Lars Hörmander and the theory of L^2 estimates for the $\overline{\partial}$ operator

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I met Lars Hörmander for the first time in the early 1980's, on the occasion of one of the "Komplexe Analysis" conferences held in Oberwolfach under the direction of Hans Grauert and Michael Schneider. My early mathematical education had already been greatly influenced by Hörmander's work on L^2 estimates for the $\overline{\partial}$-operator in several complex variables. The most basic statement is that one can solve an equation of the form $\overline{\partial}u = v$ for any given (n, q)-form v on a complex manifold X such that $\overline{\partial}v = 0$, along with a fundamental L^2 estimate of the form $\int_X |u|^2 e^{-\varphi} dV_\omega \le \int_X \gamma_q^{-1} |v|^2 e^{-\varphi} dV_\omega$. This holds true whenever φ is a plurisubharmonic function such that the right-hand side is finite, and X satisfies suitable convexity assumptions, e.g. when X possesses a weakly plurisubharmonic exhaustion function. Here dV_ω is the volume form of some Kähler metric ω on X, and $\gamma_q(x)$, at any point $x \in X$, is the sum of the q-smallest eigenvalues of $i\partial\overline{\partial}\varphi(x)$ with respect to $\omega(x)$. This was in fact the main subject of a PhD course delivered by Henri Skoda in Paris during the year 1976-1977, and, to a great extent, the theory of L^2 estimates was my entry point in complex analysis of several variables. At the same time, I followed a graduate course of Serge Alinhac on PDE theory, and Lars Hörmander appeared again as the one of the main heroes. I was therefore extremely impressed to meet him in person a few years later – his tall stature and physical appearance did make for an even stronger impression. I still remember that on the occasion of the Wednesday afternoon walk in the Black Forest, Hörmander was in a group of 2 or 3 that essentially left all the rest behind when hiking on the somewhat steep slopes leading to the Glaswaldsee, a dozen of kilometers North of the Mathematisches Forschungsinstitut Oberwolfach.

It seems that Lars Hörmander himself, at least in the mid 1960's, did not consider his work on $\overline{\partial}$-estimates [Hör65] to stand out in a particular way among his other achievements; after all, these estimates appeared to him to be only a special case of Carleman's technique, which also applies to more general classes of differential operators. In his own words, "*Apart from the results involving precise bounds, this paper does not give any new existence theorems for functions of several complex variables. However, we believe that it is justified by the methods of proof*". In spite of this rather modest statement, the paper already permitted to bypass the difficult question of boundary regularity involved in the Morrey-Kohn approach [Mor58, Koh63a, Koh63b, Koh64]. For this, Hörmander observes that the Friedrichs regularization lemma applies to the particular situation he considers. Also, and perhaps more importantly, Hörmander's technique gives a new proof of the existence of solutions of $\overline{\partial}$-equations on pseudoconvex or q-convex domains, thus recovering the results of Andreotti-Grauert [AG62] by a more direct analytic approach. We should mention here that Andreotti-Vesentini [AG65] independently obtained similar results in the context of complete Hermitian manifolds, through a refinement of the Bochner-Kodaira technique [Boc48], [Kod53, Kod54]. One year after the publication of [Hör65], Lars Hörmander published his tremendously influential book "An introduction to Complex Analysis in several variables" [Hör66], which is now considered to be one of the foundational texts in complex analysis and geometry.

It took only a few years to realize that the very precise L^2 estimates obtained by Hörmander for solutions of $\overline{\partial}$-equations had terrific applications to other domains of mathematics. Chapter VII of [Hör66] already derives a deep existence theorem for solutions of PDE equations with constant coefficients. More surprisingly, there are also striking applications in number theory. In 1970, Enrico Bombieri extended in this way an earlier result of Serge Lang concerning algebraic values of meromorphic maps of finite order: if a system of such functions has transcendance degree larger than the dimension and satisfies algebraic differential equations, then the set of points where they simultaneously take values in an algebraic number field is contained in a certain algebraic hypersurface of bounded degree. The proof combines a use of Lelong's theory of positive currents with L^2 estimates for arbitrary singular plurisubharmonic weights, cf. [Bom70]. It is crucial here to allow φ to have poles, e.g. logarithmic poles of the form $\log \sum |g_j|^2$ where the g_j' are holomorphic functions sharing common zeroes (the degree bounds were later refined by Henri Skoda [Sko76] and [Lel79]). Using Hörmander's techniques in a fundamental way, Henri Skoda gave further major applications to complex analysis and geometry. In [Sko72a], it is shown that analytic subsets of $\mathbb{C}^n$ possessing a given growth of the area at infinity can be defined by holomorphic functions with a precise control of the growth; such quantitative versions of Cartan's theorems A and B had been also anticipated in [Hör66] (see e.g. section 7.6, "Cohomology with bounds"). The article [Sko72b], on the other hand, gives an almost optimal L^2 estimate for the solutions of Bézout equations $\sum g_j h_j = f$ in the ring of holomorphic functions (if f and the g_j's are given so that a certain quotient $|f|^2|g|^{-2(n+1+\varepsilon)}e^{-\varphi}$ is integrable, then h can be found such that $|h|^2|g|^{-2(n+\varepsilon)}e^{-\varphi}$ is integrable). This result, in its turn, yields profound results of local algebra, e.g. in the form of a fine control of the integral closure of ideals in the ring of germs of holomorphic functions [BS74]; see also [Sko78] for related more geometric statements. It is remarkable that algebraic proofs of such results were only found after the discovery of the analytic proof. About the same time, Yum-Tong Siu [Siu74] gave a proof of the analyticity of sublevel sets of Lelong numbers of closed positive currents; Siu's main argument again involves the Hörmander-Bombieri technique.

But the story does not stop there, Hörmander's L^2 estimates also have intimate connections with fundamental questions of algebraic geometry. In fact, Takeo Ohsawa and Kensho Takegoshi discovered in 1987 a very deep L^2 extension theorem: every L^2 holomorphic function defined on a subvariety Y of a complex manifold X can be extended as a holomorphic function F on X satisfying an L^2 bound $\int_X |F|^2 e^{-\varphi} dV_X \leq C \int_Y |f|^2 e^{-\varphi} dV_Y$. This holds true provided φ is plurisubharmonic and suitable curvature assumptions are satisfied ([OT87]). The initial proof used a complicated twisted Bochner-Kodaira-Nakano formula, but it was recently discovered by Bo-Yong Chen [Cby11, Cby13] that in fact the Morrey-Kohn-Hörmander estimates were in fact sufficient to prove it, while improving the estimates along the way. The Ohsawa-Takegoshi L^2 extension theorem itself has quite remarkable consequences in the theory of analytic singularities, for instance a basic regularization theorem for closed positive currents [Dem92], or a proof of the semicontinuity of complex singularity exponents [DK01]. Finally, [OT87] can be used to confirm the conjecture on the invariance of plurigenera in a deformation of projective algebraic varieties ([Siu00, Pau07]): this basic statement of algebraic geometry still has no algebraic proof at this date! Another very strong link with algebraic geometry occurs through the concept of multiplier ideal sheaves: if φ is an arbitrary plurisubharmonic function, then the ideal sheaf of germs of holomorphic functions such that $|f|^2 e^{-\varphi}$ is integrable is a coherent analytic sheaf; the proof is essentially a straightforward consequence of the Hörmander-Bombieri technique. In general, if $(L, e^{-\varphi})$ is a singular hermitian line bundle on a compact Kähler manifold X, one has $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$ as soon as the curvature of φ is positive definite (a generalization of the Kawamata-Viehweg vanishing theorem [Kaw83, Vie83], cf. [Dem89], [Nad89], [Dem93]). In case $i\partial\overline{\partial}\varphi$ is merely semipositive, one gets instead a surjective Lefschetz morphism $\omega^q \wedge \bullet : H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(\varphi)) \to H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi))$ ([DPS03]). All the arguments use Hörmander's theory of L^2 estimates in one way or the other. Contrary to the exceedingly modest words of Hörmander, the L^2 existence theorem for solutions of $\overline{\partial}$-equations appears to be one of the most powerful theorems of contemporary mathematics!

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