $M = \sup_B U_K^\star$; then $T(K) = e^{-M}$ by Siciak's theorem. Since $u = U_K^\star/M \in P(B)$ and $0 \leqslant u \leqslant 1$ on B, we get

$$c(K,B) \geqslant M^{-n} \int_K (dd^c U_K^\star)^n = M^{-n}$$

by theorem 10.3. This inequality is equivalent to 11.1 (a). $\square$

Proof of 11.1 (b). — Let $u_K^\star$ be the extremal function for K relative to the ball $B' = B(0,e) \supset\supset B$. For any $v \in P_{\log}(\mathbb{C}^n)$ such that $v \leqslant 0$ on K, we have $v \leqslant U_K^\star \leqslant M+1$ on B', hence the function

$$w = \frac{v - M - 1}{M+1}$$

satisfies $w \leqslant 0$ on B' and $w \leqslant -1$ on K. We infer $w \leqslant u_K^\star$; by taking the supremum over all choices of v, we get

$$u_K^\star \geqslant \frac{U_K^\star - M - 1}{M+1}.$$

Now, there is a point $z_0 \in \overline{B}$ such that $U_K^\star(z_0) = M$, thus

$$u_K^\star(z_0) \geqslant -\frac{1}{M+1}.$$

As $u_K^\star \leqslant 0$ on B', the mean value and Harnack inequalities show that

$$u_K^\star(z_0) \leqslant C_1 \int_{B'} u_K^\star d\lambda \Longrightarrow \|u_K^\star\|_{L^1(B')} \leqslant -\frac{1}{C_1} u_K^\star(z_0) \leqslant \frac{C_2}{M}.$$

The Chern-Levine-Nirenberg inequalities 1.3 and 1.4 (a) imply now

$$c(K, B') = \int_B (dd^c u_K^\star)^n \leqslant C_3 \|u_K^\star\|_{L^1(B')} \, \|u_K^\star\|_{L^\infty(B')}^{n-1} \leqslant \frac{C_4}{M} \, .$$

As $K \subset B(0,r) \subset\subset B$, theorem 6.5 (b) gives

$$c(K, B) \leqslant C_5(r) c(K, B') \leqslant A(r) M^{-1}$$

and inequality 1.11 (b) follows. □

We now prove El Mir's theorem [E–M]. This result is an effective version of Josefson's theorem : given a psh function in the ball, a subextension can be found with prescribed singularities of poles and slow growth at infinity.

El Mir's theorem 11.4. — *Let $v \in P(B)$ with $v \leqslant -1$, $\varepsilon \in \,]0, 1/n[$ and $r < 1$. Then there exists $u \in P_{\log}(\mathbb{C}^n)$ such that $u \leqslant -|v|^{\frac{1}{n}-\varepsilon}$ on $B(0,r)$.*

Proof. — For $t \geqslant 1$, set $G_t = \{z \in B(0,r); v(z) < -t\}$ and let $U_t^\star \in P_{\log}(\mathbb{C}^n)$ be the Siciak extremal function of G_t. Since G_t is open, we have $U_t^\star = 0$ on G_t. We set $M(t) = \sup_B U_t^\star$ and

$$u(z) = \varepsilon^{-1} \int_1^{+\infty} t^{-1-\varepsilon} (U_t^\star(z) - M(t)) dt.$$

Theorem 6.6 shows that $c(G_t, B) \leqslant C_1/t$, therefore

$$M(t) = -\log T^\star(G_t) \geqslant c(G_t, B)^{-1/n} \geqslant C_2 t^{1/n}$$

by inequality 1.11 (a). As $U_t^\star - M(t) \leqslant 0$ on B, we get $U_t^\star(z) - M(t) \leqslant \log_+|z|$ by logarithmic convexity, thus

$$u(z) \leqslant \log_+|z|.$$

For $z \in B(0,r)$ we have $U_t^\star(z) = 0$ as soon as $G_t \ni z$, *i.e.* $t < -v(z)$. Hence

$$u(z) \leqslant -\varepsilon^{-1} \int_1^{|v(z)|} t^{-1-\varepsilon} M(t) dt \leqslant -C_3 \int_1^{|v(z)|} t^{-1-\varepsilon+\frac{1}{n}} dt = -C_4 |v(z)|^{\frac{1}{n}-\varepsilon}.$$

Starting if necessary with a smaller value of ε and subtracting a constant to u, we can actually get

$$u \leqslant -|v|^{\frac{1}{n}-\varepsilon} \quad \text{on } B(0,r).$$

It remains to check that u is not identically $-\infty$. By logarithmic convexity again, we have

$$\sup_{\overline{B}(0,1/2)} U_t^\star \geqslant M(t) - \log 2$$

and there exists $z_0 \in S(0;1/2)$ such that $U_t^\star(z_0) - M(t) \geqslant -\log 2$. The Harnack inequality shows that

$$\frac{1-1/4}{(1+1/2)^{2n}} \int_{\partial B} \big(U_t^\star(z) - M(t)\big) d\sigma(z) \geqslant U_t^\star(z_0) - M(t) \geqslant -\log 2$$

and integration with respect to t yields

$$\int_{\partial B} u(z) d\sigma(z) \geqslant -4/3(3/2)^{2n} \log 2 > -\infty. \ \square$$

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