Hans Grauert and the foundations of modern complex analysis

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Hans Grauert was born in 1930 in Haren, a town in Niedersachsen, Germany. He passed away on September 4, 2011, leaving the mathematical world with an extraordinary legacy in complex analysis and analytic geometry. I would like to share here a few recollections of my encounters with him during the last decades, and some connections of his work with my own research. I first met Hans Grauert at the end of the 70's, on the occasion of a conference in Paris, and still remember very well a discussion we had then. I was a young student at that time, and on the detour of a naive question I had raised, he had to explain the concept of a meromorphic map $X \to Y$ between complex manifolds ... Of course, I had been already somewhat acquainted with Grauert's major contributions to the theory of analytic spaces [G-Re58], [Gra62] and the Levi problem [Gra55, Gra58] – and as a consequence was very impressed to exchange a few words with him.

In 1986, Hans Grauert had heard more of my work, and he invited me to spend a couple of weeks in Göttingen; my stay actually took place in November 1986. At that time, Grauert was interested in the study of Kobayashi hyperbolicity, especially in view of his recently published paper "Hyperbolicity of the complement of plane curves" [G-Pe85], in collaboration with his daughter Ulrike Grauert-Peternell. He raised on the occasion a number of tough questions about hyperbolicity, and at that time I could not even think of any possible attempt to investigate them. Anyway, the discussion was to have a profound influence on my thinking years later – I will give a few more details on this below. It was also around that period that Grauert had received a private copy of Grothendieck's writing "Récoltes et semailles" [Gro85], a very personal account of an important period of Grothendieck's mathematical life. Grauert had been in close contact with Grothendieck already at the end of the 50's, culminating with their work on the direct image theorem for coherent sheaves (in the algebraic and analytic settings, [Gro58] and [Gra60] respectively). Athough he was obviously not at all targeted, I remember that Grauert was a bit upset about some of the controversial sentences contained in Grothendieck's testimony ...

Coming back to Kobayashi hyperbolicity theory, Grauert introduced in [Gra90] the important concept of a jet metric, following previous work by Green-Griffiths [Gr-G79] on jet differentials. If X is a projective nonsingular variety, let us consider J_kX to be the bundle of k-jets of holomorphic curves $f : (\mathbb{C}, 0) \to X$, together with the $\mathbb{C}^*$-action $(\lambda \cdot f)(t) = f(\lambda t)$ obtained by reparametrizing the curve with a linear change of parameter. One can consider the projectivized jet bundle $X_k := J_kX/\mathbb{C}^*$ whose fibers are weighted projective spaces, and the corresponding $\mathcal{O}_{X_k}(1)$ tautological sheaf. A *k-jet metric* is then just a hermitian metric on that sheaf; in other words, this is a nonnegative function $\rho : J_kX \to \mathbb{R}^+$ such that $\rho(\lambda \cdot f) = |\lambda|\,\rho(f)$, in Grauert's own notation. By looking at all holomorphic curve $f : \Delta \to X$ on the unit disk possessing a prescribed k-jet at 0 up to the $\mathbb{C}^*$ action, and trying to maximize the multiple, one defines in canonical way a k-jet metric $\rho_{k,\mathrm{can}}$ which is just the k-jet analogue of the Kobayashi infinitesimal metric in the case $k = 1$. Grauert realized that under a suitable negativity assumption for the curvature of the k-jet metric ρ, the Ahlfors-Schwarz lemma would imply the Kobayashi hyperbolicity of X; he then asked what type of curvature estimates the k-jet metric $\rho_{k,\mathrm{can}}$ should satisfy.

This question, which was further explained to me by Grauert's younger collaborators Gerd Dethloff and Siegmund Kosarew, is still unsolved at present. In fact, it is convenient to introduce a variant of these jet bundles (replacing the $\mathbb{C}^*$ action by the group $\mathbb{G}_k$ of k-jets of biholomorphisms $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, cf. [Dem95]), and then the conjecture is expressed by saying that *X is Kobayashi hyperbolic if and only if there exists a k-jet metric with strictly negative curvature* (*in a suitable sense*), *with poles contained in the set of k-jets that are singular at the origin, for all $k \geq k_0$ large enough.* In this statement, it can be shown, starting with hyperbolic complex surfaces, that k_0 may have to be taken arbitrary large. In the positive direction, it can be derived from a recent result of [Dem11a] that a projective manifold of general type always possesses a negatively curved k-jet metric for k large, if one forgets about the demands on the set of poles, thus proving only some sort of weak generic hyperbolicity of X. This is done by studying the cohomology of the bundles of jet differentials, and inferring from there that every entire holomorphic curve $f : \mathbb{C} \to X$ has to satisfy global algebraic differential equations $P(f; f', \ldots, f^{(k)}) = 0$; in fact f has to satisfy a large number of them when k increases.

Another foundational result is the Grauert-Riemenschneider theorem [G-Ri70] : *if X is a projective or Moishezon manifold, then $H^q(X, K_X \otimes L) = 0$ for every $q \geq 1$ and every semi-ample line bundle $L \to X$ of maximal Kodaira dimension.* The result is often used in its relative form, stating that *if $\mu : X \to Y$ is a projective birational morphism over some base Y, then all higher direct images $R^q\mu_*\mathcal{O}(K_X)$ vanish* ($q \geq 1$). In the same paper, Grauert and Riemenschneider conjectured that a compact analytic space is Moishezon if and only if it carries an almost positive coherent sheaf of rank 1, cf. also [Rie71]. Pursuing these ideas, Oswald Riemenschneider solved the Kähler case in [Rie73], by showing that a compact Kähler manifold carrying a line bundle whose curvature is semipositive and strictly positive at one point is actually Moishezon. The general case (removing the Kähler assumption) was finally settled by Yum-Tong Siu in [Siu84], using very clever bounds on Čech cohomology classes and their harmonic counterparts. These results served as the main motivation and as a strong guide for the discovery of holomorphic Morse inequalities in [Dem85] (probably the reason for Grauert's interest in my work in 1986, and, incidentally, also one of the main ingredients for the above mentioned result of [Dem11a]). These inequalities can be stated as follows: *for every compact complex manifold X and every holomorphic line bundle L, we have as $k \to +\infty$ the asymptotic estimate of cohomology groups*

$$(*) \qquad \frac{n!}{k^n}\sum_{j=0}^{q}(-1)^{q-j}h^j(X, L^{\otimes k}) \leq \int_{X(u,\leq q)}(-1)^q u^n + o(1), \qquad n = \dim_{\mathbb{C}} X,$$

where u is a smooth closed $(1,1)$-form in $c_1(L)$, and $X(u, \leq q)$ is the open set of points $x \in X$ where $u(x)$ is non degenerate with at most q negative eigenvalues. Assuming $\int_{X(u,\leq 1)} u^n > 0$, the estimate implies that $L^{\otimes k}$ has many sections, hence that L is big and that the base manifold X is Moishezon. Using a slight improvement due to L. Bonavero [Bon93] ("singular holomorphic Morse inequalities"), one concludes that *a compact complex manifold is Moishezon if and only if it carries a holomorphic line bundle possessing a singular hermitian metric h with analytic singularities, such that the curvature current $u = \Theta_{L,h} \in c_1(L)$ satisfies $\int_{X(u,\leq 1)\smallsetminus Z} u^n > 0$, on the complement of the set of poles $Z \subset X$.* This characterization strengthens Siu's solution of the Grauert-Riemenschneider conjecture.

Among Grauert's other fundamental contributions, the Andreotti-Grauert theorem [A-Gr62] stands out as one of the most important finiteness theorems of analytic geometry. Let us recall that *a complex n-dimensional complex manifold X is said to be q-convex (resp. q-complete) if X possesses a smooth exhaustion function φ such that the Levi form $i\partial\overline{\partial}\varphi$ has at least $n-q+1$ strictly positive eigenvalues on the complement $X \smallsetminus K$ of a compact set* (*resp. on X itself*); an appropriate definition can also be given for arbitrary complex spaces. Along with many other results, [A-Gr62] proves that *the cohomology groups $H^j(X,\mathcal{F})$ of a coherent analytic sheaf $\mathcal{F}$ on X are finite dimensional (resp. vanish) if X is q-convex (resp. q-complete), and $j \geq q$.* This finiteness statement implies a very interesting corollary also in the compact case. Let (L,h) be a hermitian line bundle on a compact complex manifold X, such that the curvature form $\Theta_{L,h} = -\frac{i}{2\pi}\partial\overline{\partial}\log h$ has at least $n-q+1$ positive eigenvalues at every point. Then the total space of the dual line bundle L^* is q-convex, and one can derive easily from this that $H^j(X,L^{\otimes k}) = 0$ for $j \geq q$ and k large enough. The holomorphic Morse inequalities would yield here the related (but somewhat less precise) result that $h^j(X,L^{\otimes k}) = o(k^n)$ for $j \geq q$, whenever $\Theta_{L,h}$ has at least $n-q+1$ semipositive eigenvalues at every point. An important unsolved question is whether a converse of the Andreotti-Grauert theorem holds true: *assuming that X (resp. L) is cohomologically q-convex, in the sense that the relevant cohomology groups $H^j(X,\mathcal{F})$ are finite dimensional* (*resp. vanish, resp. $H^j(X,L^{\otimes k}) = 0$, $k \gg 0$*) *for $j \geq q$, does it follow that X is q-convex* (*resp. q-complete*), *resp. does it follow that L possesses a hermitian metric h with the required signature*? The latter question has been analyzed in more depth in [DPS96]. In general, the answer is unknown except in the strictly pseudoconvex case ($q = 1$). Very recently, a partial converse was settled for line bundles L over compact complex surfaces ($n = \dim_{\mathbb{C}} X = 2$), in the form of an *asymptotic Morse equality*

$$(**) \qquad \limsup_{k\to+\infty} \frac{n!}{k^n}\sum_{j=0}^{q}(-1)^{q-j}h^j(X,L^{\otimes k}) = \inf_{u\in c_1(L)}\int_{X(u,\leq q)}(-1)^q u^n, \qquad 0 \leq q \leq n$$

(cf. [Dem11b]), the case of complex curves ($n = 1$) being also easy to check. It would be interesting to know whether such a result holds true when $n \geq 3$.

Hans Grauert has contributed many more fundamental results than those briefly discussed here. He can be considered as one of the founders of modern analytic geometry, and his achievements will certainly provide very strong guidelines for future research in the field. I have been myself deeply influenced by the research directions he initiated, and also strongly encouraged by the role he played for the recognition of my work as a member of the Mathematisches Institut and of the Akademie der Wissenschaften zu Göttingen.

References

[A-Gr62] Andreotti, A., Grauert, H., *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962) 193–259.

[Bon93] Bonavero, L.:, *Inégalités de Morse holomorphes singulières*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993) 1163-1166.

[Dem85] Demailly, J.-P., *Champs magnétiques et inégalités de Morse pour la d"-cohomologie*, Ann. Inst. Fourier (Grenoble) **35** (1985) 189–229.

[Dem95] Demailly, J.-P., *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*, AMS Summer School on Algebraic Geometry, Santa Cruz 1995, Proc. Symposia in Pure Math., ed. by J. Kollár and R. Lazarsfeld, Amer. Math. Soc., Providence, RI (1997), 285-360.

[Dem11a] Demailly, J.-P., *Holomorphic Morse Inequalities and the Green-Griffiths-Lang Conjecture*, Pure and Applied Math. Quarterly **7** (2011), 1165–1208.

[Dem11b] Demailly, J.-P., *A converse to the Andreotti-Grauert theorem*, Colloque "Analyse Complexe et Applications" en l'honneur de Nguyen Thanh Van (octobre 2008), Annales Faculté Sciences Toulouse Math., **20** (2011), 123–135.

[DPS96] Demailly, J.-P., Peternell, Th., Schneider, M., *Holomorphic line bundles with partially vanishing cohomology*, Conf. in honor of F. Hirzebruch, Israel Mathematical Conference Proceedings Vol. **9** (1996), 165–198.

[Gra55] Grauert, H., *Charakterisierung der holomorphvollstandigen komplexe Räume*, Math. Ann. **129** (1955) 233–259.

[Gra58] Grauert, H., *On Levi's problem and the embedding of real-analytic manifolds*, Ann. Math. **68** (1958) 460–472.

[Gra60] Grauert, H., *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen*, Publ. Math. I.H.E.S. **5** (1960) 233–292.

[Gra62] Grauert, H., *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962) 331–368.

[G-Re58] Grauert, H., Remmert, R., *Komplexe Räume*, Math. Ann. **136** (1958) 245–318.

[G-Re77] Grauert, H., Remmert, R., *Theorie der Steinschen Räume*, Grundlehren der Mathematischen Wissenschaften, vol. 227, Springer-Verlag, Berlin-New York, 1977.

[G-Ri70] Grauert, H., Riemenschneider, O., *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970) 263–292.

[G-Pe85] Grauert, H.; Peternell, U., *Hyperbolicity of the complement of plane curves*, Manuscripta Math. **50** (1985) 429–441.

[Gra90] Grauert, H., *Jet Metriken und hyperbolische Geometrie*, Math. Zeitschrift **200** (1989) 149–168.

[Gr-G79] Green, M., Griffiths, P., *Two applications of algebraic geometry to entire holomorphic mappings.*, The Chern Symposium 1979, Proc. Internal. Sympos. Berkeley, CA, 1979, Springer-Verlag, New York (1980), 41–74.

[Gro58] Grothendieck, A., *The cohomology theory of abstract algebraic varieties*, Proceedings of the Intern. Congress of Math., 14-21 August 1958, Edinburgh, Cambridge Univ. Press, 1960.

[Gro85] Grothendieck, A., *Récoltes et semailles*, Unpublished manuscript, available from `http://www.math.jussieu.fr/~leila/grothendieckcircle/RetS.pdf`.

[Rie71] Riemenschneider O., *Characterizing Moishezon spaces by almost positive coherent analytic sheaves*, Math. Zeitschrift **123** (1971) 263–284.

[Rie73] Riemenschneider O., *A generalization of Kodaira's embedding theorem*, Math. Ann. **200** (1973) 99–102.

[Siu84] Siu, Y.T., *A vanishing theorem for semipositive Une bundles over non-Kähler manifolds*, Diff. Geom. **19** (1984) 431–452.

[Siu85] Siu, Y.T., *Some recent results in complex manifold theory related to vanishing theorems for the semi-positive case*, Proceedings of the Math. Arbeitstagung held in Bonn (june 1984), Max Planck Inst. für Math., Lecture Notes in Math. no 1111, Springer-Verlag, 1985.