Two Counterexamples Concerning the Pluri-Complex Green Function in $\mathbb{C}^n$

*Eric Bedford and Jean-Pierre Demailly**
Indiana University and Université de Grenoble I

Given a domain $\Omega \subset \mathbb{C}^n$ and a point $z \in \Omega$, we consider the function

$$u_z(\zeta) = \sup\{v(\zeta) : v \text{ is psh on } \Omega, v < 0, \text{ and } v(\zeta) \leqslant \log|\zeta - z| + O(1)\},$$

which is the *pluri-complex Green function* on Ω with logarithmic pole at z (cf. [2,3,4]). For a large class of domains Ω, (e.g. if $\partial\Omega$ is strongly pseudoconvex and bounded), then $u_z \in C^\infty(\bar{\Omega} - \{z\})$, $u_z = 0$ on $\partial\Omega$, and $u_z(\zeta) = \log|z - \zeta| + O(1)$. Several further properties of u_z were derived in [2], where the following questions were also raised:

1. Is $u_z \in C^2(\bar{\Omega} - \{z\})$?

2. Is u_z symmetric, i.e. $u_z(\zeta) = u_\zeta(z)$?

Lempert [4] has shown that if $\Omega \subset \mathbb{C}^n$ is strictly convex and smoothly bounded, then the answer to these questions is "Yes." The convex situation, however, is quite special, and the point of this note is to show that for strongly pseudoconvex domains the answer to both these questions is "No."

Proposition **1**. — *Let $\Omega \subset \mathbb{C}^2$ be a bounded, connected, strongly pseudoconvex domain with C^2 boundary. Let Ω be invariant under the transformation $(z, w) \mapsto (z, -w)$, and suppose that $\Omega \cap \{w = 0\} = D_1 \cup \ldots \cup D_s$, where $\bar{D}_i \cap \bar{D}_j = \emptyset$, and D_j is a smoothly bounded disk. Then if $p \in D_1$, the function u_p is not C^2 in a neighborhood of the set $\bar{D}_j$ for any $2 \leqslant j \leqslant s$.*

Remark. — For a specific example, we may let Ω be a small neighborhood of the unit circle $\{(x_1, x_2) \in \mathbb{R}^2 \subset \mathbb{C}^2 : x_1^2 + x_2^2 = 1\}$.

Proof. — Since both p and the domain Ω are invariant under the mapping $(z, w) \mapsto (z, -w)$, it follows that u_p is also invariant. If u_p is of class C^2 on a neighborhood of the set $\bar{D}_2$, then we have an invariant solution of $\det(\partial^2 u_p / \partial z_i \partial \bar{z}_j) = 0$ on $\bar{D}_2$, with $u_p = 0$ on ∂D_2. But this is a contradiction to the Proposition of [1], which completes the proof.

Proposition **2**. — *There is a bounded, strongly pseudoconvex domain Ω in $\mathbb{C}^2$ with real analytic boundary for which the Green function is not symmetric.*

* This paper was written while the authors were visiting the Mittag-Leffler Institute, and they wish to thank the Institute for its hospitality.

Proof. — Let $u(z,w) = \max\left(\frac{1}{2}\log(|w^2 - z^2(z-a)|/\varepsilon^2), \log|z|\right)$, where $|a| < 1$. We will show that for $\varepsilon > 0$ small, the Green function for the domain $\Omega = \{u(z,w) < 0\}$, is not symmetric. To prove Proposition 2, we consider an increasing sequence of strongly pseudoconvex domains Ω_j with real analytic boundary, which increase to Ω. If the Green function $u_z(\zeta, \Omega_j)$ is symmetric for all j, then so is $\lim_{j\to\infty} u_z(\zeta, \Omega_j) = u_z(\zeta)$. Thus one of the sets Ω_j must give us the set desired in Proposition 2.

We note that u is the Green function for the domain Ω with pole at $(0,0)$. Thus $u_{(0,0)}(a,0) = u(a,0) = \log|a|$. On the other hand, the pole of $u_{(a,0)}$ is estimated by the function

$$v(z,w) = \begin{cases} \log|w|/2 \quad \text{if} \quad |z-a| \leqslant \varepsilon\ , \\ \max\left(\log|w|/2, \left(\dfrac{1}{2} + \dfrac{1}{\sqrt{\log 1/\varepsilon}}\right)\log\left|\dfrac{z-a}{1-\bar{a}z}\right|\right) \quad \text{if} \quad |z-a| \geqslant \varepsilon\ , \end{cases}$$

Indeed, we have $|w| < 2$ on Ω, and thus $v < 0$. Moreover the inequalities

$$|w|^2 \geqslant |z^2(z-a)| - \varepsilon^2 \geqslant (|a| - \varepsilon)\varepsilon - \varepsilon^2$$

on $\Omega \cap \{|z-a| = \varepsilon\}$ show that v is psh. As $v(z,w) \leqslant \log(|w| + |z-a|) + O(1)$ near $(a,0)$, it follows that $v(z,w) \leqslant u_{(a,0)}(z,w)$. Thus if ε is small, we have

$$u_{(a,0)}(0,0) \geqslant \left(\frac{1}{2} + \frac{1}{\sqrt{\log 1/\varepsilon}}\right)\log|a| > u_{(0,0)}(a,0),$$

and so the Green function is not symmetric.

Remark. In general, the Green function $u_z(\zeta)$ cannot even be expected to be psh in z. In fact, for any domain Ω, the symmetry property is equivalent to the fact that $v(z) = u_z(\zeta)$ is psh in z for every fixed ζ. To see this, observe that $v < 0$ and $v(z) = \log|z-\zeta| + O(1)$ near ζ. the plurisubharmonicity of v implies therefore $u_z(\zeta) = v(z) \leqslant u_\zeta(z)$, and similarly $u_\zeta(z) \leqslant u_z(\zeta)$.

References

[1] E. Bedford and J.-E. Fornæss. — *Counterexamples to regularity for the complex Monge-Ampère equation*, Inv. Math., **50** (1979), 129–134.

[2] J.-P. Demailly. — *Mesures de Monge-Ampère et mesures pluriharmoniques*, Math. Z., **194** (1987), 519–564.

[3] M. Klimek. — *Extremal plurisubharmonic functions and invariant pseudodistances*, Bull. Soc. Math. France, **113** (1985), 123–142.

[4] L. Lempert. — *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France, **109** (1981), 427–474.