Pierre Lelong : a foundational work in complex analysis and in analytic geometry

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My first encounter with Pierre Lelong goes back to 1977, a year during which I started to attend his "Séminaire d'Analyse" in Paris, coorganized in collaboration with Henri Skoda since 1976. During the same year, I also benefitted from a series of lectures that Pierre Lelong gave on the theory of plurisubharmonic functions and positive currents, following a PhD course presented a few months earlier by Henri Skoda at Université Paris VI. These early contacts have had a strong and lasting influence on my scientific career, that's the very least I can say. In fact, my later scientific investigations almost never departed from the fundamental theories initiated by Pierre Lelong during the decades 1940–50 and 1950–60 ([Lel42], [Lel57]): these theories have wonderful applications to vast subdomains of mathematics, e.g. in Number Theory or Algebraic Geometry. Even though many mathematicians throughout the world have continued exploring these directions in the following decades, most experts would certainly agree that a lot remains to be done today.

A few years later, when I defended my "Thèse de Doctorat d'État" in 1982, I had the privilege of being invited at Pierre Lelong's private dwelling in Paris. This was the occasion for me to realize another important aspect of his past activities, namely, his deep commitment to politics, and the guidance he exerted in 1959–1961 during a major reform of higher education and research, as one of the scientific advisors of Géneral de Gaulle, then the President of the French Republic. Especially impressive were Pierre Lelong's private library and the unusual number of books and documents dealing with politics and political science. At present, the French system of higher education is faced with severe difficulties, and I cannot refrain from thinking what a benefit my country had during the 60's, a period of course economically much more favorable, when the scientific policy was guided by such enlightened minds as Pierre Lelong. In fact, France enjoyed at that time a sustained scientific and technological development, as well as a very strong increase of the number of students at universities. One would like to see clearer signs today that the European governments are ready to invest in science and to give it again a prominent role in the evolution of society – and, as a consequence, to rely extensively on the expertise of the scientific community rather than on technocrats!

Even though this is probably not the most central part of Pierre Lelong's scientific work, I would like to discuss here one contribution that has in my view shed light on several important problems. This is a paper entitled *"Éléments extrémaux sur le cône des courants positifs fermés"* (*"Extremal elements in the cone of closed positive currents"*), published in the "Séminaire d'Analyse" in 1971/1972 [Lel71]. The first main statement of the paper is:

Theorem 1. *Let M be an analytic set of complex codimension p, assumed to be irreducible, in a connected complex analytic analytic X (countable at infinity). Then the current of integration $[M]$ is extremal in the cone of closed positive currents of bidegree (p, p) in X.*

After stating this result, Pierre Lelong observed that many other examples of closed positive currents that had then been investigated were not extremal, especially those arising from smooth convex functions or smooth plurisubharmonic exhaustion functions like $\log |z|$, and he concluded: "*It is likely that Theorem* 1 *does not produce all extremal elements in the cone of closed positive currents; this seems to be an important unsolved question of complex geometry*". I still remember a private discussion we had on this issue at Jussieu. Strongly stimulated by these observations and by a further exchange with Jean-Louis Verdier, I realized a couple of weeks later, around end of 1981, that such a restrictive property concerning extremal elements could not hold. In fact, it would have implied via Choquet's represention theorem a formulation of Hodge conjecture that was much too strong to be true. Therefore, some of the extremal elements must be more complicated than currents of integration on analytic sets, and I found shortly afterwards an explicit example of such an extremal closed positive current of bidegree $(1,1)$ in the complex projective plane [De82]. As first noticed by Eric Bedford, further examples appear in a natural way in complex dynamics of several variables; many invariant closed positive currents produced by complex dynamical systems are actually extremal currents, although their support is in general a fractal set, and therefore is not analytic. This is for instance what happens for a current of the form $\lim_{k\to+\infty} d^{-k}\frac{i}{\pi}\partial\overline{\partial}\log|P^k(z)|$, where P^k is the k-th iteration of a polynomial endomorphism of degree $d > 1$ on projective space, the support of such a current being a Julia set [Sib99]. The dynamical study of "hyperbolic" endomorphisms of certain algebraic surfaces, e.g. K3 surfaces, also leads to such extremal invariant currents [Ca03].

Another fundamental statement contained in the above cited article [Lel71] is the following.

Theorem 2. *if G is a pseudoconvex domain in $\mathbb{C}^n$, the positive cone generated over rational coefficients by functions of the form* $\log|f|$, *where f is holomorphic in G, is dense in the cone of plurisubharmonic functions on G.*

Corollary. *If moreover $H^2(G,\mathbb{R}) = 0$, the positive cone generated over rational coefficients by currents of integration $[D]$ on irreducible divisors of G is dense in the cone of closed positive currents of type* $(1,1)$ *on G.*

The original proof of Lelong rests upon a use of complex function theory on Hartogs domains of the type $|w| < e^{-\varphi(z)}$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. If φ is plurisubharmonic and if z is taken in a pseudoconvex domain G, it is known that the corresponding Hartogs domain in $G \times \mathbb{C}$ is again pseudoconvex ; as a consequence, there exists a holomorphic function $F(z,w)$, the domain of existence of which is precisely the Hartogs domain $|w| < e^{-\varphi(z)}$. The approximation of the function φ by logarithms of holomorphic functions $f_j(z)$ is then obtained by applying Hadamard's formula to compute the radius of convergence of the power series $\sum_{k\in\mathbb{N}} a_k(z)w^k$ of $F(z,w)$. The corollary is then derived by means of the fundamental "Lelong-Poincaré" equation, stating that for every holomorphic function F the current $\frac{i}{\pi}\partial\overline{\partial}\log|F|$ coincides with the current of integration $[Z_F]$ on the zero divisor of F.

These approximation results for currents are now a central ingredient of modern analytic geometry. By replacing the qualitative existence theorem of defining holomorphic functions and Hadamard's formula with deeper results such as the Ohsawa-Takegoshi L^2 extension theorem ([OT87]), one can obtain more precise statements in which the multiplicities of the approximating $\mathbb{Q}$-divisors converge uniformly to the "Lelong numbers" of the given closed positive $(1,1)$-current. In that way, one gets a very strong analytic tool that allows in particular to prove numerous geometric results – for instance, Siu's theorem on the analyticity of level sets associated with Lelong numbers of closed positive currents [Siu74]. Another consequence of such techniques in algebraic geometry is the proof of the conjecture on the invariance of plurigenera for deformations of arbitrary non singular projective algebraic varieties ([Siu00], [Pa07]) ; the latter result relies again on the Ohsawa-Takegoshi theorem and on a compactness argument for closed positive currents of type $(1,1)$; it comes as a surprise that no algebraic proof is known at this point in time, although the statement involves only algebraic objects !

Finally, among applications to number theory, one should mention Bombieri's theorem on algebraic values of meromorphic functions of several variables satisfying algebraic differential equations [Bo70], [Sk76], [Lel79]. The proof, here again, exploits in an essential way the compactness properties of closed positive currents of type $(1,1)$ in classes of currents of finite order, in conjunction with Hörmander's L^2 estimates for the $\overline{\partial}$ operator.

Pierre Lelong's clever use of "flexible objects"(*), such as plurisubharmonic functions and positive currents, has permitted the emergence of various important techniques that have led to strong effective formulations of many results in algebraic, analytic or arithmetic geometry, especially in areas where previously known techniques could only produce qualitative results. Pierre Lelong was perfectly aware of the philosophical dimension of the contributions he made, and he has very early set up their most fundamental consequences.

References

[Bo70] Bombieri, E., *Algebraic values of meromorphic maps*, Invent. Math. **10** (1970) 267–287 ; Addendum, Invent. Math. **11** (1970) 163–166.

[Ca03] Cantat, S., *Dynamique des automorphismes des surfaces* K3, Acta Math. **187** (2001) 1–57.

[De82] Demailly, J.-P., *Courants positifs extrêmaux et conjecture de Hodge*, Invent. Math. **69** (1982) 347–374.

[Lel42] Lelong, P., *Définition des fonctions plurisousharmoniques*, C. R. Acad. Sci. Paris **215** (1942) p. 398 et p. 454.

[Lel57] Lelong, P., *Intégration sur un ensemble analytique complexe*, Bull. Soc. Math. France **85** (1957) 239–262.

[Lel68] Lelong, P., *Fonctions plurisousharmoniques et formes différentielles positives*, Dunod, Paris, Gordon & Breach, New York (1968).

[Lel71] Lelong, P., *Éléments extrémaux sur le cône des courants positifs fermés*, Springer Lecture Notes in Math. **332** (1973) 112–130.

[Lel73] Lelong, P., *Notice sur les titres et travaux scientifiques*, site de l'Académie des Sciences, `http://www.academie-sciences.fr/academie/membre/Lelong_notice_1973.pdf`.

[Lel79] Lelong, P., *Sur les cycles holomorphes à coefficients positifs dans* $\mathbb{C}^n$ *et un complément au théorème de E. Bombieri*, C. R. Math. Acad. Sc. Canada, vol. 1, n°4 (1979) 211–213.

[OT87] Ohsawa, T., Takegoshi, K., *On the extension of* L^2 *holomorphic functions*, Math. Zeitschrift **195** (1987) 197–204.

[Pa07] Păun, M., *Siu's invariance of plurigenera: a one-tower proof*, J. Differential Geom. **76** (2007) 485–493.

[Sib99] Sibony, N., *Dynamique des applications rationnelles de* $\mathbb{P}^k$, Dynamique et géométrie complexes (Lyon, 1997), ix–x, xi–xii, 97–185, Panor. Synthèses **8**, Soc. Math. France, Paris (1999).

[Siu74] Siu, Y.T., *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. **27** (1974) 53–156.

[Siu00] Siu, Y.T., *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type*, Complex Geometry (Göttingen, 2000), Springer, Berlin (2002) 223–277.

[Sko76] Skoda, H., *Estimations* L^2 *pour l'opérateur* $\overline{\partial}$ *et applications arithmétiques*, Séminaire P. Lelong (Analyse), année 1975/76, Lecture Notes in Math. no **538**, Springer-Verlag, Berlin (1977) 314–323.

(*) "Objets souples", according to the terminology employed by Pierre Lelong himself in his notice of scientific work sent to Académie des Sciences [Lel73].