HOLOMORPHIC MORSE INEQUALITIES

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1. Introduction

Let M be a compact C^∞ manifold, $\dim_{\mathbb{R}} M = m$, and h a Morse function, *i.e.* a function such that all critical points are non degenerate. The standard Morse inequalities relate the Betti numbers $b_q = \dim H^q_{DR}(M, \mathbb{R})$ and the numbers

$$s_q = \#\ \text{critical points of index } q\ ,$$

where the index of a critical point is the number of negative eigenvalues of the Hessian form $(\partial^2 h/\partial x_i \partial x_j)$. Specifically, the following "strong Morse inequalities" hold :

$$b_q - b_{q-1} + \cdots + (-1)^q b_0 \leqslant s_q - s_{q-1} + \cdots + (-1)^q s_0 \tag{1.1}$$

for each integer $q \geqslant 0$. As a consequence, one recovers the "weak Morse inequalities" $b_q \leqslant s_q$ and the expression of the Euler-Poincaré characteristic

$$\chi(M) = b_0 - b_1 + \cdots + (-1)^m b_m = s_0 - s_1 + \cdots + (-1)^m s_m\ . \tag{1.2}$$

The purpose of these lectures is to explain what are the complex analogues of these inequalities for $\overline{\partial}-$cohomology groups with values in holomorphic line (or vector) bundles, and to present a few applications.

Let X be a compact complex manifold, $n = \dim_{\mathbb{C}} X$ and E, F holomorphic vector bundles over X with

$$\operatorname{rank} E = 1\ ,\ \operatorname{rank} F = r\ .$$

We denote here $h^q(F) = \dim H^q(X, \mathcal{O}(F))$.

Assume that E is endowed with a C^∞ hermitian metric and denote by $c(E)$ its curvature form. Then $ic(E)$ is a real $(1,1)-$form on X (*cf.* §2). Finally, consider the $q-$index sets

$$X(q,E)=\left\{\begin{matrix} x\in X\ ;\ ic(E)_x \text{ has } & q \quad \text{negative eigenvalues} \\ & n-q \text{ positive eigenvalues}\end{matrix}\right\}\ ,$$
$$X(\leqslant q,E)=\bigcup_{1\leqslant j\leqslant q} X(j,E)\ .$$

Observe that $X(q,E)$ and $X(\leqslant q,E)$ are open subsets of X .

Main Theorem. — *The sequence of tensor powers $E^k\otimes F$ satisfy the following asymptotic estimates as $k\to+\infty$:*

(1.3) Weak Morse inequalities *:*

$$h^q(E^k\otimes F)\leqslant r\frac{k^n}{n!}\int_{X(q,E)}(-1)^q\big(\frac{i}{2\pi}c(E)\big)^n+o(k^n)\ .$$

(1.4) Strong Morse inequalities *:*

$$\sum_{0\leqslant j\leqslant q}(-1)^{q-j}h^j(E^k\otimes F)\leqslant r\frac{k^n}{n!}\int_{X(\leqslant q,E)}(-1)^q\big(\frac{i}{2\pi}c(E)\big)^n+o(k^n)\ .$$

(1.5) Asymptotic Riemann-Roch formula *:*

$$\chi(E^k\otimes F)=r\frac{k^n}{n!}\int_X\big(\frac{i}{2\pi}c(E)\big)^n+o(k^n)\ .$$

Observe that (1.5) is in fact a weak consequence of the Hirzebruch-Riemann-Roch formula.

The above theorem was first proved in [De 2] in 1985. It was largely motivated by Siu's solution of the Grauert-Riemenschneider conjecture ([Siu 2], 1984), which we will reprove below as a special case of a stronger statement. The basic tool is a spectral theorem which describes the eigenvalue distribution of the complex Laplace-Beltrami operators. The original proof of [De 2] was based partly on Siu's techniques and partly on an extension of Witten's analytic proof [Wi, 1982] for the standard Morse inequalities. Somewhat later Bismut [Bi] and quite recently Getzler [Ge 3] gave new proofs, both relying on an analysis of the heat kernel in the spirit of Atiyah-Bott-Patodi's proof of the Atiyah-Singer index theorem [A-B-P]. Although the basic idea is simple, Bismut used deep results arising from probability theory (the Malliavin calculus), while Getzler relied on his supersymmetric symbolic calculus for spin pseudodifferential operators [Ge 1].

We will try to present here a simple heat equation proof, based essentially on Mehler's formula and elementary asymptotic estimates (*cf.* §3 and §4). The next sections deal with various generalizations and applications :

- in §5, we obtain as a consequence a strong characterization of Moishezon varieties in terms of the existence of line bundles satisfying some integral positivity hypothesis of the curvature. In particular, this gives a solution of the Grauert-Riemenschneider conjecture. We also give a generalization to all projective algebraic manifolds of G. Kempf's distortion inequalities for ample line bundles over abelian varieties [Kem].
- in §6, the case of q-convex manifolds is considered. As shown by Thierry Bouche [Bou 1], Morse inequalities then hold in degrees $\geqslant q$. By an argument of Siu [Siu 3,4], these inequalities imply a general a priori estimate for the Monge-Ampère operator.
- in §7, we present (with a different and more elementary proof) Getzler's extension [Ge 1] of holomorphic Morse inequalities to the vector bundle case.
- §8 briefly discusses the degenerate case when $ic(E)$ has rank $< \dim X$ everywhere, as well as related open problems.

2. Hermitian connections, curvature and Laplace-Beltrami operators

Let F be a complex vector bundle of rank r over a smooth differentiable manifold M. We denote by $C^\infty_q(M,F)$ the space of smooth $q-$forms with values in F, *i.e.* smooth sections of $\Lambda^q T^\star M \otimes F$. To fix notations, we recall here the basic terminology used.

A *connection* D on F is a linear differential operator

$$D : C^\infty_q(M,F) \to C^\infty_{q+1}(M,F)$$

such that

$$D(f \wedge u) = df \wedge u + (-1)^{\deg f} f \wedge Du \tag{2.1}$$

for all forms $f \in C^\infty_p(X,\mathbb{C})$, $u \in C^\infty_q(X,F)$. On an open set $\Omega \subset M$ where F is trivial, $F_{\restriction\Omega} \simeq \Omega \times \mathbb{C}^r$, a connection D can be written

$$Du = du + \Gamma \wedge u$$

where $\Gamma \in C^\infty_1(\Omega, \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^r))$ is an arbitrary matrix of $1-$forms and d acts componentwise. It is then easy to check that

$$D^2u = (d\Gamma + \Gamma \wedge \Gamma) \wedge u \text{ on } \Omega \ ,$$

so that $D^2u = c(D) \wedge u$ for some global $2-$form

$$c(D) \in C^\infty_2(M, \mathrm{Hom}(F,F))$$

called the curvature of D.

Assume now that F is endowed with a C^∞ hermitian metric along the fibers and that the isomorphism $F_{\restriction\Omega} \simeq \Omega \times \mathbb{C}^r$ is given by a C^∞ frame (e_λ). We then have a canonical sesquilinear pairing

$$\begin{array}{ccc} C^\infty_p(M,F) & \longrightarrow & C^\infty_{p+q}(M,\mathbb{C}) \\ (u,v) & \longmapsto & \{u,v\} \end{array}$$

given by

$$\{u,v\} = \sum_{\lambda,\mu} u_\lambda \wedge \overline{v}_\mu \langle e_\lambda, e_\mu\rangle \quad \text{for } u = \sum u_\lambda \otimes e_\lambda \ , \ v = \sum v_\mu \otimes e_\mu \ .$$

The connection D is said to be *hermitian* if it satisfies the additional property

$$d\{u,v\} = \{Du,v\} + (-1)^{\deg u}\{u,Dv\} \ . \tag{2.2}$$

Assuming that (e_λ) is orthonormal, one easily checks that D is hermitian if and only if $\Gamma^* = -\Gamma$. In this case $c(D)^* = -c(D)$, thus

$$ic(D) \in C_2^\infty(M, \mathrm{Herm}(F,F)) \ .$$

(2.3) *Special case.* For a bundle E of rank 1, the connection form Γ of a hermitian connection D can be written $\Gamma = -iA$ where A is a *real* 1−form. Then we have $c(D) = d\Gamma = -idA$ and we will denote

$$B = ic(D) = dA \ .$$

A phase change $u = ve^{i\theta}$ in the isomorphism $E_{|\Omega} \simeq \Omega \times \mathbb{C}$ replaces A by the new connection form $A + d\theta$.

(2.4) *Complex analytic case.* If M is a complex manifold X, every connection D can be splitted in a unique way as a sum of a $(1,0)$ and of a $(0,1)$−connection :

$$D = D' + D'' \ .$$

In a local trivialization given by a C^∞ frame, one can write

$$D'u = d'u + \Gamma' \wedge u \ ,$$
$$D''u = d''u + \Gamma'' \wedge u \ ,$$

with $\Gamma = \Gamma' + \Gamma''$. The connection is hermitian if and only if $\Gamma' = -(\Gamma'')^*$ in any orthonormal frame. Thus there exists a unique hermitian connection corresponding to a prescribed $(0,1)$ part D''.

Assume now that the bundle E itself has a *holomorphic* structure. The unique hermitian connection for which $D'' = \overline{\partial}$ is called the *Chern connection* of F. In a local holomorphic frame (e_λ) of $E_{|\Omega}$, the metric is given by some hermitian matrix $H = (h_{\lambda\mu})$ where $h_{\lambda\mu} = \langle e_\lambda, e_\mu\rangle$. Easy computations yield the expression of the Chern connection :

$$\begin{cases} D'u &= \partial u + \overline{H}^{-1}\partial\overline{H} \wedge u \\ d''u &= \overline{\partial} u \\ c(F) &\underset{\text{def}}{=} c(D) = -\overline{\partial}(\overline{H}^{-1}\partial\overline{H}) \ . \end{cases}$$

For a rank 1 bundle E, the matrix H is simply a positive weight function $e^{-\varphi}$, $\varphi \in C^\infty(\Omega, \mathbb{R})$, and we find

$$c(E) = \partial\overline{\partial}\varphi \ .$$

(2.5) *Hodge theory*. Assume now that M is a riemannian manifold with metric $g = \sum g_{ij} dx_i \otimes dx_j$. Given a $q-$form u on M with values in F , we consider the global L^2 norm

$$\|u\|^2 = \int_M |u(x)|^2 d\sigma(x)$$

where $|u|$ is the pointwise hermitian norm and $d\sigma$ the riemannian volume form. The Laplace Beltrami operator associated to the connection D is

$$\Delta = DD^* + D^*D$$

where D is the (formal) adjoint of D; the complex Laplace operators Δ' and Δ'' are defined similarly. In degree 0 we simply have $\Delta = D^*D$.

When M is compact, the elliptic operator Δ has a discrete spectrum

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots$$

and corresponding eigenfunctions $\psi_j \in C_q^\infty(M, F) \cap L^2$. Our main goal is to obtain asymptotic formulae for the eigenvalues. For that, we make use of the *heat operator* $e^{-t\Delta}$. In the above setting, the heat operator is the bounded hermitian operator associated to the *heat kernel*

$$K_t(x, y) = \sum_{j=1}^{+\infty} e^{-\lambda_j t} \psi_j(x) \otimes \psi_j^*(y) \ ,$$

and $K_t \in C^\infty(]0, +\infty[\times M \times M, \mathrm{Hom}(F, F))$. The asymptotic distribution of eigenvalues can be recovered from the well known (obvious) formula

$$\sum_{j=1}^{+\infty} e^{-\lambda_j t} = \int_M \mathrm{tr}_F K_t(x, x) d\sigma(x) \ .$$

3. Asymptotic formulas for the heat kernel and the eigenvalue distribution

Let $E, F \to M$ be complex vector bundles equipped with hermitian connections, D_k the associated connection on $E^k \otimes F$ and $\Delta_k = D_k^* D_k$ the Laplace-Beltrami operator acting on sections of $E^k \otimes F$ (degree 0). Finally, let $V \in C^\infty(M, \mathrm{Herm}\ (F, F))$; we still denote V the operator $\mathrm{Id}_{E^k} \otimes V$ acting on $E^k \otimes F$.

If $\Omega \subset\subset M$ is a smoothly bounded relatively compact open subset of M, we consider the quadratic form

$$Q_{k,\Omega}(u) = \int_\Omega \frac{1}{k} |D_k u|^2 - \langle Vu, u \rangle$$

with domain equal to the Sobolev space $W_0^1(\Omega, E^k \otimes F)$ = closure of the space of smooth sections with compact support in Ω, in the space $W_{\mathrm{loc}}^1(M, E^k \otimes F)$ of sections which have $L^2_{\rm loc}$ coefficients as well as their first derivatives. That is, we consider the self adjoint operator

$$\Box_k = \frac{1}{k} D_k^* D_k - V$$

densely defined in the Hilbert space $W^1_0(\Omega, E^k \otimes F)$ (Dirichlet boundary conditions).

We want to study the asymptotic eigenvalue distribution of $\Box_k$ as $k \to +\infty$, as well as an asymptotic formula for the heat kernel $e^{-t\Box_k}$. The basic idea is extremely simple and reduces the proof into two steps :

- show that the asymptotic estimates are purely local (up to error terms of lower order) and can be obtained by freezing the coeffiicents of the operators involved at any given point.
- compute explicitly the heat kernel in the case of connections with constant curvature, assuming that M has a flat metric.

In order to see that the situation is local, let (ψ_j) be a partition of unity relative to an arbitrarily fine covering of $\overline{\Omega}$, such that $\sum \psi_j^2 = 1$ near $\overline{\Omega}$. As $\sum \psi_j d\psi_j = 0$ on Ω, we then find

$$\sum_j Q_{k,\Omega}(\psi_j u) - Q_{k,\Omega}(u) = \frac{1}{k} \int_\Omega (\sum |d\psi_j|^2)|u|^2 \leqslant O(\frac{1}{k})|u|^2 \ .$$

By the minimax principle, it follows that the eigenvalues are shifted by at most $O(1/k)$, thus the asymptotic distribution is not modified as $k \to +\infty$.

Now, let $x^0 \in M$ be a given point. We can choose coordinates $(x_1, \ldots, x_m)$ centered at x^0 such that $(\partial/\partial x_1, \ldots, \partial/\partial x_m)$ is orthonormal at x^0 and such that $B = ic(E)$ is "diagonal" at x^0 :

$$B(x^0) = \sum_{j=1}^{s} B_j dx_j \wedge dx_{j+s} \ ;$$

here $2s \leqslant m$ is the rank of the skew-symmetric $2-$form $B(x^0)$, which may depend on x^0, and $B_1 \geqslant B_2 \geqslant \cdots \geqslant B_s \geqslant 0$ are its non-zero eigenvalues.

Let us consider the operators obtained by "freezing" the coefficients at x^0. More specifically, we assume that

- E has constant curvature $B = \sum_{j=1}^s B_j dx_j \wedge dx_{j+s}$. Then there is a local trivialization in which
$$D_E u = du - iA \wedge u \ ,$$
$$A = \sum_{j=1}^{s} B_j x_j dx_{j+s} \ .$$
- D_F is flat.
- $\Omega \simeq \mathbb{R}^n$ and the metric g is flat : $g = \sum dx_j \otimes dx_j$.

- the hermitian form V is constant. We choose an orthonormal frame of F in which V is diagonal, *i.e.*

$$\langle Vu, u\rangle = \sum_{1\leqslant\lambda\leqslant r} V_\lambda |u_\lambda|^2 \ .$$

The connection D_k on $E^k \otimes F$ can then be written $D_k u = du - ikA\wedge u$ and the quadratic form $Q_{k,\Omega}$ is given by

$$Q_{k,\Omega}(u) = \int_{\mathbb{R}^m} \frac{1}{k}\left[\sum_{\substack{1\leqslant j\leqslant s\\ 1\leqslant\lambda\leqslant r}} \left(\Big|\frac{\partial u_\lambda}{\partial x_j}\Big|^2 + \Big|\frac{\partial u_\lambda}{\partial x_{j+s}} - ikB_j x_j u_\lambda\Big|^2\right) + \sum_{\substack{j>2s\\ 1\leqslant\lambda\leqslant r}} \Big|\frac{du_\lambda}{dx_j}\Big|^2\right] - \sum_{1\leqslant\lambda\leqslant r} V_\lambda |u_\lambda|^2 \ .$$

In this situation, $Q_{k,\Omega}$ is a direct sum of quadratic forms acting on each component u_λ and the computation of $e^{-t\Box_k}$ is reduced to the following simple cases $(a), (b)$:

(a)
$$Q(f) = \int_{\mathbb{R}} \Big|\frac{df}{dx}\Big|^2 \ , \quad \Box f = -\frac{d^2 f}{dx^2}$$

As is well known the heat kernel is given in this case by

$$K_t(x,y) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} \ .$$

(b)
$$Q(f) = \int_{\mathbb{R}^2} \Big|\frac{df}{dx_1}\Big|^2 + \Big|\frac{df}{dx_2} - iax_1 f\Big|^2 \ .$$

A partial Fourier transform in the x_2 variable gives

$$Q(f) = \int_{\mathbb{R}^2} \Big|\frac{d\hat{f}}{dx_1}(x_1,\xi_2)\Big|^2 + a^2(x_1 - \frac{\xi_2}{a})^2 |\hat{f}(x_1,\xi_2)|^2$$

and the change of variables $x_1' = x_1 - x_2/a$, $x_2' = \xi_2$ leads to the so called "harmonic oscillator" energy functional

$$q(g) = \int_{\mathbb{R}} \Big|\frac{dg}{dx}\Big|^2 + a^2x^2|g|^2 \ ,$$
$$\Box = -\frac{d^2}{dx^2} + a^2x^2 \ .$$

The heat kernel of this operator is given by Mehler's formula :

$$k_t(x,y) = \sqrt{\frac{a}{2\pi\sinh 2at}} \exp\Big(-\frac{a}{2}(\coth 2at)(x-y)^2 - a(\tanh at)xy\Big) \ .$$

One way of obtaining this relation is to observe that the eigenfunctions of $\Box$ are

$$(2^p p! \sqrt{\frac{\pi}{a}})^{-1/2} \Phi_p(\sqrt{a}x) \ , \ p = 0,1,2,\ldots \ ,$$

with associated eigenvalues $(2p+1)a$, where (Φ_p) is the sequence of functions associated to Hermite polynomials :

$$\Phi_p(x) = e^{x^2/2}\frac{d^p}{dx^p}(e^{-x^2}) \ .$$

Therefore we have

$$k_t(x,y) = \sqrt{\frac{a}{\pi}}e^{a(x^2+y^2)/2}\sum_{p=0}^{+\infty}\frac{e^{-(2p+1)at}}{2^p p! a^p}\frac{d^p}{dx^p}(e^{-ax^2})\frac{d^p}{dy^p}(e^{-ay^2})$$

and the summation $\Sigma(x,y)$ can be computed from its Fourier transform

$$\widehat{\Sigma}(\xi,\eta) = e^{-at}\exp(-\frac{1}{2a}e^{-2at}\xi\eta)\cdot\frac{1}{\sqrt{2a}}e^{-(\xi^2+\eta^2)/4a} \ .$$

The heat kernel operator of Q is thus given by

$$(e^{-t\Box}f)^\wedge(x_1,\xi_2) = \int_{\mathbb{R}} k_t\big(x_1-\frac{\xi_2}{a}, y_1-\frac{\xi_2}{a}\big)\hat{f}(y_1,\xi_2)dy_1 \ .$$

By an inverse Fourier transform we obtain the desired heat kernel :

$$k_t(x_1,x_2;y_1,y_2) + \frac{a}{4\pi\sinh at}\exp\Big(-\frac{a}{4}(\coth at)\big((x_1-y_1)^2+(x_2-y_2)^2\big)\Big) \\ \times\exp\big(\frac{i}{2}a(x_1+y_1)(x_2-y_2)\big) \ .$$

The heat kernel associated to a sum of (pairwise commuting) operators $\Box_1,\ldots,\Box_m$ acting on disjoint sets of variables is the product of all heat kernels $e^{-t\Box_j}$. Let K_t^λ be the heat kernel of $Q_{k,\Omega}$ acting on a single component u_λ. The factor in the heat kernel corresponding to the pair of variables (x_j, x_{j+s}), $1\leqslant j\leqslant s$) is obtained when substituting kB_j to a and t/k to t. Therefore

$$K_t^\lambda(x,y) = \prod_{j=1}^{s}\frac{kB_j}{4\pi\sinh B_j t}\exp\Big(-\frac{kB_j}{4}(\coth B_j t)\big((x_{2j-1}-y_{2j-1})^2+(x_{2j}-y_{2j})^2\big) \\ +\frac{i}{2}kB_j(x_{2j-1}+y_{2j-1})(x_{2j}-y_{2j})\Big) \\ \times e^{tV_\lambda}\times\frac{1}{(4\pi t/k)^{m-2s}/2}\exp\big(-k\sum_{j>2s}(x_j-y_j)^2/4t\big) \ .$$

On the diagonal of M, the global heat kernel K_t is thus given by

$$K_t(x,x) = k^{m/2}\frac{e^{tV}}{(4\pi)^{m/2}t^{m/2-s}}\prod_{j=1}^{s}\frac{B_j}{\sinh B_j t} \ .$$

THEOREM 3.1. — *In the variable coefficient case, the heat kernel of $\Box_k$ admits the asymptotic estimate*

$$K_t^k(x,x) \sim k^{m/2}\frac{e^{tV(x)}}{(4\pi)^{m/2}t^{m/2-s}}\prod_{j=1}^{s}\frac{B_j(x)}{\sinh B_j(x)t}$$

as $k \to +\infty$, where $\sim$ is uniform with respect to $x \in M$ and t in a bounded interval $[t_0, t_1] \subset]0, +\infty[$.

Proof. — The only thing one has to get convinced of is that $e^{-t\square_k} - e^{-t\square_k^0}$ is $o(k^{m/2})$ at the point $(x^0, x^0) \in M \times M$, where $\square_k^0$ is the operator $\square_k$ "freezed" at x^0. This can be cheecked by means of the well known formula

$$e^{-t\square_k} - e^{-t\square_k^0} = \int_0^1 e^{-(1-u)t\square_k}(\square_k^0 - \square_k)e^{-ut\square_k^0} du \ ,$$

once the singularity of $e^{-t\square_k}$ along the diagonal is known. We also use the estimate

$$\square_k - \square_k^0 = O\Big(\frac{1}{k}|x - x^0|\nabla^2 + \big(\frac{1}{k} + |x - x^0|^2\big)\nabla + |x - x^0| + \frac{1}{k}\Big) \ . \square$$

By the localization argument already discussed, we obtain as a consequence the following estimate for the eigenvalues :

COROLLARY 3.2. — *The eigenvalues $\lambda_j^{k,\Omega}$ of $Q_{k,\Omega}$ satisfy for every $t > 0$ the estimate*

$$\sum_{j=1}^{+\infty} e^{-t\lambda_j^{k,\Omega}} \sim k^{m/2} \int_\Omega \frac{\mathrm{tr}(e^{tV(x)})}{(4\pi)^{m/2}t^{m/2-s}} \prod_{j=1}^{s} \frac{B_j(x)}{\sinh B_j(x)t} d\sigma(x) \ .$$

This result can be also interpreted in terms of the counting function

$$N_{k,\Omega}(\lambda) = \#\{j \ ; \ \lambda_j^{k,\Omega} \leqslant \lambda\}$$

and of the spectral density measure (a sum of Dirac measures)

$$\mu_{k,\Omega} = k^{-m/2}\frac{d}{d\lambda}N_{k,\Omega}(\lambda) \ .$$

In these notations, corollary 3.2 can be restated :

$$\lim_{k\to+\infty} \int_{-\infty}^{+\infty} e^{-t\lambda} d\mu_{k,\Omega}(\lambda) = \int_\Omega \frac{\mathrm{tr}(e^{tV(x)})}{(4\pi)^{m/2}t^{m/2-s}} \prod_{j=1}^{s} \frac{B_j(x)}{\sinh B_j(x)t} d\sigma(x) \ .$$

We thus see that the sequence of measures $\mu_{k,\Omega}$ converges weakly to a measure μ_Ω whose Laplace transform is given by the right hand side. Inverting the formula, one obtains :

COROLLARY 3.3. — *For almost all $\lambda \in \mathbb{R}$*

$$\lim_{k\to+\infty} k^{-m/2}N_{k,\Omega}(\lambda) = \mu_\Omega(]-\infty, \lambda]) = \int_\Omega \sum_{j=1}^{r} \nu_{B(x)}(V_j(x) + \lambda) d\sigma(x)$$

where $\nu_{B(x)}(\lambda)$ is the function on $M \times \mathbb{R}$ defined by

$$\nu_B(\lambda) = \frac{2^{s-m}\pi^{-m/2}}{\Gamma(\frac{m}{2} - s + 1)} B_1 \cdots B_s \sum_{(p_1,\ldots,p_s)\in\mathbb{N}^s} [\lambda - \Sigma(2p_j + 1)B_j]_+^{\frac{m}{2}-s} \ .$$

4. Proof of the holomorphic Morse inequalities

Let X be a compact complex manifold, E and F holomorphic hermitian vector bundles of rank 1 and r over X. If X is endowed with a hermitian metric ω, Hodge theory shows that the Dolbeault cohomology group $H^q(X, E^k \otimes F)$ can be identified with the space of harmonic $(0,q)-$forms with respect to the Laplace-Beltrami operator $\Delta''_k = \overline{\partial}_k \overline{\partial}_k^* + \overline{\partial}_k^* \overline{\partial}_k$ acting on $E^k \otimes F$. We thus have to estimate the zero-eigenspace of Δ''_k .

In order to apply corollary 3.2, we first have to compute Δ''_k in terms of the hermitian connection ∇_k on $E^k \otimes F \otimes \Lambda^{0,q} T^* X$ deduced from the Chern connections of E, F, TX. What plays now the role of F is the bundle $F \otimes \Lambda^{0,q} T^* X$.

The relationship between Δ''_k and ∇_k is most easily obtained by means of the Bochner-Kodaira-Nakano identity. In order to simplify the exposition, we assume here that the metric ω on X is *Kähler*. For any hermitian holomorphic line bundle G on X, the operators Δ' and Δ'' of G are related by the B-K-N identity

$$\Delta'' = \Delta' + [ic(G), \Lambda] \ . \tag{4.1}$$

Here we have $c(E^k \otimes F) = kc(E) \otimes \mathrm{id}_F + c(F)$, thus

$$\Delta''_k = \Delta'_k + k[ic(E), \Lambda] + [ic(F), \Lambda] \ .$$

At a given point $z^0 \in X$, we can find a coordinate system $(z_1, \ldots, z_n)$ such that $(\partial/\partial z_j)$ is an orthonormal basis of TX and

$$ic(E) = \frac{i}{2} \sum_{1 \leqslant j \leqslant n} \alpha_j dz_j \wedge d\overline{z}_j$$

where $\alpha_1, \ldots, \alpha_n$ are the curvature eigenvalues of $c(E)$ at z^0. A standard formula gives the expression of the curvature term $[ic(E), \Lambda]u$ for any $(p,q)-$form u. With $u = \sum u_{I,J,\lambda} dz_I \wedge d\overline{z}_J \otimes e_\lambda$, we have

$$\langle [ic(E), \Lambda]u, u \rangle = \sum_{I,J,\lambda} (\alpha_J - \alpha_{\complement I}) |u_{I,J,\lambda}|^2$$

where $\alpha_J = \sum_{j \in J} \alpha_j$. In the case of a $(O,q)-$form $u = \sum u_{J,\lambda} d\overline{z}_J \otimes e_\lambda$ we simply have $\Delta'_k u = D'^*_k D'_k u = \nabla'^*_k \nabla'_k u$ and

$$\Delta''_k = \nabla'^*_k \nabla'_k - kV' + [ic(F), \Lambda] \ , \tag{4.2'}$$

$$\langle V'u, u \rangle = \sum_{J,\lambda} \alpha_{\complement J} |u_{J,\lambda}|^2 \quad \text{(here } I = \emptyset) \ .$$

This is not yet what was needed, since only the $(1,0)$ part ∇'_k appears. To get the $(0,1)$ component, we consider u as a (n,q) form with values in $E^k \otimes F \otimes \Lambda^n TX$. We then get $\Delta'_k u = D'_k D'^*_k u$ where

$$D'^*_k u = -\sum \partial u_{I,J,\lambda} / \partial \overline{z}_j dz_1 \wedge \cdots \widehat{dz_j} \cdots \wedge dz_n \wedge d\overline{z}_J \otimes e_\lambda$$

in normal coordinates. Thus $\Delta'_k u = \nabla''^*_k \nabla''_k u$ and

$$\Delta''_k = \nabla''^*_k \nabla''_k + kV'' + [ic(F \otimes \Lambda^n TX), \Lambda] \ , \tag{4.2''}$$

$$\langle V''u,u\rangle=\sum_{J,\lambda}\alpha_J|u_{J,\lambda}|^2\qquad(\text{here } I=\{1,\ldots,n\})\ .$$

If the metric ω is non Kähler, we get additional torsion terms, but these terms are independent of k. A combination of $(4.2')$ and $(4.2'')$ yields

$$(4.3)\qquad \frac{2}{k}\Delta''_k=\frac{1}{k}\nabla_k^*\nabla_k-V+\frac{1}{k}\Theta$$

where Θ is a hermitian form independent of k and

$$\langle Vu,u\rangle=\sum_{J,\lambda}(\alpha_{\complement J}-\alpha_J)|u_{J,\lambda}|^2\ .$$

Now apply theorem 3.1 and observe that Θ does not give any significant contribution to the heat kernel as $k\to+\infty$. We write here $z_j=x_j+iy_j$, so that

$$B=ic(E)=\sum_{1\leqslant j\leqslant n}\alpha_j dx_j\wedge dy_j\ .$$

The curvature eigenvalues are given by $B_j=|\alpha_j|$. We denote $s=s(x)$ the rank of $B(x)$ and order the eigenvalues so that

$$|\alpha_1|\geqslant\cdots\geqslant|\alpha_s|>0=\alpha_{s+1}=\cdots=\alpha_n\ .$$

The eigenvalues of V acting on $F\otimes\Lambda^nT^*X$ are the coefficients $\alpha_{\complement J}-\alpha_J$, counted with multiplicity r . Therefore

THEOREM 4.4. — *The heat kernel associated to $e^{-\frac{2t}{k}\Delta''_k}$ in bidegree $(0,q)$ satisfies*

$$K_t^k(x,x)\sim k^n\frac{r\sum_{|J|=q}e^{t(\alpha_{\complement J}(x)-\alpha_J(x))}}{(4\pi)^nt^{n-s}}\prod_{j=1}^{s}\frac{|\alpha_j(x)|}{\sinh|\alpha_j(x)|t}$$

as $k\to+\infty$. In particular, if $\lambda_1^{k,q}\leqslant\lambda_2^{k,q}\leqslant\cdots$ are the eigenvalues of $\frac{1}{k}\Delta''_k$ in bidegree $(0,q)$, we have

$$\sum_{j=1}^{+\infty}e^{-2t\lambda_j^{k,q}}\sim rk^n\sum_{|J|=q}\int_X\frac{e^{t(\alpha_{\complement J}(x)-\alpha_J(x))}}{(4\pi)^nt^{n-s}}\prod_{j=1}^{s}\frac{|\alpha_j(x)|}{\sinh|\alpha_j(x)|t}$$

for every $t>0$.

At this point, the main idea is to use the eigenspaces in order to construct a finite dimensional subcomplex of the Dolbeault complex with the same cohomology groups. This was already the basic idea in Witten's analytic proof of the standard Morse inequalities. We denote by

$$\mathcal{H}_\lambda^{k,q}\quad,\quad\text{resp.}\ \mathcal{H}_{\leqslant\lambda}^{k,q}$$

the $\lambda-$eigenspace of $\frac{1}{k}\Delta''_k$ acting on $C^\infty_{0,q}(X,E^k\otimes F)$, resp. the direct sum of eigenspaces corresponding to all eigenvalues $\leqslant\lambda$. As $\overline{\partial}_k$ and Δ''_k commute, we see that $\overline{\partial}(\mathcal{H}_\lambda^{k,q})\subset\mathcal{H}_\lambda^{k,q+1}$, thus $\mathcal{H}_\lambda^{k,\bullet}$ and $\mathcal{H}_{\leqslant\lambda}^{k,\bullet}$ are finite dimensional subcomplexes of the Dolbeault complex

$$\overline{\partial}\ :C^\infty_{0,\bullet}(X,E^k\otimes F)\ .$$

Since $\overline{\partial}_k\overline{\partial}_k^* + \overline{\partial}_k^*\overline{\partial}_k = \Delta_k'' = k\lambda\,\mathrm{id}$ on $\mathcal{H}_\lambda^{k,\bullet}$, we see that $\mathcal{H}_\lambda^{k,\bullet}$ has trivial cohomology for $\lambda \neq 0$. Since $\mathcal{H}_0^{k,\bullet}$ is the space of harmonic forms, we see that $\mathcal{H}_{\leqslant\lambda}^{k,\bullet}$ has the same cohomology as the Dolbeault complex for $\lambda > 0$. We will call this complex the Witten $\overline{\partial}-$complex. We need an elementary lemma of linear algebra.

Lemma 4.5. — *Set $h_k^q = \dim H^q(X, E^k \otimes F)$. Then for every $t > 0$*

$$h_k^q - h_k^{q-1} + \cdots + (-1)^q h_k^0 \leqslant \sum_{\ell=0}^{q} (-1)^{q-\ell} \sum_{j=1}^{+\infty} e^{-t\lambda_j^{k,\ell}} .$$

Proof. — The left hand side is the contribution of the 0 eigenvalues in the right hand side. All we have to check is that the contribution of the other eigenvalues is $\geqslant 0$. The contribution of the eigenvalues such that $\lambda_j^{k,\ell} = \lambda > 0$ is

$$e^{-t\lambda} \sum_{\ell=0}^{q} (-1)^{q-\ell} \dim \mathcal{H}_\lambda^{k,\ell} .$$

As $\mathcal{H}_\lambda^{k,\bullet}$ is exact, one easily sees that the last sum is equal to the dimension of $\overline{\partial}\mathcal{H}_\lambda^{k,q} \subset \mathcal{H}_\lambda^{k,q+1}$, hence $\geqslant 0$. $\square$

Combining theorem 4.4 with lemma 4.5, we get

$$h_k^q - h_k^{q-1} + \cdots + (-1)^q h_k^0 \leqslant o(k^n) + rk^n \sum_{\ell=0}^{q} (-1)^{q-\ell} \sum_{|J|=\ell} \int_X \frac{\prod_{j\leqslant s} |\alpha_j| \cdot e^{t(\alpha_{\complement J} - \alpha_J - \sum |\alpha_j|)}}{2^{2n-s}\pi^n t^{n-s} \prod_{j\leqslant s}(1 - e^{-2t|\alpha_j|})} .$$

This inequality is valid for any $t > 0$, so we can let t tend to $+\infty$. It is clear that $\alpha_{\complement J} - \alpha_J - \sum |\alpha_j|$ is always $\leqslant 0$, thus the integrand tends to 0 at every point where $s < n$. When $s = n$, we have $\alpha_{\complement J}(x) - \alpha_J x) - \sum |\alpha_j(x)| = 0$ if and only if $\alpha_j(x) > 0$ for every $j \in \complement J$ and $\alpha_j(x) < 0$ for every $j \in J$. This implies $x \in X(\ell, E)$; in this case there is only one multi-index J satisfying the above conditions and the limit is $(2\pi)^{-n}|\alpha_1 \cdots \alpha_n|$. By the monotone convergence theorem, our sum of integrals converges to

$$\sum_{\ell=0}^{q} (-1)^{q-\ell} \int_{X(\ell,E)} (2\pi)^{-n} |\alpha_1 \cdots \alpha_n| d\sigma = \frac{1}{n!} \int_{X(\leqslant q,E)} (-1)^q \left(\frac{i}{2\pi} c(E)\right)^n .$$

5. Applications to algebraic geometry

Let E be a holomorphic line bundle over a compact connected complex manifold X of dimension n and $V_k = H^0(X, E^k)$. If $Z(V_k)$ denotes the set of common zeroes of all sections in V_k, there is a natural holomorphic map

$$\Phi_k \ : X \backslash Z(V_k) \longrightarrow \mathbb{P}(V_k^*)$$

which sends a point $x \in X \backslash Z(V_k)$ to the hyperplane $H \subset V_k$ of sections $\sigma \in V_k$ such that $\sigma(x) = 0$.

When E is > 0, one can construct many sections of high tensor powers E^k (e.g. by Hörmander's L^2 estimates). For $k \geqslant k_0$ large enough, the "base locus" $Z(V_k)$ is empty, the sections in V_k separate any two points of X and generate all 1-jets at any point. Then Φ_k gives an embedding of X in some projective space $\mathbb{P}^N$.This gives the famous *Kodaira embedding theorem* : a compact complex manifold X is projective algebraic if and only if X has a hermitian line bundle E with positive curvature.

The Grauert-Riemenschneider conjecture [G-R] was an attempt to characterize the more general class of Moishezon varieties in terms of semi-positive line bundles. Let us first recall a few definitions. The algebraic dimension $a(X)$ is the transcendence degree of the field $\mathcal{M}(X)$ of meromorphic functions on X. A well-known theorem of Siegel asserts that $0 \leqslant a(X) \leqslant n$. A manifold (or variety) X is said to be *Moishezon* if $a(X) = n$.

Grauert-Riemenschneider conjecture (1970). — *A compact complex variety Y is Moishezon if and only if there is a proper non singular modification $X \to Y$ and a line bundle E over X such that the curvature is > 0 on a dense open subset.*

When Y is Moishezon, it is well known that there exists a projective algebraic modification X; therefore E can even be taken > 0 everywhere on such an X. Siu [Siu 2] solved the conjecture by proving the converse statement in 1984; he even showed that X is Moishezon as soon as $ic(E) \geqslant 0$ everywhere and $ic(E) > 0$ in at least one point. We will see that Morse inequalities give a still stronger criterion, requiring only the positivity of some curvature integral.

Since $\mathcal{M}(Y) \simeq \mathcal{M}(X)$, we only have to show that X itself is Moishezon. This will be done by producing many sections of E^k. For $x = 1$, the strong Morse inequality (1.4) gives

$$h^1(E^k) - h^0(E^k) \leqslant -\frac{k^n}{n!} \int_{X(\leqslant 1, E)} \Big(\frac{i}{2\pi} c(E)\Big)^n + o(k^n) \ .$$

In particular, we get the lower bound

$$h^0(E^k) \geqslant \frac{k^n}{(2\pi)^n n!} \int_{X(\leqslant 1, E)} (ic(E))^n - o(k^n) \ . \tag{5.1}$$

By definition, the *Kodaira dimension* $\kappa(E)$ is the supremum of the dimension of the images $Y_k = \Phi_k(X \backslash Z(V_k)) \subset \mathbb{P}(V_k^*)$ for all integers $k > 0$. Since the field of meromorphic functions on X obtained by restriction of rational functions of $\mathbb{P}(V_k^*)$ to Y_k has transcendence degree $\dim Y_k$, we infer that $\kappa(E) = \sup \dim Y_k \leqslant a(X)$. The following elementary lemma is needed.

Lemma 5.2. — *For every line bundle E, there is a constant $C > 0$ such that*

$$\dim H^0(X, E^k) \leqslant C k^{\kappa(E)} \ .$$

The proof proceeds as follows : select a hermitian metric on E and a family of balls $B_j = B(z_j, r_j) \subset B'_j = B(z_j, 2r_j)$ covering X, on which E is trivial. If E^k had too many sections, one could solve a linear system in many unknowns to get a section s vanishing at a high order m at all centers z_j. Then Schwarz'lemma gives

$$\|s\|_\infty = \sup \|s\|_{B_j} \leqslant 2^{-m} C^k \sup \|s\|_{B'_j} \leqslant 2^{-m} C^k \|s\|$$

where C is the oscillation of the metric on B'_j. Thus $m \leqslant k \log C / \log 2$ if $s \neq 0$. Since the sections of E^k are essentially constant along the fibers of Φ_k, only $m^{\dim Y_k} \#\{z_j\}$ equations are needed to make s vanish at order m . Therefore we can choose $m \simeq C_1 h^0(E^k)^{1/\dim Y_k}$, so that

$$h^0(E^k) \leqslant C_2 m^{\dim Y_k} \leqslant C_3 k^{\kappa(E)} \ . \square$$

Combining (5.1) and lemma 5.2, we get the following result which implies the Grauert-Riemenschneider conjecture.

THEOREM 5.3. — *If a hermitian line bundle E verifies the integral condition $\int_{X(\leqslant 1, E)} (ic(E))^n > 0$, then $\kappa(E) = n$, in particular X is Moishezon.* $\square$

Another application of the heat kernel estimates is a generalization of G. Kempf's distortion inequalities ([Kem], [Ji]) to all projective algebraic manifolds.

Let E be a positive hermitian line bundle over a projective manifold X, equipped with a hermitian metric ω. Then $V_k = H^0(X, E^k)$ has a natural hermitian metric given by the global L^2 norm of sections. For $k \geqslant k_0$ large enough, Φ_k is an embedding and E^k can be identified to the pull-back $\Phi_k^* O(1)$. We want to compare the original metric $|\ \ |$ of E and the metric $|\ \ |_{FS}$ induced by the Fubini-Study metric of $O(1)$.

Let $(s_1, \ldots, s_N)$ be an orthonormal basis of $H^0(X, E^k)$. It is not difficult to check that

$$|\xi|^2_{FS} = \frac{|\xi|^2}{|s_1(x)|^2 + \cdots + |s_N(x)|^2} \text{ for } \xi \in E^k_x \ ,$$

thus all that we need is to get an estimate of $\sum |s_j(x)|^2$. However, this sum is the contribution of the 0 eigenvalue in the heat kernel

$$K^k_t(x, x) = \sum_{j=1}^{+\infty} e^{-2t\lambda^k_j} |\psi_j(x)|^2$$

associated to $\frac{2}{k}\square''_k$ in bidegree $(0, 0)$. We observe that non zero eigenvalues λ^k_j are also eigenvalues in bidegree $(0, 1)$, since $\overline\partial$ is injective on the corresponding eigenspaces. The associated eigenfunctions are $\overline\partial\psi_j / \sqrt{k\lambda^k_j}$, for

$$\|\overline\partial\psi_j\|^2 = \langle \Delta''_k \psi_j, \psi_j \rangle = k\lambda^k_j \ .$$

Thus the summation

$$\sum_{j=1}^{+\infty} e^{-2t\lambda^k_j} |\overline\partial\psi_j(x)|^2$$

is bounded by the heat kernel in bidegree $(0,1)$, which is itself bounded by $k^n e^{-ct}$ with $c > 0$ (note that $\alpha_{\complement J} - \alpha_J - \sum |\alpha_j| < 0$ on X for $|J| = 1$). Taking $t = k^\varepsilon$ with ε small, one can check that all estimates remain uniformly valid and that the contribution of the non zero eigenfunctions in $K_t^k(x,x)$ becomes negligible in C^0 norm. Then theorem 4.4 shows that

$$\sum |s_j(x)|^2 \sim K_t^k(x,x) \sim k^n (2\pi)^{-n} |\alpha_1(x) \cdots \alpha_n(x)|$$

as $t = k^\varepsilon \to +\infty$. For $\xi \in E_x^k$ we get therefore the C^0 convergence

$$\frac{|\xi|^2}{|\xi|_{FS}^2} \sim \Big(\frac{k}{2\pi}\Big)^n |\alpha_1(x) \cdots \alpha_n(x)| \quad \text{as } k \to +\infty .$$

As a consequence, the Fubini-Study metric on E induced by Φ_k converges to the original metric. G. Tian [Ti] has proved that this last convergence statement holds in fact in norm C^4.

6. The case of q-convex manifolds

Thierry Bouche [Bou 1] has obtained an extension of the holomorphic Morse inequalities to the case of strongly q-convex manifolds. We explain here the main ideas used.

A complex (non compact) manifold X of dimension n is strongly q-convex in the sense of Andreotti and Grauert [A-G] if there exists a C^∞ exhaustion function ψ on X such that $i\partial\overline{\partial}\psi$ has at least $n - q + 1$ positive eigenvalues outside a compact subset of X. In this case, the Andreotti-Grauert theorem shows that all cohomology groups $H^m(X, \mathcal{F})$ with values in a coherent analytic sheaf are finite dimensional for $m \geqslant q$.

THEOREM 6.1. — *Let E, F be holomorphic vector bundles over X with rank $E = 1$, rand $F = r$. Assume that X is strongly q-convex and that E has a metric for which $ic(E)$ has at least $n - p + 1$ nonnegative eigenvalues outside a compact subset. Then for all $m \geqslant p + q - 1$ the following strong Morse inequalities hold :*

$$\sum_{\ell=m}^{n} (-1)^{\ell-m} \dim H^\ell(X, E^k \otimes F) \leqslant r \frac{k^n}{n!} \int_{X(\geqslant m, E)} (-1)^m \Big(\frac{i}{2\pi} c(E)\Big)^n + o(k^n) .$$

Proof. — For every $c \in \mathbb{R}$, we consider the sublevel sets

$$X_c = \{x \in X \; ; \; \psi(x) < c\} .$$

Select c_0 such that $i\partial\overline{\partial}\psi$ has $n-q+1$ positive eigenvalues on $X \backslash X_c$. One can choose a hermitian metric ω_0 on X in such a way that the eigenvalues $\gamma_1^0 \leqslant \cdots \leqslant \gamma_n^0$ of $i\partial\overline{\partial}\psi$ with respect to ω_0 satisfy

$$(6.2) \qquad -\frac{1}{n} \leqslant \gamma_1^0 \leqslant \cdots \leqslant \gamma_{q-1}^0 \leqslant 1 \quad \text{and} \quad \gamma_q^0 = \cdots = \gamma_n^0 = 1 \text{ on } X \backslash X_{c_0} \; ;$$

this can be achieved by taking ω_0 equal to $i\partial\overline{\partial}\psi$ on a C^∞ subbundle of TX of rank $n-q+1$ on which $i\partial\overline{\partial}\psi$ is positive, and ω_0 very large on the orthogonal complement. We set $\omega = e^\rho\omega_0$ where ρ is a function increasing so fast at infinity that ω will be complete.

More important, we multiply the metric of E by a weight $e^{-\chi\circ\psi}$ where χ is a convex increasing function. The resulting hermitian line bundle is denoted E_χ. For any $(0,m)$ form u with values in $E^k\otimes F$, viewed as an (n,m) form with values in $E^k\otimes F\otimes\Lambda^n TX$, the Bochner-Kodaira-Nakano formula implies an inequality

$$\langle\Delta''_k u,u\rangle \geqslant \int_X k\langle[ic(E_\chi),\Lambda]u,u\rangle + \langle\Theta u,u\rangle$$

where Θ depends only on the curvature of $F\otimes\Lambda^n TX$ and the torsion of ω. By the formulas of §4, we have

$$\langle[ic(E_\chi),\Lambda]u,u\rangle \geqslant (\alpha_1+\cdots+\alpha_m)|u|^2$$

where $\alpha_1\leqslant\cdots\leqslant\alpha_n$ are the eigenvalues of

$$ic(E_\chi) = ic(E) + i\partial\overline{\partial}(\chi\circ\psi) \geqslant ic(E) + (\chi'\circ\psi)i\partial\overline{\partial}\psi\ .$$

If β is the lowest eigenvalue of $ic(E)$ with respect to ω, we find

$$\alpha_j \geqslant \beta + (\chi'\circ\psi)\gamma_j^0/e^\rho\ ,$$
$$\alpha_1+\cdots+\alpha_m \geqslant m\beta + (\chi'\circ\psi)(\gamma_1^0+\cdots+\gamma_m^0)/e^\rho\ ,$$

and by (6.2) we get for all $m\geqslant q$:

$$\alpha_1+\cdots+\alpha_m \geqslant m\beta + \frac{1}{n}e^{-\rho}\chi'\circ\psi \text{ on } X\backslash X_{c_0}\ .$$

It follows that one can choose χ increasing very fast in such a way that the Bochner inequality becomes

$$(6.3)\qquad \langle\Delta''_k u,u\rangle \geqslant k\int_{X\backslash C_{c_0}} A(x)|u(x)|^2 - C\int_X |u(x)|^2$$

where $A\geqslant 1$ is a function tending to $+\infty$ at infinity on X and $C\geqslant 0$. Now, Rellich's lemma easily shows that Δ''_k has a compact resolvent. Hence the spectrum of Δ''_k is discrete and its eigenspaces are finite dimensional. Standard arguments also show the following :

Lemma 6.4. — *When χ increases sufficiently fast at infinity, the space $\mathcal{H}^m(X,E^k_\chi\otimes F)$ of L^2-harmonic forms of bidegree $(0,m)$ for Δ''_k is isomorphic to the cohomology group $H^m(X,E^k\otimes F)$ for all $k\in\mathbb{N}$ and $m\geqslant q$.*

For a domain $\Omega\subset\subset X$, we consider the quadratic form

$$Q^{k,m}_\Omega(u) = \frac{1}{k}\int_\Omega |\overline{\partial}_k u|^2 + |\overline{\partial}^*_k u|^2$$

with Dirichlet boundary conditions on $\partial\Omega$. We denote by $\mathcal{H}^{k,m}_{\leqslant\lambda,\Omega}$ the direct sum of all eigenspaces of $Q^{k,m}_\Omega$ corresponding to eigenvalues $\leqslant\lambda$ (*i.e.* $\leqslant k\lambda$ for Δ''_k).

LEMMA 6.5. — *For every $\lambda \geqslant 0$ and $\varepsilon > 0$, there exists a domain $\Omega \subset\subset X$ and an integer k_0 such that*

$$\dim \mathcal{H}^{k,m}_{\leqslant\lambda,\Omega} \leqslant \dim \mathcal{H}^{k,m}_{\leqslant\lambda,X} \leqslant \dim \mathcal{H}^{k,m}_{\leqslant\lambda+\varepsilon,\Omega} \text{ for } k \geqslant k_0 \ .$$

Proof. — The left hand inequality is a straightforward consequence of the minimax principle, because the domain of the global quadratic form $Q^{k,m}_\Omega$ is contained in the domain of $Q^{k,m}_X$.

For the other inequality, let $u \in \mathcal{H}^{k,m}_{\leqslant\lambda,X}$. Then (6.3) gives

$$k\int_{X\setminus X_{c_0}} A|u|^2 - C\int_{X_{c_0}} |u|^2 \leqslant k\lambda \int_X |u|^2 \ .$$

Choose $c_2 > c_1 > c_0$ so that $A(x) \geqslant a$ on $X\setminus X_{c_1}$ and a cut-off function φ with compact support in X_{c_2} such that $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on X_{c_1}. Then we find

$$\int_{X\setminus X_{c_1}} |u|^2 \leqslant \frac{C + k\lambda}{ka}\int_X |u|^2 \ .$$

For a large enough, we get $\int_{X\setminus X_{c_1}} |u|^2 \leqslant \varepsilon\|u\|^2$. Set $\Omega = X_{c_2}$. Then

$$\begin{aligned} Q^{k,m}_\Omega(\varphi u) &= \frac{1}{k}\int_\Omega |\overline{\partial}\varphi \wedge u + \varphi\overline{\partial}_k u|^2 + |\varphi\overline{\partial}{}^*_k u - \partial\varphi \lrcorner u|^2 \\ &\leqslant (1+\varepsilon)Q^{k,m}_X(u) + \frac{C'}{k}\Big(1+\frac{1}{\varepsilon}\Big)\|u\|^2 \\ &\leqslant (1+\varepsilon)(\lambda + \frac{C'}{k\varepsilon})\|u\|^2 \ . \end{aligned}$$

As $\|\varphi u\|^2 \geqslant \int_{X_{c_1}} |u|^2 \geqslant (1-\varepsilon)\|u\|^2$, we infer

$$Q^{k,m}_\Omega(\varphi u) \leqslant \frac{1+\varepsilon}{1-\varepsilon}\Big(\lambda + \frac{C'}{k\varepsilon}\Big)\|\varphi u\|^2 \ .$$

If ε is replaced by a suitable smaller number and k taken large enough, we obtain $Q^{k,m}_\Omega(v) \leqslant (\lambda+\varepsilon)\|v\|^2$ for all $v \in \varphi\mathcal{H}^{k,m}_{\leqslant\lambda,X}$. Then the right hand inequality in lemma 6.5 follows by the minimax principle. □

Now, corollary 3.3 easily computes the counting function $N^{k,m}_\Omega$ for the eigenvalues :

$$\lim_{\lambda\to 0_+}\lim_{k\to+\infty} k^{-n} N^{k,m}_\Omega(\lambda) = \frac{r}{n!}\int_{X(m,E_\chi)} (-1)^m(\frac{i}{2\pi}c(E_\chi))^n \ .$$

Applying this to the Witten complex $\mathcal{H}^{k,\bullet}_{\leqslant\lambda,X}$, we easily infer the inequality of theorem 6.1, except that $c(E)$ is replaced by $c(E_\chi)$. However, up to now, the inequality is valid for all $m \geqslant q$. Take the convex function χ equal to 0 on $]-\infty, c_0]$. Then

$$ic(E_\chi) = ic(E) + i\partial\overline{\partial}(\chi\circ\psi)$$

coincides with $ic(E)$ on X_{c_0} and has at most $(p-1)+(q-1)$ negative eigenvalues on $X\backslash X_{c_0}$. Hence $X(m,E_\chi)=X(m,E)$ for $m\geqslant p+q-1$ and $ic(E_\chi)=ic(E)$ on these sets. Theorem 6.1 is proved. □

As a corollary, one obtains a general a priori estimate for the Monge-Ampère operator $(i\partial\overline{\partial})^n$ on q-convex manifolds.

COROLLARY 6.6. — *Let X be a strongly q-convex manifold and φ a C^∞ function on X, weakly p-convex outside a compact subset of X. For $\ell=0,1,\ldots,n$, let G_ℓ be the open set of points where $i\partial\overline{\partial}\varphi$ is non degenerate and admits ℓ negative eigenvalues. Then for all $m\geqslant p=q-1$*

$$\sum_{\ell=m}^{n}\int_{G_\ell}(i\partial\overline{\partial}\varphi)^m \text{ has the sign of } (-1)^m\ .$$

This result has been first obtained by Y.T. Siu [Siu 4] for q-convex domains in a Stein manifold. At that time, the q-convex case of the inequalities was not yet available and Siu had to rely on a rather sophisticated approximation argument of Stein manifolds by algebraic varieties; the proof could then be reduced to the compact case.

The general statement given above is in fact a direct consequence of theorem 6.1 : take for E the trivial bundle $E=\mathcal{O}_X$ equipped with the metric defined by the weight $e^{-\varphi}$ and $F=\mathcal{O}_X$. Since $H^m(X,E^k)=H^m(X,\mathcal{O}_X)$ is independent of k and finite dimensional, theorem 6.1 implies

$$k^n\sum_{\ell=m}^{n}\int_{G_\ell}(-1)^m(i\partial\overline{\partial}\varphi)^n\geqslant \text{ constant } -o(k^n)$$

for all $k\geqslant k_0$ and $m\geqslant p+q-1$, whence the result. □

7. Holomorphic Morse inequalities for vector bundles

A natural question arising in connection with our Morse inequalities is whether one can extend the inequalities for high tensor powers of a vector bundle E of rank $\geqslant 2$. Since $E^{\otimes k}$ is decomposable for $k\geqslant 2$ (e.g. $E^{\otimes 2}=S^2E\oplus\Lambda^2E$) we are led to consider only irreducible tensor powers of E, *i.e.* the irreducible representations of the linear group $G\ell(E)$. This is done by Getzler [Ge 2], in the general framework of Lie group theory and representations. As we are only dealing with the case of the full linear group, we will give here an elementary presentation. We first recall some ideas from Borel-Weil's theory and a special case of Bott's formula [Bot].

Let V be a complex vector space of dimension r and $M(V)$ the flag manifold of V, *i.e.* the set of all $(r+1)$-tuples $z=(V_0,V_1,\ldots,V_r)$ with $V=V_0\supset V_1\supset$

$\cdots \supset V_r = \{0\}$ and $\mathrm{codim} V_j = j$. On $M(V)$ we have canonical line bundles Q_j such that

$$Q_{j,z} = V_{j-1}/V_j \ , \ 1 \leqslant j \leqslant r^{\cdot}$$

For any r-tuple $(a_1, \ldots, a_r) \in \mathbb{Z}^r$, we set

$$Q^a = Q_1^{a_1} \otimes \cdots \otimes Q_r^{a_r} \ .$$

As $G\ell(V)$ acts equivariantly on $Q^a \to M(V)$, the spaces of sections

$$\Gamma^a V = H^0\big(M(V), Q^a\big) \tag{7.1}$$

are equipped with a natural $G\ell(V)$ action. Observe that $Q^{(1,\ldots,1)}$ is isomorphic to the trivial bundle $M(V) \times \det V$ (but of course the action of $G\ell(V)$ on $\det V$ is non trivial). To describe $\Gamma^a V$, we can therefore assume that all a_j are nonnegative. Then any section $\sigma \in \Gamma^a V$ can be viewed as a polynomial $P_\sigma(\xi_1, \ldots, \xi_r)$ on $(V^*)^r$ as follows : if $\xi_1, \ldots, \xi_r \in V^*$ are linearly independent, one can associate to (ξ_j) the flag $z = (V_j)$ defined by $V_j = \xi_1^{-1}(0) \cap \cdots \cap \xi_j^{-1}(0)$. Then ξ_j induces a well defined linear form $\tilde{\xi}_j$ on $Q_{j,z} = V_{j-1}/V_j$ and we set

$$P_\sigma(\xi_1, \ldots, \xi_r) = (\tilde{\xi}_1^{a_1} \otimes \cdots \otimes \tilde{\xi}_r^{a_r}) \cdot \sigma(z) \ .$$

It is clear that P_σ remains locally bounded on a neighborhood of the hypersurface $\det(\xi_1, \ldots, \xi_r) = 0$; therefore P_σ extends to a polynomial on $(V^*)^r$ that is homogeneous of degree a_j in the variable ξ_j. Also, neither the flag z nor the linear forms $\tilde{\xi}_j$ are modified if we replace ξ_j by $\xi_j + \sum_{k<j} \lambda_{jk}\xi_k$. It follows that P_σ satisfies the relation

$$P_\sigma(\xi_j + \sum_{k<j} \lambda_{jk}\xi_k) = P_\sigma(\xi_1, \ldots, \xi_r) \ , \ \forall \lambda_{jk} \in \mathbb{C} \ ,$$

and conversely any polynomial P of multidegree $(a_1, \ldots, a_r)$ satisfying this condition yields a (unique) section $\sigma \in \Gamma^a V$. Hence $\Gamma^a V$ is the subspace of tensors in $S^{a_1}V \otimes \cdots \otimes S^{a_r}V$ enjoying the above additional antisymmetry properties. In particular we have

$$\begin{aligned} &S^k V = \Gamma^{(k,0,\ldots,0)} V \ , \\ &\Lambda^k V = \Gamma^{(1,\ldots,1,0,\ldots,0)} V \ , \ (k \text{ first integers } = 1) \ . \end{aligned}$$

We will see soon that $\Gamma^a V = \{0\}$ unless $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r$. The spaces $\Gamma^a V$ $(a_1 \geqslant \cdots \geqslant a_r)$ can be seen to be *irreducible* representations of $G\ell(V)$. As is well known in representation theory, $(\Gamma^a V)$ is in fact the complete list of irreducible representations of $G\ell(V)$ up to isomorphism.

Assume now that V is equipped with a hermitian metric. Then any flag $z^0 \in M(V)$ is represented by an orthonormal basis $(e_1, \ldots, e_r)$ such that $V_j^0 = \mathrm{Vect}(e_{j+1}, \ldots, e_r)$. Now z^0 is contained in the affine chart of points $z = (V_j)$ with

$$V_j = \mathrm{Vect}(v_{j+1}, \ldots, v_r) \ , \ v_k = e_k + \sum_{j<k} z_{jk} e_j$$

where $(z_{jk}) \in \mathbb{C}^{n(n-1)/2}$ are the affine coordinates of z . The canonical metric on Q^a induced by V has curvature

$$c(Q^a)_{z^0} = \sum_{1\leqslant j<j\leqslant n} (a_j - a_k) dz_{jk} \wedge d\overline{z}_{jk} \ ; \tag{7.2}$$

we will omit the easy (and standard) computation. By homogeneity, we see that Q^a is positive as soon as $a_1 > a_2 > \cdots > a_r$. On the other hand, when $a_{j-1} < a_j$, we see that Q^a is negative along the $\mathbb{P}^1$ line in $M(V)$ obtained by fixing all $V_k = V_k^0$ except V_j. Therefore $\Gamma^a V = H^0\big(M(V), Q^a\big) = \{0\}$ in this case.

Assume from now on that $a_1 \geqslant \cdots \geqslant a_r$, and more specifically that

$$a_1 = \cdots = a_{s_1} > a_{s_1+1} = \cdots = a_{s_2} > \cdots > a_{s_{m-1}+1} = \cdots = a_{s_m} \ ,$$

where $s_m = r$. As $Q_{j+1,z} \otimes \cdots \otimes Q_{k,z} \simeq \det(V_j/V_k)$, we see that Q^a is the pull back of the bundle

$$Q_s^a = \det(V/V_{s_1})^{a_{s_1}} \otimes \cdots \otimes \det(V_{s_{m-1}}/V_{s_m})^{a_{s_m}}$$

over the manifold $M_s(V)$ of partial flags

$$V \supset V_{s_1} \supset \cdots \supset V_{s_m} = \{0\} \ ,$$

via the obvious projection $\pi_s : M(V) \to M_s(V)$. On $M_s(V)$ we have a formula completely analogous to (7.2), where the only indices (j,k) involved are those for which $a_j > a_k$. Thus Q_s^a is ample and $Q^a = \pi_s^\star Q_s^a$, in particular

$$H^0\big(M_s(V), Q_s^a\big) = H^0\big(M(V), Q^a\big) = \Gamma^a V \ . \tag{7.3}$$

Now let E, F be holomorphic vector bundles over a compact manifold X and let

$$n = \dim_{\mathbb{C}} X \quad , \quad r = \text{rank } E \quad , \quad r' = \text{rank } F \ .$$

We want to get asymptotic estimates for the dimension of cohomology groups $H^q(X, \Gamma^{ka}E \otimes F)$ as $k \to +\infty$. For that, we introduce the flag bundle

$$M_s(E) \to X \ ,$$

where $s = (s_1, \ldots, s_m)$ is defined as above, and we consider the universal line bundle Q_s^a over $M_s(E)$. As Q_s^a is ample along the fibers of $\pi_s \ : M_s(E) \to X$, the higher direct images

$$R^q(\pi_s)_\star(Q_s^{ka} \otimes \pi_s^\star F), \qquad q \geqslant 1$$

vanish for $k \geqslant k_0$. By (7.3) we get

$$(\pi_s)_\star(Q_s^{ka} \otimes \pi_s^\star F) = \Gamma^{ka}E \otimes F \ .$$

The Leray spectral sequence gives the isomorphism

$$H^q(X, \Gamma^{ka}E \otimes F) \simeq H^q(Y, Q_s^{ka} \otimes \pi_s^\star F) \ , \qquad Y = M_s(E) \ ,$$

and we are reduced to applying Morse inequalities to tensor powers of the line bundle Q_s^a. We still need a formula for the curvature of Q_s^a with the metric induced by a given hermitian metric on E. Let $z^0 \in M_s(E_{x^0})$ be a point in $M_s(E)$. Choose a holomorphic frame $(e_1, \ldots, e_r)$ of E such that the flag z^0 is given by the basis $(e_1(x^0), \ldots, e_r(x^0))$ (supposed to be orthonormal). Assume also (e_λ) chosen such that $De_\lambda(x^0) = 0$ and consider the curvature tensor

$$c(E)_{x^0} = \sum_{\substack{1\leqslant j,k\leqslant n\\ 1\leqslant \lambda,\mu\leqslant r}} c_{jk\lambda\mu}\, dx_j \wedge d\overline{x}_k \otimes e_\lambda^\star \otimes e_\mu.$$

It can be shown that the associated curvature of Q_s^a is

$$c(Q_s^a)_{z^0} = \sum_{\substack{1\leqslant j,k\leqslant n\\ 1\leqslant \lambda\leqslant r}} a_\lambda c_{jk\lambda\lambda}\, dx_j \wedge d\overline{x}_k + \sum_{a_\lambda > a_\mu} (a_\lambda - a_\mu)\, dz_{\lambda\mu} \wedge d\overline{z}_{\lambda\mu}$$

where $(z_{\lambda\mu})$ are the affine coordinates along the fiber $M_s(E_{x^0})$. Finally, let $N(s)$ be the dimension of the fibers $M_s(E_x)$. Using the isomorphism (7.4), the strong Morse inequality becomes

$$\sum_{m=0}^{q} (-1)^{q-m} \dim H^m(X, \Gamma^{ka}E \otimes F) \leqslant r' \frac{k^{n+N(s)}}{(n+N(s))!} \int_{Y(\leqslant q, Q_s^a)} \Big(\frac{i}{2\pi} c(Q_s^a)\Big)^{n+N(s)}.$$

The most interesting case is the case of symmetric powers S^kE. Then we simply have $M_s(E) = \mathbb{P}(E^\star)$, $N(s) = r-1$, $Q_s^a = O_E(1)$.

8. Related questions and open problems

(a) As in the case of the Riemann-Roch Hirzebruch formula, it would be extremely interesting to get some insight on the error term $o(k^n)$ of the estimates. However, this problem encounters two major difficulties :

• First, the lower order terms in the asymptotic expansion of the heat kernel involve derivatives of the curvature of E, as well as terms coming from the curvature of F or of the manifold X. These terms are very difficult to compute.

• Second, the open sets $X(\leqslant q, E)$ involved in the estimates may be extremely irregular, even when the metric of E is smooth. This leaves very little hope of being able to handle the error term.

One way to avoid the difficulties of the first point is to assume that the dominant term is identically zero, *i.e.* that $ic(E)$ has rank $< n$ everywhere and is generically of a given rank s. Concerning the second point, a reasonable hypothesis is the following : assume that there is a smooth foliation $\mathcal{F}$ of codimension s in X such that $ic(E)$ vanishes along the leaves. With these hypotheses, Th. Bouche [Bou2] has shown that when $\operatorname{rk} E = \operatorname{rk} F = 1$, $\ell \to +\infty$, $k/\ell \to +\infty$, then

$$\begin{aligned}&\sum_{0\leqslant m\leqslant q} (-1)^{q-m} \dim H^m(X, E^k \otimes F^\ell \otimes G)\\ &\leqslant \frac{k^s}{s!}\frac{\ell^{n-s}}{(n-s)!}(\operatorname{rk} G) \int_{X(\leqslant q, E, F)} \Big(\frac{i}{2\pi}c(E)\Big)^s \wedge \Big(\frac{i}{2\pi}c(F)\Big)^{n-s} + o(k^s\ell^{n-s}),\end{aligned}$$

where the index set $X(m,E,F)$ is the set where $(ic(E))^s \wedge (ic(F))^{n-s}$ has sign $(-1)^m$. It would be very interesting to get rid of the tensor powers F^ℓ, but then the difficulty is that the estimates cannot be localized along the leaves of the foliation.

(b) A natural problem is of course to extend the Morse inequalities to cohomology groups associated to other operators than $\overline{\partial}$. This has been carried out by Getzler [Ge3] for the operator $\overline{\partial}_b$ on a compact strongly pseudoconvex CR manifold.

(c) Jean Varouchas has drawn my attention on the following interesting question of Fujiki related to the Grauert-Riemenschneider conjecture. The G-R conjecture was an attempt to characterize Moishezon manifolds, *i.e.* manifolds which have a projective algebraic modification.

Another interesting class of manifolds is the Fujiki class $\mathcal{C}$, that is, the class of compact complex manifolds which have a Kähler modification (for instance, such manifolds possess Hodge decomposition). Assume that X is compact and has a closed semi-positive $(1,1)$-form ω such that ω is positive definite on a dense open subset (say, on the complement of an analytic subset). Then, is X in the Fujiki class $\mathcal{C}$?

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