# Appendix to I. Cheltsov and C. Shramov's article "Log canonical thresholds of smooth Fano threefolds" : On Tian's invariant and log canonical thresholds

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The goal of this appendix is to relate log canonical thresholds with the $\alpha$ invariant introduced by G. Tian [Tia87] for the study of the existence of Kähler-Einstein metrics. The approximation technique of closed positive $(1,1)$-currents introduced in [Dem92] is used to show that the $\alpha$ invariant actually coincides with the log canonical threshold.

Algebraic geometers have been aware of this fact after [DK01] appeared, and several papers have used it consistently in the latter years (see e.g. [JK01], [BGK05]). However, it turns out that the required result is stated only in a local analytic form in [DK01], in a language which may not be easily recognizable by algebraically minded people. Therefore, we will repair here the lack of a proper reference by stating and proving the statements required for the applications to projective varieties, e.g. existence of Kähler-Einstein metrics on Fano varieties and Fano orbifolds.

Usually, in these applications, only the case of the anticanonical line bundle $L = -K_X$ is considered. Here we will consider more generally the case of an arbitrary line bundle $L$ (or $\mathbb{Q}$-line bundle $L$) on a complex manifold $X$, with some additional restrictions which will be stated later.

Assume that $L$ is equipped with a *singular hermitian metric* $h$ (see e.g. [Dem90]). Locally, $L$ admits trivializations $\theta : L_{|U} \simeq U \times \mathbb{C}$, and on $U$ the metric $h$ is given by a weight function $\varphi$ such that

$$\|\xi\|_h^2 = |\xi|^2 e^{-2\varphi(z)}, \qquad z \in U, \quad \xi \in L_z$$

when $\xi \in L_z$ is identified with a complex number. We are interested in the case where $\varphi$ is (at the very least) a locally integrable function for the Lebesgue measure, since it is then possible to compute the curvature form

$$\Theta_{L,h} = \frac{i}{\pi}\partial\overline{\partial}\varphi$$

in the sense of distributions. We have $\Theta_{L,h} \geqslant 0$ as a $(1,1)$-current if and only if the weights $\varphi$ are plurisubharmonic functions. In the sequel we will be interested only in that case. Let us first introduce the concept of complex singularity exponent, following e.g. [Var82, 83], [ArGV85] and [DK01].

**(A.1) Definition.** *If $K$ is a compact subset of $X$, we define the complex singularity exponent $c_K(h)$ of the metric $h$, written locally as $h = e^{-2\varphi}$, to be the supremum of all positive numbers $c$ such that $h^c = e^{-2c\varphi}$ is integrable in a neighborhood of every point $z_0 \in K$, with respect to the Lebesgue measure in holomorphic coordinates centered at $z_0$.*

Now, we introduce a generalized version of Tian's invariant $\alpha$, as defined in [Tia87] (see also [Siu88]).

**(A.2) Definition.** *Assume that $X$ is a compact manifold and that $L$ is a pseudo-effective line bundle, i.e. $L$ admits a singular hermitian metric $h_0$ with $\Theta_{L,h_0} \geqslant 0$. If $K$ is a compact subset of $X$, we put*

$$\alpha_K(L) = \inf_{\{h,\,\Theta_{L,h}\geqslant 0\}} c_K(h)$$

*where $h$ runs over all singular hermitian metrics on $L$ such that $\Theta_{L,h} \geqslant 0$.*

In algebraic geometry, it is more usual to look instead at linear systems defined by a family of linearly independent sections $\sigma_0, \sigma_1, \ldots \sigma_N \in H^0(X, L^{\otimes m})$. We denote by $\Sigma$ the vector subspace generated by these sections and by

$$|\Sigma| := P(\Sigma) \subset |mL| := P(H^0(X, L^{\otimes m}))$$

the corresponding linear system. Such an $(N+1)$-tuple of sections $\sigma = (\sigma_j)_{0\leqslant j\leqslant N}$ defines a singular hermitian metric $h$ on $L$ by putting in any trivialization

$$\|\xi\|_h^2 = \frac{|\xi|^2}{\big(\sum_j |\sigma_j(z)|^2\big)^{1/m}} = \frac{|\xi|^2}{|\sigma(z)|^{2/m}}, \qquad \xi \in L_z,$$

hence $h(z) = |\sigma(z)|^{-2/m}$ with $\varphi(z) = \frac{1}{m}\log|\sigma(z)| = \frac{1}{2m}\log\sum_j |\sigma_j(z)|^2$ as the associated weight function. Therefore, we are interested in the number $c_K(|\sigma|^{-2/m})$. In the case of a single section $\sigma_0$ (corresponding to a one-point linear system), this is the same as the log canonical threshold $\mathrm{lct}_K(X, \frac{1}{m}D)$ of the associated divisor $D$, in the notation of Section 1 of [CS08]. We will also use the formal notation $\mathrm{lct}_K(X, \frac{1}{m}|\Sigma|)$ in the case of a higher dimensional linear system $|\Sigma| \subset |mL|$.

Now, recall that the line bundle $L$ is said to be *big* if the Kodaira-Iitaka dimension $\kappa(L)$ equals $n = \dim_{\mathbb{C}} X$. The main result of this appendix is

**(A.3) Theorem.** *Let $L$ be a big line bundle on a compact complex manifold $X$. Then for every compact set $K$ in $X$ we have*

$$\alpha_K(L) = \inf_{\{h,\,\Theta_{L,h}\geqslant 0\}} c_K(h) = \inf_{m\in\mathbb{Z}_{>0}} \inf_{D\in|mL|} \mathrm{lct}_K\Big(X, \frac{1}{m}D\Big).$$

Observe that the inequality

$$\inf_{m\in\mathbb{Z}_{>0}} \inf_{D\in|mL|} \mathrm{lct}_K\Big(X, \frac{1}{m}D\Big) \geq \inf_{\{h,\,\Theta_{L,h}\geqslant 0\}} c_K(h)$$

is trivial, since any divisor $D \in |mL|$ gives rise to a singular hermitian metric $h$. The converse inequality will follow from the approximation technique of [Dem92] and some elementary analysis. The proof is parallel to the proof of Theorem 4.2 of [DK01], although the language used there was somewhat different. In any case, we use again the crucial concept of multiplier ideal sheaves : if $h$ is a singular hermitian metric with local plurisubharmonic weights $\varphi$, the multiplier ideal sheaf $\mathcal{I}(h) \subset \mathcal{O}_X$ (also denoted by $\mathcal{I}(\varphi)$) is the ideal sheaf defined by

$$\mathcal{I}(h)_z = \Big\{ f \in \mathcal{O}_{X,z}\,;\ \exists \text{ a neighborhood } V \ni z \text{ such that } \int_V |f(x)|^2 e^{-2\varphi(x)} d\lambda(x) < +\infty \Big\}$$

where $\lambda$ is the Lebesgue measure. By Nadel [Nad89], this is a coherent analytic sheaf on $X$. Theorem (A.3) has a more precise version which can be stated as follows.

**(A.4) Theorem.** *Let $L$ be a line bundle on a compact complex manifold $X$ possessing a singular hermitian metric $h$ with $\Theta_{L,h} \geqslant \varepsilon\omega$ for some $\varepsilon > 0$ and some smooth positive definite hermitian $(1,1)$-form $\omega$ on $X$. For every real number $m > 0$, consider the space $\mathcal{H}_m = H^0(X, L^{\otimes m} \otimes \mathcal{I}(h^m))$ of holomorphic sections $\sigma$ of $L^{\otimes m}$ on $X$ such that*

$$\int_X |\sigma|^2_{h^m} dV_\omega = \int_X |\sigma|^2 e^{-2m\varphi} dV_\omega < +\infty,$$

*where $dV_\omega = \frac{1}{m!}\omega^m$ is the hermitian volume form. Then for $m \gg 1$, $\mathcal{H}_m$ is a non zero finite dimensional Hilbert space and we consider the closed positive $(1,1)$-current*

$$T_m = \frac{i}{2\pi}\partial\overline{\partial}\Big(\frac{1}{2m}\log\sum_k |g_{m,k}|^2\Big) = \frac{i}{2\pi}\partial\overline{\partial}\Big(\frac{1}{2m}\log\sum_k |g_{m,k}|^2_h\Big) + \Theta_{L,h}$$

*where $(g_{m,k})_{1\leqslant k\leqslant N(m)}$ is an orthonormal basis of $\mathcal{H}_m$. Then :*

(i) *For every trivialization $L_{|U} \simeq U \times \mathbb{C}$ on a cordinate open set $U$ of $X$ and every compact set $K \subset U$, there are constants $C_1, C_2 > 0$ independent of $m$ and $\varphi$ such that*

$$\varphi(z) - \frac{C_1}{m} \leqslant \psi_m(z) := \frac{1}{2m}\log\sum_k |g_{m,k}(z)|^2 \leqslant \sup_{|x-z|<r} \varphi(x) + \frac{1}{m}\log\frac{C_2}{r^n}$$

*for every $z \in K$ and $r \leqslant \frac{1}{2}d(K, \partial U)$. In particular, $\psi_m$ converges to $\varphi$ pointwise and in $L^1_{\rm loc}$ topology on $\Omega$ when $m \to +\infty$, hence $T_m$ converges weakly to $T = \Theta_{L,h}$.*

(ii) *The Lelong numbers $\nu(T,z) = \nu(\varphi,z)$ and $\nu(T_m,z) = \nu(\psi_m,z)$ are related by*

$$\nu(T,z) - \frac{n}{m} \leqslant \nu(T_m,z) \leqslant \nu(T,z) \qquad \text{for every } z \in X.$$

(iii) *For every compact set $K \subset X$, the complex singularity exponents of the metrics given locally by $h = e^{-2\varphi}$ and $h_m = e^{-2\psi_m}$ satisfy*

$$c_K(h)^{-1} - \frac{1}{m} \leqslant c_K(h_m)^{-1} \leqslant c_K(h)^{-1}.$$

*Proof.* The major part of the proof is a minor variation of the arguments already explained in [Dem92] (see also [DK01] Theorem 4.2). We give them here in some detail for the convenience of the reader.

(i) We note that $\sum |g_{m,k}(z)|^2$ is the square of the norm of the evaluation linear form $\sigma \mapsto \sigma(z)$ on $\mathcal{H}_m$, hence

$$\psi_m(z) = \sup_{\sigma \in B(1)} \frac{1}{m} \log |\sigma(z)|$$

where $B(1)$ is the unit ball of $\mathcal{H}_m$. For $r \leqslant \frac{1}{2} d(K, \partial\Omega)$, the mean value inequality applied to the plurisubharmonic function $|\sigma|^2$ implies

$$\begin{aligned} |\sigma(z)|^2 &\leqslant \frac{1}{\pi^n r^{2n}/n!} \int_{|x-z|<r} |\sigma(x)|^2 d\lambda(x) \\ &\leqslant \frac{1}{\pi^n r^{2n}/n!} \exp\Big(2m \sup_{|x-z|<r} \varphi(x)\Big) \int_\Omega |\sigma|^2 e^{-2m\varphi} d\lambda. \end{aligned}$$

If we take the supremum over all $\sigma \in B(1)$ we get

$$\psi_m(z) \leqslant \sup_{|x-z|<r} \varphi(x) + \frac{1}{2m} \log \frac{1}{\pi^n r^{2n}/n!}$$

and the right hand inequality in (i) is proved. Conversely, the Ohsawa-Takegoshi extension theorem [OhT87], [Ohs88] applied to the 0-dimensional subvariety $\{z\} \subset U$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function $f$ on $U$ such that $f(z) = a$ and

$$\int_U |f|^2 e^{-2m\varphi} d\lambda \leqslant C_3 |a|^2 e^{-2m\varphi(z)},$$

where $C_3$ only depends on $n$ and $\operatorname{diam} U$. Now, provided $a$ remains in a compact set $K \subset U$, we can use a cut-off function $\theta$ with support in $U$ and equal to 1 in a neighborhood of $a$, and solve the $\overline{\partial}$-equation $\overline{\partial} g = \overline{\partial}(\theta f)$ in the $L^2$ space associated with the weight $2m\varphi + 2(n+1)|\log|z-a|$, that is, the singular hermitian metric $h(z)^m |z-a|^{-2(n+1)}$ on $L^{\otimes m}$. For this, we apply the standard Andreotti-Vesentini-Hörmander $L^2$ estimates (see e.g. [Dem82] for the required version). This is possible for $m \geqslant m_0$ thanks to the hypothesis that $\Theta_{L,h} \geqslant \varepsilon\omega > 0$, even if $X$ is non Kähler ($X$ is in any event a Moishezon variety from our assumptions). The bound $m_0$ depends only on $\varepsilon$ and the geometry of a finite covering of $X$ by compact sets $K_j \subset U_j$, where $U_j$ are coordinate balls (say); it is independent of the point $a$ and even of the metric $h$. It follows that $g(a) = 0$ and therefore $\sigma = \theta f - g$ is a holomorphic section of $L^{\otimes m}$ such that

$$\int_X |\sigma|^2_{h^m} dV_\omega = \int_X |\sigma|^2 e^{-2m\varphi} dV_\omega \leqslant C_4 \int_U |f|^2 e^{-2m\varphi} dV_\omega \leqslant C_5 |a|^2 e^{-2m\varphi(z)},$$

in particular $\sigma \in \mathcal{H}_m = H^0(X, L^{\otimes m} \otimes \mathcal{I}(h^m))$. We fix $a$ such that the right hand side is 1. This gives the inequality

$$\psi_m(z) \geqslant \frac{1}{m} \log |a| = \varphi(z) - \frac{\log C_5}{2m}$$

which is the left hand part of statement (i).

(ii) The first inequality in (i) implies $\nu(\psi_m, z) \leqslant \nu(\varphi, z)$. In the opposite direction, we find

$$\sup_{|x-z|<r} \psi_m(x) \leqslant \sup_{|x-z|<2r} \varphi(x) + \frac{1}{m}\log\frac{C_2}{r^n}.$$

Divide by $\log r < 0$ and take the limit as $r$ tends to 0. The quotient by $\log r$ of the supremum of a psh function over $B(x, r)$ tends to the Lelong number at $x$. Thus we obtain

$$\nu(\psi_m, x) \geqslant \nu(\varphi, x) - \frac{n}{m}.$$

(iii) Again, the first inequality (in (i) immediately yields $h_m \leqslant C_6 h$, hence $c_K(h_m) \geqslant c_K(h)$. For the converse inequality, since we have $c_{\cup K_j}(h) = \min_j c_{K_j}(h)$, we can assume without loss of generality that $K$ is contained in a trivializing open patch $U$ of $L$. Let us take $c < c_K(\psi_m)$. Then, by definition, if $V \subset X$ is a sufficiently small open neighborhood of $K$, the Hölder inequality for the conjugate exponents $p = 1 + mc^{-1}$ and $q = 1 + m^{-1}c$ implies, thanks to equality $\frac{1}{p} = \frac{c}{mq}$,

$$\begin{aligned}
\int_V e^{-2(m/p)\varphi} dV_\omega &= \int_V \Big(\sum_{1\leqslant k\leqslant N(m)} |g_{m,k}|^2 e^{-2m\varphi}\Big)^{1/p} \Big(\sum_{1\leqslant k\leqslant N(m)} |g_{m,k}|^2\Big)^{-c/mq} dV_\omega \\
&\leqslant \left(\int_X \sum_{1\leqslant k\leqslant N(m)} |g_{m,k}|^2 e^{-2m\varphi} dV_\omega\right)^{1/p} \left(\int_V \Big(\sum_{1\leqslant k\leqslant N(m)} |g_{m,k}|^2\Big)^{-c/m} dV_\omega\right)^{1/q} \\
&= N(m)^{1/p} \left(\int_V \Big(\sum_{1\leqslant k\leqslant N(m)} |g_{m,k}|^2\Big)^{-c/m} dV_\omega\right)^{1/q} < +\infty.
\end{aligned}$$

From this we infer $c_K(h) \geqslant m/p$, i.e., $c_K(h)^{-1} \leqslant p/m = 1/m + c^{-1}$. As $c < c_K(\psi_m)$ was arbitrary, we get $c_K(h)^{-1} \leqslant 1/m + c_K(h_m)^{-1}$ and the inequalities of (iii) are proved. □

*Proof of Theorem (A.3).* Given a big line bundle $L$ on $X$, there exists a modification $\mu : \widetilde{X} \to X$ of $X$ such that $\widetilde{X}$ is projective and $\mu^* L = \mathcal{O}(A + E)$ where $A$ is an ample divisor and $E$ an effective divisor with rational coefficients. By pushing forward by $\mu$ a smooth metric $h_A$ with positive curvature on $A$, we get a singular hermitian metric $h_1$ on $L$ such that $\Theta_{L,h_1} \geqslant \mu_*\Theta_{A,h_A} \geqslant \varepsilon\omega$ on $X$. Then For any $\delta > 0$ and any singular hermitian metric $h$ on $L$ with $\Theta_{L,h} \geqslant 0$, the interpolated metric $h_\delta = h_1^\delta h^{1-\delta}$ satisfies $\Theta_{L,h_\delta} \geqslant \delta\varepsilon\omega$. Since $h_1$ is bounded away from 0, it follows that $c_K(h) \geqslant (1-\delta)c_K(h_\delta)$ by monotonicity. By theorem (A.4, iii) applied to $h_\delta$, we infer

$$c_K(h_\delta) = \lim_{m\to+\infty} c_K(h_{\delta,m}),$$

and we also have

$$c_K(h_{\delta,m}) \geqslant \mathrm{lct}_K\Big(\frac{1}{m}D_{\delta,m}\Big)$$

for any divisor $D_{\delta,m}$ associated with a section $\sigma \in H^0(X, L^{\otimes m} \otimes \mathcal{I}(h_\delta^m))$, since the metric $h_{\delta,m}$ is given by $h_{\delta,m} = (\sum_k |g_{m,k}|^2)^{-1/m}$ for an orthornormal basis of such sections. This clearly implies

$$c_K(h) \geqslant \liminf_{\delta\to 0}\ \liminf_{m\to+\infty}\ \mathrm{lct}_K\Big(\frac{1}{m}D_{\delta,m}\Big) \geqslant \inf_{m\in\mathbb{Z}_{>0}}\ \inf_{D\in|mL|}\ \mathrm{lct}_K\Big(\frac{1}{m}D\Big). \qquad \square$$

In the applications, it is frequent to have a finite or compact group $G$ of automorphisms of $X$ and to look at $G$-invariant objects, namely $G$-equivariant metrics on $G$-equivariant line bundles $L$ ; in the case of a reductive algebraic group $G$ we simply consider a compact real form $G^{\mathbb{R}}$ instead of $G$ itself.

One then gets an $\alpha$ invariant $\alpha_{G,K}(L)$ by looking only at $G$-equivariant metrics in Definition A.2. All contructions made are then $G$-equivariant, especially $\mathcal{H}_m \subset |mL|$ is a $G$-invariant linear system. For every $G$-invariant compact set $K$ in $X$, we thus infer

$$\text{(A.5)}\qquad \alpha_{G,K}(L) := \inf_{\{h\ G\text{-equiv.},\,\Theta_{L,h}\geqslant 0\}} c_K(h) = \inf_{m\in\mathbb{Z}_{>0}} \ \inf_{|\Sigma|\subset|mL|,\ \Sigma^G=\Sigma} \ \mathrm{lct}_K\Big(\frac{1}{m}|\Sigma|\Big).$$

When $G$ is a finite group, one can pick for $m$ large enough a $G$-invariant divisor $D_{\delta,m}$ associated with a $G$-invariant section $\sigma$, possibly after multiplying $m$ by the order of $G$. One then gets the slightly simpler equality

$$\text{(A.6)}\qquad \alpha_{G,K}(L) := \inf_{\{h\ G\text{-equiv.},\,\Theta_{L,h}\geqslant 0\}} c_K(h) = \inf_{m\in\mathbb{Z}_{>0}} \ \inf_{D\in|mL|^G} \ \mathrm{lct}_K\Big(\frac{1}{m}D\Big).$$

In a similar manner, one can work on an orbifold $X$ rather than on a non singular variety. The $L^2$ techniques work in this setting with almost no change ($L^2$ estimates are essentially insensitive to singularities, since one can just use an orbifold metric on the open set of regular points).

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(June 12, 2008; printed on June 12, 2008)