HOLOMORPHIC MORSE INEQUALITIES ON q-CONVEX MANIFOLDS

after the work of Thierry Bouche

by Jean-Pierre DEMAILLY

Université Joseph Fourier Grenoble I,
Institut Fourier, BP 74,
Laboratoire associé au C.N.R.S. n°188,
F-38402 Saint-Martin d'Hères

1. Statement of results and applications.

This note is a report about a recent work of Thierry Bouche (Univ. Joseph Fourier, Grenoble). Bouche [Bo] obtained an extension of the holomorphic Morse inequalities proved in Demailly [D3] for strongly q–convex manifolds.

Let X be a complex manifold of dimension n . We assume that X is strongly q–convex in the sense of Andreotti-Grauert [A-G], i.e. that there exists a C^∞ exhaustion function ψ on X such that the complex Hessian $id'd''\psi$ has at least $n-q+1$ positive eigenvalues outside a compact subset of X . In this case, Andreotti-Grauert's theorem [A-G] asserts that all cohomology groups $H^m(X,\mathcal{F})$ with values in a coherent analytic sheaf are finite dimensional for $m \geqslant q$.

Let $L \longrightarrow X$ be a holomorphic line bundle with a hermitian metric of class C^∞ and $E \longrightarrow X$ a holomorphic vector bundle of rank r . Let $D = D' + D''$ be the Chern connection of L and $c(L) = D^2$ the associated curvature form. We denote by $X(m,L)$ the set of points of X where the real $(1,1)$–form $ic(L)$ is non degenerate and has exactly m negative eigenvalues. We also set

$$X(\leqslant m, L) = \bigcup_{\nu \leqslant m} X(\nu, L) \ , \qquad X(\geqslant m, L) = \bigcup_{\nu \geqslant m} X(\nu, L) \ .$$

Then the cohomology groups $H^m(X, E \otimes L^k)$ are finite dimensional for $m \geqslant q$, and the following theorem gives an asymptotic estimate of $\dim H^m(X, E \otimes L^k)$ as k tends to $+\infty$.

1.1. Theorem. — *Assume that X is strongly q–convex and that the curvature form $ic(L)$ has at least $n-p+1$ nonnegative eigenvalues outside a compact subset of X . Then for all $m \geqslant p+q-1$ the following asymptotic inequalities hold :*

(a_m) *Weak Morse inequalities* :

$$\dim H^m(X, E\otimes L^k) \leqslant r\frac{k^n}{n!}\int_{X(m,L)} (-1)^m\Big(\frac{i}{2\pi}c(L)\Big)^n + \mathrm{o}(k^n)\ .$$

(b_m) *Strong Morse inequalities* :

$$\sum_{m\leqslant\nu\leqslant n} (-1)^{\nu-m}\dim H^\nu(X, E\otimes L^k) \leqslant r\frac{k^n}{n!}\int_{X(\geqslant m,L)} (-1)^m\Big(\frac{i}{2\pi}c(L)\Big)^n + \mathrm{o}(k^n)\ .$$

Observe that (a_m) is obtained simply by adding (b_m) and (b_{m+1}). The special case when X is compact ($p=q=0$) has been first proved in [D3]. Then (a_m) and (b_m) are valid for all $m \geqslant -1$. As the strong Morse inequalities (b_0) , (b_{-1}) are identical except for the sign of their terms, we see that the case $m=0$ is an asymptotic equality; this equality is in fact a weak form of the Riemann-Roch-Hirzebruch formula for the Euler-Poincaré characteristic $\chi(X, E\otimes L^k)$. Subtracting inequality (b_2) from (b_0) , we get the lower bound

(1.2)

$$\dim H^0(X, E\otimes L^k) - \dim H^1(X, E\otimes L^k) \geqslant r\frac{k^n}{n!}\int_{X(\leqslant 1,L)} \Big(\frac{i}{2\pi}c(L)\Big)^n + \mathrm{o}(k^n)\ .$$

This result is used in [D3] in order to obtain a new proof of the Grauert-Riemenschneider conjecture [G-R], which had been solved in the affirmative by Y.T. Siu [S2] soon before. More precisely, we get the following stronger version of the conjecture.

1.3. Theorem. — *Let X be a connected n–dimensional compact manifold. If X carries a hermitian line bundle L such that $\int_{X(\leqslant 1,L)}\big(ic(L)\big)^n > 0$, then X is Moišezon.*

Let us recall that the *algebraic dimension* $a(X)$ is by definition the transcendence degree of the field of meromorphic functions of X . This degree always verifies $0 \leqslant a(X) \leqslant n$ and X is said to be a *Moišezon* variety if $a(X) = n$. Inequality (1.2) shows that $\dim H^0(X, L^k) \geqslant ck^n$ for some constant $c > 0$ and $k \geqslant k_0$; by a standard method of Poincaré, Serre [Se] and Siegel [Si], it follows that the field of meromorphic functions generated by quotients of sections in $H^0(X, L^k)$, $k \in \mathbb{N}$, has transcendence degree $\geqslant n$, hence $a(X) = n$. $\square$

Another consequence of Morse inequalities is a general a priori estimate for Monge-Ampère operator $(id'd'')^n$ on q–convex manifolds.

1.4. Theorem. — *Let X be a strongly q–convex manifold and φ a $\mathcal{C}^\infty$ function on X , weakly p-convex outside a compact subset $K \subset X$. For $0 \leqslant \nu \leqslant n$,* *let G_ν be the set of points of X where $id'd''\varphi$ is non degenerate and admits ν negative eigenvalues. Then for all $m \geqslant p+q-1$ one has*

$$\sum_{m\leqslant\nu\leqslant n}\int_{G_\nu}(-1)^m(id'd''\varphi)^n \geqslant 0\ .$$

This result has been first obtained by Y.T. Siu [S4] for q–convex domains Ω in a Stein manifold X . Siu's method is rather sophisticated and rests merely upon the compact case of theorem 1.1 : X is "approximated" by affine algebraic manifolds $X_{\rm alg}$, and the proof is then reduced to an application of Morse inequalities on a compactification of $X_{\rm alg}$. The general statement given above appears in fact as a straightforward consequence of the q–convex case of theorem 1.1 : take for L the trivial bundle $X\times\mathbb{C}$ endowed with the curvature form defined by the weight $e^{-\varphi}$. Then $ic(L) = id'd''\varphi$. Since $H^m(X,L^k) = H^m(X,\mathbb{C})$ is independent of k and finite dimensional for $m\geqslant q$, theorem 1.1 (b) implies

$$k^n\sum_{m\leqslant\nu\leqslant n}\int_{G_\nu}(-1)^m(id'd''\varphi)^n \geqslant \text{constant}$$

for all $k\geqslant k_0$ and $m\geqslant p+q-1$, whence the result. $\square$

Following Th. Bouche [Bo], we give now an account of the main ideas occurring in the proof of theorem 1.1. We refer to [Bo] and to [D3] for the details.

2. The Bochner technique.

The cohomology groups $H^m(X,E\otimes L^k)$ are computed by means of the Dolbeault complex D'' : $C^\infty_{0,\bullet}(X,E\otimes L^k)$, where $C^\infty_{s,t}(X,E)$ denotes the space of C^∞ sections of the bundle $\Lambda^{s,t}T^\star X\otimes E$. This complex is isomorphic to the complex $C^\infty_{n,\bullet}(X,\widetilde{E}\otimes L^k)$ with $\widetilde{E} = \Lambda^nTX$, thus after changing E into $\widetilde{E}$, we can work with forms of bidegree (n,m) instead of $(0,m)$ (note that the estimates of theorem 1.1 do not depend on E except for the rank r).

Assume that the bundle E is endowed with a C^∞ hermitian metric, and let $D = D'+D''$, $c(E) = D^2$ be the corresponding Chern connection and curvature form. Let ω be a complete hermitian metric on X (to be chosen later in a suitable way), $\langle ?,?\rangle$ the pointwise inner product of differential forms on X with values in E and $\langle\!\langle ?,?\rangle\!\rangle$ the global L^2 inner product obtained after integration with respect to the volume form $dV = \omega^n/n!$. The corresponding norms are denoted $|\,?\,|$ and $||\,?\,||$, and Λ stands as usual for the (pointwise) adjoint of the wedge multiplication by ω . Finally, we denote by $\delta = \delta'+\delta''$ the formal adjoint of D .

If A and B are endomorphisms of respective degrees a,b of the graded module $C^\infty_\bullet(X,E)$, we set $[A,B] = AB-(-1)^{ab}BA$. The holomorphic and conjugate-holomorphic Laplace-Beltrami operators are then defined by

$$\Delta' = [D',\delta']\ ,\qquad \Delta'' = [D'',\delta'']\ .$$

These operators are related by a formula of Weitzenböck type, known as the Bochner-Kodaira-Nakano identity; we refer to [D1] for the explicit formulation given below.

2.1. Proposition. — *Let* $\tau = [\Lambda, d'\omega]$ *, let* $\tau^\star$ *be its pointwise adjoint,* $\Delta'_\tau = [D'+\tau, \delta'+\tau^\star]$ *and* $T_\omega = [\Lambda, [\Lambda, \frac{i}{2}d'd''\omega]] - [d'\omega, (d'\omega)^\star]$ *. Then*

$$\Delta'' = \Delta'_\tau + [ic(E), \Lambda] + T_\omega \ .$$

Now, assume that ψ is an exhaustion function on X that is strongly q–convex on $X \setminus K$ and that $ic(L)$ has at least $(n-p+1)$ nonnegative eigenvalues on $X \setminus K$, for some compact set K in X . In order to take into account the q–convexity of X in formula 2.1, we multiply the metric of L by a weight $e^{-\chi\circ\psi}$ where χ is a convex increasing function. The resulting hermitian line bundle is denoted L_χ . Formula 2.1 is now applied to the bundle $E \otimes L_\chi^k$, whose curvature form is $c(E \otimes L_\chi^k) = c(E) + kc(L_\chi) \otimes \mathrm{Id}_E$. For every compactly supported smooth form with values in $E \otimes L^k$ one gets

$$\langle\!\langle \Delta'' u, u\rangle\!\rangle = \langle\!\langle \Delta'_\tau u, u\rangle\!\rangle + \int_X \Big(k\langle [ic(L_\chi), \Lambda]u, u\rangle + \langle \Theta u, u\rangle \Big) dV \tag{2.2}$$

where $\Theta = [ic(E), \Lambda] + T_\omega$ is a C^∞ endomorphism independent of χ and k . Now, we have $\langle\!\langle \Delta'' u, u\rangle\!\rangle = ||D''u||^2 + ||\delta'' u||^2$ and a similar formula for Δ'_τ , in particular $\langle\!\langle \Delta'_\tau u, u\rangle\!\rangle$ is nonnegative.

For every $c \in \mathbb{R}$, we consider the sublevel sets

$$X_c = \{x \in X \,;\, \psi(x) < c\} \ .$$

Select c_0 such that $K \subset X_{c_0}$. One can choose a hermitian metric ω^0 on X in such a way that the eigenvalues $\gamma_1^0 \leqslant \ldots \leqslant \gamma_n^0$ of $id'd''\psi$ with respect to ω^0 verify

$$-1/n \leqslant \gamma_1^0 \leqslant \ldots \leqslant \gamma_{q-1}^0 \leqslant 1 \quad \text{and} \quad \gamma_q^0 = \ldots = \gamma_n^0 = 1 \quad \text{on} \quad X \setminus X_{c_0} \ ;$$

this can be achieved by taking ω^0 equal to $id'd''\psi$ on a C^∞ subbundle of TX of rank $n-q+1$ on which $id'd''\psi$ is positive, and ω^0 large on the orthogonal complement. We select $\omega = e^\rho \omega_0$ where ρ is a function increasing fast enough at infinity so that ω is complete. Let α be the lowest eigenvalue of $ic(L)$ with respect to ω . The eigenvalues $\beta_1 \leqslant \ldots \leqslant \beta_n$ of

$$ic(L_\chi) = ic(L) + id'd''(\chi\circ\psi) \geqslant ic(L) + \chi'\circ\psi\, id'd''\psi$$

with respect to ω verify the inequality

$$\beta_j \geqslant \alpha + \chi'\circ\psi\, \gamma_j^0/e^\rho \ .$$

Let u be a (n,m)–form. A standard computation gives

$$\langle [ic(L_\chi), \Lambda]u, u\rangle \geqslant (\beta_1 + \ldots + \beta_m)|u|^2 \ . \tag{2.3}$$

Moreover, we have

$$\beta_1 + \ldots + \beta_m \geqslant m\alpha + e^{-\rho}\chi'\circ\psi(\gamma_1^0 + \ldots + \gamma_m^0) \geqslant \alpha + \frac{1}{n}e^{-\rho}\chi'\circ\psi \quad \text{on} \quad X \setminus X_{c_0}$$

because $\gamma_j^0 \geqslant -1/n$ and $\gamma_m^0 = 1$ $(m \geqslant q)$. It follows that one can choose χ increasing very fast in such a way that (2.2) and (2.3) imply

$$\langle\!\langle \Delta'' u, u\rangle\!\rangle \geqslant k \int_{X\setminus X_{c_0}} A|u|^2 dV - C\int_{X_{c_0}} |u|^2\, dV \ , \tag{2.4}$$

where $A \geqslant 1$ is a function tending to $+\infty$ at infinity on X and where C is a large constant. The completeness of the metric ω shows that inequality (2.4) is still valid for any u in the domain of Δ'' acting on the Hilbert space $L^2_{n,m}(X, E \otimes L^k_\chi)$.

2.5. Lemma. — *When χ increases sufficiently fast at infinity, the space $\mathcal{H}^{n,m}(X, E \otimes L^k_\chi)$ of Δ''–harmonic forms in $L^2_{n,m}(X, E \otimes L^k_\chi)$ is isomorphic to the cohomology group $H^m(X, \Lambda^n T^\star X \otimes E \otimes L^k)$ for all $k \in \mathbb{N}$ and $m \geqslant q$.*

As A tends to $+\infty$ in inequality (2.4), Rellich's lemma easily shows that Δ'' has a compact resolvent. Hence the spectrum of Δ'' is discrete and its eigenspaces are finite dimensional. When X is compact, lemma 2.5 is true for all $m \geqslant 0$ and for any choice of χ (a very well-known fact of Hodge theory). The q–convex case is slightly more involved; it rests essentially on the above finiteness result and on an approximation theorem of Runge type deduced from (2.4) (cf. for example [D2]). $\square$

3. Witten's complex.

We come now to one of the main ideas introduced by E. Witten [Wi]. This idea led Witten to an analytic new proof of the standard Morse inequalities for Betti numbers of a compact differentiable manifold. In our case, we consider the Dolbeault complex instead of the De Rham complex, but the method is similar by many aspects.

Since D'' commutes with Δ'' , the sequence of eigenspaces $E^m(\lambda) \subset L^2_{n,m}(X, E \otimes L^k_\chi)$ corresponding to a given eigenvalue λ is a *subcomplex* of the Dolbeault complex $D'' : C^\infty_{n,\bullet}(X, E \otimes L^k)$. Moreover we have $D''\delta'' + \delta'' D'' = \Delta'' = \lambda\,\mathrm{Id}$ on $E^\bullet(\lambda)$, thus $\frac{1}{\lambda}\delta''$ is a homotopy operator for D'' and $E^\bullet(\lambda)$ is acyclic for $\lambda \neq 0$. For any $\lambda \geqslant 0$, let

$$\mathcal{H}^m_{X,k}(\lambda)\ , \qquad m \geqslant q\ , \tag{3.1}$$

be the sum of the eigenspaces of Δ'' corresponding to eigenvalues $\leqslant k\lambda$ (the reason for this choice will become apparent later). By what we have said at the end of §2, the sequence $\mathcal{H}^m_{X,k}(\lambda)$, $m \geqslant q$, is a finite dimensional subcomplex of the Dolbeault complex. Its cohomology groups are equal to the desired groups $H^m(X, \Lambda^n T^\star X \otimes E \otimes L^k)$, except perhaps for $m = q$. In order to get the correct group for $m = q$, one possibility is to add an extra term

$$\mathcal{H}^{q-1}_{X,k}(\lambda) = \delta''\Big(\text{cycles in } \ \mathcal{H}^q_{X,k}(\lambda)\Big)\ .$$

The following elementary lemma of homological algebra is then involved.

3.2. Lemma. — *Let $0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \longrightarrow C^n \xrightarrow{d^n} 0$ be a complex of finite dimensional vector spaces of dimensions c^q over a field K . Let $h^q = \dim H^q(C^\bullet)$. Then for every index m the following "strong Morse* *inequalities" hold :*

$$h^m - h^{m+1} + \cdots + (-1)^{n-m} h^n \leqslant c^m - c^{m+1} + \cdots + (-1)^{n-m} c^n \ .$$

In our case, $C^m = \mathcal{H}^m_{X,k}(\lambda)$ and $H^m(C^\bullet) = H^m(X, \Lambda^n T^\star X \otimes E \otimes L^k)$ for $m \geqslant q$. The proof of theorem 1.1 is therefore reduced to finding an asymptotic estimation of the dimensions $\dim \mathcal{H}^m_{X,k}(\lambda)$. The first step is to show that this can be done after replacing X by relatively compact open sets $\Omega \subset\subset X$.

For a domain $\Omega \subset\subset X$, we consider the quadratic form

$$Q^m_{\Omega,k}(u) = \frac{1}{k} \int_\Omega \big(|D''u|^2 + |\delta''u|^2\big)\, dV$$

associated to $\frac{1}{k}\Delta''$ with Dirichlet conditions on $\partial\Omega$; this means that the domain of $Q^m_{\Omega,k}(u)$ is the Sobolev space $W^1_0(\Omega, \Lambda^{n,m} T^\star X \otimes E \otimes L^k)$ of sections which are in $L^2(\Omega)$ so as their first derivatives, and which can be approximated in this topology by smooth sections with compact support in Ω . We denote by $\mathcal{H}^m_{\Omega,k}(\lambda)$ the direct sum of all eigenspaces of $Q^m_{\Omega,k}$ corresponding to eigenvalues $\leqslant \lambda$ (i.e. $\leqslant k\lambda$ for Δ'').

3.3. Lemma. — *For every $\lambda \geqslant 0$ and $\varepsilon > 0$, there exists a domain $\Omega \subset\subset X$ and an integer k_0 such that*

$$\dim \mathcal{H}^m_{\Omega,k}(\lambda) \leqslant \dim \mathcal{H}^m_{X,k}(\lambda) \leqslant \dim \mathcal{H}^m_{\Omega,k}(\lambda + \varepsilon) \quad \text{ for } \ k \geqslant k_0 \ .$$

Proof. — As we work on a complete manifold, the eigenvalues of $\frac{1}{k}\Delta''$ on $L^2_{m,n}(X, E \otimes L^k_\chi)$ are the same as those of the quadratic form $Q^m_{X,k}$. The left hand inequality is then a consequence of the minimax principle, because the domain of $Q^m_{\Omega,k}$ is contained in that of $Q^m_{X,k}$ (the sections of $W^1_0(\Omega)$ can be extended by 0 on $X \setminus \Omega$).

For the other inequality, we proceed in several steps. Let $u \in \mathcal{H}^m_{X,k}(\lambda)$. Then

$$\langle\!\langle \Delta'' u, u \rangle\!\rangle = ||D''u||^2 + ||\delta''u||^2 \leqslant k\lambda ||u||^2 \ ,$$

and a combination of this estimate with (2.4) yields

$$k \int_{X \setminus X_{c_0}} A|u|^2\, dV - C \int_{X_{c_0}} |u|^2\, dV \leqslant k\lambda \int_X |u|^2\, dV \ .$$

Consider the compact set $P = \overline{X_{c_0}} \cup \{A(x) \leqslant a\}$. Then $A(x) > a$ on $X \setminus P$ and we find

$$\int_{X \setminus P} |u|^2\, dV \leqslant \frac{C + k\lambda}{ka} \int_X |u|^2\, dV \ .$$

If we choose a large enough, we get $\int_{X\setminus P} |u|^2\, dV \leqslant \varepsilon ||u||^2$. Let $\varphi \in \mathcal{D}(X)$ be a cut-off function equal to 1 on P and $\Omega \subset\subset X$ an open set containing the support of φ . The section φu is in the domain of $Q^m_{\Omega,k}$ and we have

$$\begin{aligned} Q^m_{\Omega,k}(\varphi u) &= \frac{1}{k} \int_\Omega \big(|d''\varphi \wedge u + D''u|^2 + |\delta''u - d'\varphi \,\lrcorner\, u|^2\big)\, dV \\ &\leqslant (1+\varepsilon) Q^m_{X,k}(u) + \frac{C'}{k}(1 + 1/\varepsilon)||u||^2 \\ &\leqslant \Big((1+\varepsilon)\lambda + \frac{C'}{k}(1+1/\varepsilon)\Big)||u||^2 \ . \end{aligned}$$

As $||\varphi u||^2 \geqslant \int_P |u|^2\,dV \geqslant (1-\varepsilon)||u||^2$, we infer

$$Q^m_{\Omega,k}(\varphi u) \leqslant \Big(\frac{1+\varepsilon}{1-\varepsilon}\lambda + \frac{C(\varepsilon)}{k}\Big)||\varphi u||^2 \ .$$

If ε is replaced by a smaller number and k taken large enough, we obtain $Q^m_{\Omega,k}(v) \leqslant (\lambda+\varepsilon)||v||^2$ when v describes the space $\varphi\mathcal{H}^m_{X,k}(\lambda)$. The right hand inequality in lemma 3.3 follows by the minimax principle. □

In order to handle properly the quadratic form $Q^m_{\Omega,k}$, we need another calculation derived from the Bochner-Kodaira-Nakano identity (2.2) (cf. [D3]) :

$$(3.4)\qquad 2Q^m_{\Omega,k}(u) = \int_\Omega \Big(\frac{1}{k}|\nabla u + Su|^2 - \langle Vu,u\rangle + \frac{1}{k}\langle\Theta u,u\rangle\Big)\,dV \ ,$$

where ∇ is the canonical hermitian connection on the bundle $\Lambda^{n,m}T^\star X\otimes E\otimes L^k_\chi$ deduced from the Chern connections on $\Lambda T^\star X$, E and L_χ . The operators S and Θ are hermitian endomorphisms which act only on the component $\Lambda^{n,m}T^\star X\otimes E$ and are independent of k . Finally V is a curvature endomorphism acting only on the component $\Lambda^{n,m}T^\star X$. Let $(\xi_1,\ldots,\xi_n)$, $(e_1,\ldots,e_r)$ and l be orthonormal frames of TX , E , L_χ respectively, and assume that $ic(L_\chi)$ is diagonalized in (ξ_j) . One finds

$$u = \sum_{|J|=m,\,1\leqslant h\leqslant r} u_{J,h}\,\xi_1^\star\wedge\ldots\wedge\xi_n^\star\wedge\overline{\xi}^\star_J\otimes e_h\otimes l^k \ ,$$

$$(3.5)\qquad \langle Vu,u\rangle = \sum_{|J|=m,\,1\leqslant h\leqslant r} (\beta_{\complement J}-\beta_J)|u_{J,h}|^2 \ ,$$

where $\beta_1,\ldots,\beta_n$ are the eigenvalues of $ic(L_\chi)$ and $\beta_J = \sum_{j\in J}\beta_j$.

4. A spectral theorem for Schrödinger operators.

The main tool for the proof of Morse inequalities is a spectral theorem which describes very precisely the asymptotic distribution of eigenvalues for quadratic forms similar to $Q^m_{\Omega,k}$.

Let M be a n–dimensional Riemannian manifold of class C^∞ and $F\longrightarrow M$, $L\longrightarrow M$ hermitian vector bundles of respective ranks t and 1 . We assume that F,L are endowed with hermitian connections ∇_E , ∇_L , and we let ∇_k be the resulting connection on $F\otimes L^k$. Finally let V and S be hermitian endomorphisms of F; we still denote by V and S their extensions $V\otimes\mathrm{Id}$, $S\otimes\mathrm{Id}$ to $F\otimes L^k$. For any domain $\Omega\subset\subset M$, we consider the quadratic forms

$$(4.1)\qquad Q_{\Omega,k}(u) = \int_\Omega\Big(\frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu,u\rangle\Big)\,dV \ , \qquad u\in W^1_0(\Omega, F\otimes L^k)$$

with Dirichlet condition on $\partial\Omega$. For every real number λ , we denote by $N_{\Omega,k}(\lambda)$ the number of eigenvalues of $Q_{\Omega,k}$ which are $\leqslant\lambda$; these eigenvalues are of course counted with multiplicity. We shall see soon that $N_{\Omega,k}(\lambda)$ can be estimated in terms of the curvature of L and of the potential V .

The curvature form $B = ic(L) = i\nabla_L^2$ is a real 2–form on X; at each point $x \in M$, this form can be written

$$(4.2) \qquad B(x) = \sum_{1 \leqslant j \leqslant s} B_j(x)\, \xi_{2j-1}^\star \wedge \xi_{2j}^\star \ , \qquad s \leqslant n/2$$

where $(\xi_1^\star, \ldots, \xi_n^\star)$ is an orthonormal basis of $T_x^\star M$ and $B_1(x) \leqslant \ldots \leqslant B_s(x) > 0$ the modules of the non zero eigenvalues of $B(x)$, viewed as an antisymmetric endomorphism on $T_x M$. Let

$$(4.3) \ \ \nu_B(\lambda) = \frac{2^{s-n}\pi^{-n/2}}{\Gamma\big(\frac{n}{2}-s+1\big)} B_1 \ldots B_s \sum_{(p_1,\ldots,p_s)\in\mathbb{N}^s} \Big(\lambda - \sum_{1 \leqslant j \leqslant s} (2p_j+1)B_j\Big)_+^{(n/2)-s} .$$

Here a_+ means the maximum of a and 0 , with the special convention that $a_+^0 = 0$ for $a \leqslant 0$ and $a_+^0 = 1$ for $a > 0$. It is clear that $\nu_B(\lambda)$ is a non decreasing function of λ that is continuous on the left side; one can also check that $\nu_{B(x)}(\lambda)$ is lower semi-continuous in x . Let $\overline{\nu}_B(\lambda) = \lim_{\varepsilon\to 0+} \nu_B(\lambda)$ be the associated increasing function continuous on the right side. The asymptotic eigenvalue distribution of $Q_{\Omega,k}$ is then given by the following fundamental theorem.

4.4. Theorem. — *Let $V_1(x), \ldots, V_t(x)$ be the eigenvalues of the endomorphism $V(x) \in \mathrm{End}(F_x)$. Then*

$$\liminf_{k\to+\infty} \ k^{-n/2}\, N_{\Omega,k}(\lambda) \geqslant \sum_{1 \leqslant j \leqslant t} \int_\Omega \nu_B(V_j + \lambda)\, dV \ ,$$

$$\limsup_{k\to+\infty} \ k^{-n/2}\, N_{\Omega,k}(\lambda) \leqslant \sum_{1 \leqslant j \leqslant t} \int_{\overline{\Omega}} \overline{\nu}_B(V_j + \lambda)\, dV \ .$$

Assume that $\partial\Omega$ has measure zero. As the integrals are increasing functions of λ and as the upper bound is the limit of the lower bound at $\lambda + 0$, we easily conclude :

4.5. Corollary. — *There is a countable subset $\mathcal{D} \subset \mathbb{R}$ such that*

$$N_{\Omega,k}(\lambda) \sim k^{n/2} \sum_{1 \leqslant j \leqslant t} \int_\Omega \nu_B(V_j + \lambda)\, dV \ , \qquad \forall \lambda \in \mathbb{R} \setminus \mathcal{D} \ .$$

The proof of theorem 4.4 is made in several steps. The first step is a localization procedure originated with the work of H. Weyl. Weyl [We] introduced the localization procedure and the minimax principle to get the asymptotic estimate of the distribution of eigenvalues of linear partial differential equations. Assume that Ω is partitioned into a union of small cubes plus some extra set of small measure. The intuition is that for the Dirichlet problem with zero boundary value, the wave length of all eigenfunctions tends to 0 as k tends to infinity and the Dirichlet problem looks more and more like the union of the Dirichlet problems for the small cubes. If the side of the cubes is well chosen ($k^{-1/3}$ is a suitable scale) the problem can be approximated by a simpler one in which the potential V and the curvature B are constant. At this point, an expansion of u as a Fourier series shows that the variables can be separated. Our Dirichlet problem is then reduced to the 1–dimensional harmonic oscillator problem, or to the Dirichlet problem for the operator $-d^2/dx^2$ on an interval. In these cases, the eigenvalues and eigenfunctions can be computed explicitly. The general estimate follows. $\square$

To get the eigenvalue distribution of $Q^m_{\Omega,k}$ in (3.4), we apply corollary 4.5 to the bundles $F = \Lambda^{n,m}T^\star X \times E$, $L = L_\chi$ and note that $\dim_{\mathbb{R}} X = 2n$. When the eigenvalues β_j are ordered in such a way that $|\beta_1| \geqslant \ldots \geqslant |\beta_s| > 0 = \beta_{s+1} = \ldots = \beta_n$, we get $B_j = |\beta_j|$. By (3.5), the eigenvalues V_j are the quantities $\beta_{\complement J} - \beta_J$ when J describes all multi-indices of length $|J| = m$, counted with multiplicity r . Hence the eigenvalue distribution $N^m_{\Omega,k}(\lambda)$ of $Q^m_{\Omega,k}(\lambda)$ is given by

$$N^m_{\Omega,k}(\lambda) \sim rk^n \sum_{|J|=m} \int_\Omega \nu_B(2\lambda + \beta_{\complement J} - \beta_J)\, dV \ , \qquad \forall \lambda \in \mathbb{R} \setminus \mathcal{D} \ .$$

We use this estimate when λ tends to $0+$. Each integral converges to the limit $\int_\Omega \overline{\nu}_B(\beta_{\complement J} - \beta_J)\, dV$ and we have

$$\overline{\nu}_B(\beta_{\complement J} - \beta_J) = \frac{2^{s-2n}\pi^{-n}}{\Gamma(n-s+1)} |\beta_1 \ldots \beta_s| \times \sum_{(p_1,\ldots,p_s \in \mathbb{N}^s)} \Big(\beta_{\complement J} - \beta_J - \sum_{1 \leqslant j \leqslant s} (2p_j+1)|\beta_j|\Big)_+^{n-s} \ .$$

The expression between $(\quad)_+$ is always $\leqslant 0$, and it can be zero only when $p_1 = \ldots = p_s = 0$, $\beta_j \leqslant 0$ for $j \in J$ and $\beta_j \geqslant 0$ for $j \in \complement J$. By our conventions $\lim_{\lambda\to 0+} (\lambda)^0_+ = 1$ and the result is non zero only when $s = n$. The above function is therefore equal to zero in all cases except when $\beta_j < 0$ for $j \in J$ and $\beta_j > 0$ for $j \in \complement J$; then

$$\overline{\nu}_B(\beta_{\complement J} - \beta_J) = (2\pi)^{-n} |\beta_1 \ldots \beta_n| \ .$$

This situation can only happen at a point $x \in X(m, L_\chi)$, and the corresponding multi-index J is then unique. We find therefore

$$\begin{aligned} \lim_{\lambda \to 0+} \ \lim_{k \to +\infty} \ k^{-n} N^m_{\Omega,k}(\lambda) &= r \int_{X(m,L_\chi)} (2\pi)^{-n}\, |\beta_1 \ldots \beta_n|\, dV \\ &= \frac{r}{n!} \int_{X(m,L_\chi)} (-1)^m \Big(\frac{i}{2\pi} c(L_\chi)\Big)^n\, dV \ . \end{aligned}$$

This shows that theorem 1.1 is valid for all $m \geqslant q$ when the curvature form $ic(L)$ is replaced by $ic(L_\chi)$. Let $K \subset\subset X$ be the exceptional compact set where the convexity properties of ψ and L are not satisfied. Take a convex increasing function χ such that $\chi = 0$ on $]-\infty, c_0]$ with $X_{c_0} \supset K$. Then

$$ic(L_\chi) = ic(L) + id'd''(\chi \circ \psi)$$

coincides with $ic(L)$ on K and has at least $n-(p-1)-(q-1)$ positive eigenvalues on $X \setminus K$, because $ic(L)$ and $id'd''\psi$ are both positive on subspaces of respective codimensions $p-1, q-1$ in TX . Hence $X(m, L_\chi) = X(m, L)$ for $m \geqslant p+q-1$ and $ic(L_\chi) = ic(L)$ on these sets. Theorem 1.1 is proved.

References

[A-G] A. Andreotti and H. Grauert. — *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France, **90** (1962), 193–259.

[Bi] J.-M. Bismut. — *Demailly's asymptotic inequalities : a heat equation proof*, J. Funct. Anal., **72** (1987), 263–278.

[Bo] Th. Bouche. — *Inégalités de Morse pour la d''–cohomologie sur une variété holomorphe non compacte*, Submitted for publication in Ann. Scient. Éc. Norm. Sup., 1988.

[D1] J.-P. Demailly. — *Champs magnétiques et inégalités de Morse pour la d''-cohomologie*, Ann. Inst. Fourier Grenoble, **35** (1985), 189–229.

[D2] J.-P. Demailly. — *Sur les théorèmes d'annulation et de finitude de T. Ohsawa et O. Abdelkader*, Séminaire P. Lelong – P. Dolbeault – H. Skoda (Analyse) 1985/86, Lecture Notes in Math. n° 1295, Springer-Verlag (1987), 48–58.

[D3] J.-P. Demailly. — *Sur l'identité de Kodaira-Nakano en géométrie hermitienne.* Séminaire P. Lelong – P. Dolbeault – H. Skoda (Analyse) (1983/84), 88-97, Lecture Notes in Math. **1198**, Springer, 1986.

[G-R] H. Grauert and O. Riemenschneider. — *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math., **11** (1970), 263–292.

[Se] J.-P. Serre. — *Fonctions automorphes : quelques majorations dans le cas où X/G est compact*,Séminaire Cartan 1953–1954, 2–1 to 2–9.

[Si] C.L. Siegel. — *Meromorphe Funktionen auf kompakten Mannigfaltigkeiten*, Nachrichten der Akademie der Wissenschaften in Göttingen, Math.-Phys. Klasse, **4** (1955), 71–77.

[S1] Y.T. Siu. — *A vanishing theorem for semi-positive line bundles over non-Kähler manifolds*, J. Differential Geom., **19** (1984), 431–452.

[S2] Y.T. Siu. — *Some recent results in complex manifold theory related to vanishing theorems for the semi-positive case*, Proceedings of the Bonn Arbeitstagung 1984, Lecture Notes in Math. n° 1111, Springer (1985), 169–192.

[S3] Y.T. Siu. — *Asymptotic Morse inequalities for analytic sheaf cohomology,* Sém. Bourbaki, exposé n° 666, juin 1986, 16 p.

[S4] Y.T. Siu. — *Calculus inequalities derived from holomorphic Morse inequalities*, Preprint, 1986, to appear in Math. Annalen.

[We] H. Weyl. — *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partielle Differentialgleichungen*, Math. Ann., **71** (1911), 441–469.

[Wi] E. Witten. — *Supersymmetry and Morse theory*, J. Differential Geom., **17** (1982), 661–692.