Applications of the theory of L^2 estimates and positive currents in algebraic geometry

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These notes are derived in part from the lectures "*Multiplier ideal sheaves and analytic methods in algebraic geometry*" given at the ICTP School held in Trieste, Italy, April 24 – May 12, 2000 "*Vanishing theorems and effective results in Algebraic Geometry*", as well as from a more detailed online book on Algebraic Geometry available at the author's home page
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0. Introduction

Transcendental methods of algebraic geometry have been extensively studied since a long time, starting with the work of Abel, Jacobi and Riemann in the nineteenth century. More recently, in the period 1940-1970, the work of Hodge, Hirzebruch, Kodaira, Atiyah revealed deeper relations between complex analysis, topology, PDE theory and algebraic geometry. In the last twenty years, gauge theory has proved to be a very efficient tool for the study of many important questions: moduli spaces, stable sheaves, non abelian Hodge theory, low dimensional topology ...

Our main purpose here is to describe a few analytic tools which are useful to study questions such as linear series and vanishing theorems for algebraic vector bundles. One of the early successes of analytic methods in this context is Kodaira's use of the Bochner technique in relation with the theory of harmonic forms, during the decade 1950-60. The idea is to represent cohomology classes by harmonic forms and to prove vanishing theorems by means of suitable a priori curvature estimates. The prototype of such results is the Akizuki-Kodaira-Nakano theorem (1954): if X is a nonsingular projective algebraic variety and L is a holomorphic line bundle on X with positive curvature, then $H^q(X, \Omega_X^p \otimes L) = 0$ for $p+q > \dim X$ (throughout the paper we set $\Omega_X^p = \Lambda^p T_X^\star$ and $K_X = \Lambda^n T_X^\star$, $n = \dim X$, viewing these objects either as holomorphic bundles or as locally free $\mathcal{O}_X$-modules). It is only much later that an algebraic proof of this result has been proposed by Deligne-Illusie, via characteristic p methods, in 1986.

A refinement of the Bochner technique used by Kodaira led, about ten years later, to fundamental L^2 estimates due to Hörmander [Hör65], concerning solutions of the Cauchy-Riemann operator. Not only vanishing theorems are proved, but more precise information of a quantitative nature is obtained about solutions of $\overline{\partial}$-equations. The best way of expressing these L^2 estimates is to use a geometric setting first considered by Andreotti-Vesentini [AV65]. More explicitly, suppose that we have a holomorphic line bundle L equipped with a hermitian metric of weight $e^{-2\varphi}$, where φ is a (locally defined) plurisubharmonic function; then explicit bounds on the L^2 norm $\int_X |f|^2 e^{-2\varphi}$ of solutions is obtained. The result is still more useful if the plurisubharmonic weight φ is allowed to have singularities. Following Nadel [Nad89], we define the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ to be the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally summable. Then $\mathcal{I}(\varphi)$ is a coherent algebraic sheaf over X and $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$ for all $q \geq 1$ if the curvature of L is positive (as a current). This important result can be seen as a generalization of the Kawamata-Viehweg vanishing theorem ([Kaw82], [Vie82]), which is one of the cornerstones of higher dimensional algebraic geometry (especially of Mori's minimal model program). In the dictionary between analytic geometry and algebraic geometry, the ideal $\mathcal{I}(\varphi)$ plays a very important role, since it directly converts an analytic object into an algebraic one, and, simultaneously, takes care of singularities in a very efficient way.

Another analytic tool used to deal with singularities is the theory of positive currents introduced by Lelong [Lel57]. Currents can be seen as generalizations of algebraic cycles, and many classical results of intersection theory still apply to currents. The concept of Lelong number of a current is the analytic analogue of the concept of multiplicity of a germ of algebraic variety. Intersections of cycles correspond to wedge products of currents (whenever these products are defined).

Besides the Kodaira-Nakano vanishing theorem, one of the most basic "effective result" expected to hold in algebraic geometry is expressed in the following conjecture of Fujita [Fuj87]: if L is an ample (i.e. positive) line bundle on a projective n-dimensional algebraic variety X, then $K_X+(n+1)L$ is generated by sections and $K_X+(n+2)L$ is very ample. In the last decade, a lot of effort has been brought for the solution of this conjecture – and it seems indeed that a solution might finally emerge in the first years or the third millenium – hopefully during this Summer School! The first major results are the proof of the Fujita conjecture in the case of surfaces by Reider [Rei88] (the case of curves is easy and has been known since a very long time), and the numerical criterion for the very ampleness of $2K_X+L$ given in [Dem93b], obtained by means of analytic techniques and Monge-Ampère equations with isolated singularities. Alternative algebraic techniques were developed slightly later by Kollár [Kol92], Ein-Lazarsfeld [EL93], Fujita [Fuj93], Siu [Siu95, 96], Kawamata [Kaw97] and Helmke [Hel97]. We will explain here Siu's method because it is technically the simplest method; one of the results obtained by this method is the following effective result: $2K_X+mL$ is very ample for $m\geq 2+\binom{3n+1}{n}$. The basic idea is to apply the Kawamata-Viehweg vanishing theorem, and to combine this with the Riemann-Roch formula in order to produce sections through a clever induction procedure on the dimension of the base loci of the linear systems involved.

Although Siu's result is certainly not optimal, it is sufficient to obtain a nice constructive proof of *Matsusaka's big theorem* ([Siu93], [Dem96]). The result states that there is an effective value m_0 depending only on the intersection numbers L^n and $L^{n-1}\cdot K_X$, such that mL is very ample for $m\geq m_0$. The basic idea is to combine results on the very ampleness of $2K_X+mL$ together with the theory of holomorphic Morse inequalities ([Dem85b]). The Morse inequalities are used to construct sections of $m'L-K_X$ for m' large. Again this step can be made algebraic (following suggestions by F. Catanese and R. Lazarsfeld), but the analytic formulation apparently has a wider range of applicability.

In the next sections, we pursue the study of L^2 estimates, in relation with the Nullstellenstatz and with the extension problem. Skoda [Sko72b, Sko78] showed that the division problem $f=\sum g_jh_j$ can be solved holomorphically with very precise L^2 estimates, provided that the L^2 norm of $|f|\,|g|^{-p}$ is finite for some sufficiently large exponent p ($p>n=\dim X$ is enough). Skoda's estimates have a nice interpretation in terms of local algebra, and they lead to precise qualitative and quantitative estimates in connection with the Bézout problem. Another very important result is the L^2 extension theorem by Ohsawa-Takegoshi [OT87, Ohs88], which has also been generalized later by Manivel [Man93]. The main statement is that every L^2 section f of a suitably positive line bundle defined on a subavariety $Y\subset X$ can be extended to a L^2 section $\tilde f$ defined over the whole of X. The positivity condition can be understood in terms of the canonical sheaf and normal bundle to the subvariety. The extension theorem turns out to have an incredible amount of important consequences: among them, let us mention for instance Siu's theorem [Siu74] on the analyticity of Lelong numbers, the basic approximation theorem of closed positive (1,1)-currents by divisors, the subadditivity property $\mathcal{I}(\varphi+\psi)\subset\mathcal{I}(\varphi)\mathcal{I}(\psi)$ of multiplier ideals [DEL00], the restriction formula $\mathcal{I}(\varphi_{|Y})\subset\mathcal{I}(\varphi)_{|Y}$, ... A suitable combination of these results can be used to reprove Fujita's result [Fuj94] on approximate Zariski decomposition, as we show in section 15.

In the last section 16, we show how subadditivity can be used to derive an approximation theorem for (almost) plurisubharmonic functions: any such function can be approximated by a sequence of (almost) plurisubharmonic functions which are smooth outside an analytic set, and which define the same multiplier ideal sheaves. From this, we derive a generalized version of the *hard Lefschetz theorem* for cohomology with values in a pseudo-effective line bundle; namely, the Lefschetz map is surjective when the cohomology groups are twisted by the relevant multiplier ideal sheaves.

These notes are essentially written with the idea of serving as an analytic toolbox for algebraic geometers. Although efficient algebraic techniques exist, our feeling is that the analytic techniques are very flexible and offer a large variety of guidelines for more algebraic questions (including applications to number theory which are not discussed here). We made a special effort to use as little prerequisites and to be as self-contained as possible; hence the rather long preliminary sections dealing with basic facts of complex differential geometry. I am indebted to L. Ein, J. Kollár, R. Lazarsfeld, Th. Peternell, M. Schneider and Y.T. Siu for many discussions on these subjects over a period of time of two decades or more. These discussions certainly had a great influence on my research work and therefore on the contents of the present notes. We refer to Rob Lazarsfeld's book [Laz04] for an extremely detailed algebraic exposition of the subject of positivity in algebraic geometry.

1. Preliminary Material

1.A. Dolbeault Cohomology and Sheaf Cohomology

Let X be a $\mathbb{C}$-analytic manifold of dimension n. We denote by $\Lambda^{p,q}T^\star_X$ the bundle of differential forms of bidegree (p,q) on X, i.e., differential forms which can be written as

$$u = \sum_{|I|=p,\,|J|=q} u_{I,J}dz_I \wedge d\overline{z}_J.$$

Here $(z_1,\ldots,z_n)$ denote arbitrary local holomorphic coordinates, $I=(i_1,\ldots,i_p)$, $J=(j_1,\ldots,j_q)$ are multiindices (increasing sequences of integers in the range $[1,\ldots,n]$, of lengths $|I|=p$, $|J|=q$), and

$$dz_I := dz_{i_1}\wedge\ldots\wedge dz_{i_p}, \qquad d\overline{z}_J := d\overline{z}_{j_1}\wedge\ldots\wedge d\overline{z}_{j_q}.$$

Let $\mathcal{E}^{p,q}$ be the sheaf of germs of complex valued differential (p,q)-forms with C^∞ coefficients. Recall that the exterior derivative d splits as $d = d' + d''$ where

$$d'u = \sum_{|I|=p,\,|J|=q,1\le k\le n} \frac{\partial u_{I,J}}{\partial z_k} dz_k\wedge dz_I\wedge d\overline{z}_J,$$
$$d''u = \sum_{|I|=p,\,|J|=q,1\le k\le n} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k\wedge dz_I\wedge d\overline{z}_J$$

are of type $(p+1,q)$, $(p,q+1)$ respectively. The well-known Dolbeault-Grothendieck lemma asserts that any d''-closed form of type (p,q) with $q>0$ is locally d''-exact (this is the analogue for d'' of the usual Poincaré lemma for d, see e.g. [Hör66]). In other words, the complex of sheaves $(\mathcal{E}^{p,\bullet}, d'')$ is exact in degree $q>0$; in degree $q=0$, $\operatorname{Ker} d''$ is the sheaf Ω^p_X of germs of holomorphic forms of degree p on X.

More generally, if F is a holomorphic vector bundle of rank r over X, there is a natural d'' operator acting on the space $C^\infty(X,\Lambda^{p,q}T^\star_X\otimes F)$ of smooth (p,q)-forms with values in F; if $s=\sum_{1\le\lambda\le r} s_\lambda e_\lambda$ is a (p,q)-form expressed in terms of a local holomorphic frame of F, we simply define $d''s := \sum d''s_\lambda\otimes e_\lambda$, observing that the holomorphic transition matrices involved in changes of holomorphic frames do not affect the computation of d''. It is then clear that the Dolbeault-Grothendieck lemma still holds for F-valued forms. For every integer $p=0,1,\ldots,n$, the *Dolbeault Cohomology* groups $H^{p,q}(X,F)$ are defined to be the cohomology groups of the complex of global (p,q) forms (graded by q):

$$H^{p,q}(X,F) = H^q\big(C^\infty(X,\Lambda^{p,\bullet}T^\star_X\otimes F)\big). \tag{1.1}$$

Now, let us recall the following fundamental result from sheaf theory (De Rham-Weil isomorphism theorem): let $(\mathcal{L}^\bullet, d)$ be a resolution of a sheaf $\mathcal{A}$ by acyclic sheaves, i.e. a complex of sheaves $(\mathcal{L}^\bullet,\delta)$ such that there is an exact sequence of sheaves

$$0\longrightarrow \mathcal{A} \xrightarrow{\ j\ } \mathcal{L}^0 \xrightarrow{\ \delta^0\ } \mathcal{L}^1 \longrightarrow\cdots\longrightarrow \mathcal{L}^q \xrightarrow{\ \delta^q\ } \mathcal{L}^{q+1}\longrightarrow\cdots\ ,$$

and $H^s(X,\mathcal{L}^q)=0$ for all $q\ge 0$ and $s\ge 1$. Then there is a functorial isomorphism

$$H^q\big(\Gamma(X,\mathcal{L}^\bullet)\big)\longrightarrow H^q(X,\mathcal{A}). \tag{1.2}$$

We apply this to the following situation: let $\mathcal{E}(F)^{p,q}$ be the sheaf of germs of C^∞ sections of $\Lambda^{p,q}T^\star_X \otimes F$. Then $(\mathcal{E}(F)^{p,\bullet}, d'')$ is a resolution of the locally free $\mathcal{O}_X$-module $\Omega^p_X \otimes \mathcal{O}(F)$ (Dolbeault-Grothendieck lemma), and the sheaves $\mathcal{E}(F)^{p,q}$ are acyclic as modules over the soft sheaf of rings $\mathcal{C}^\infty$. Hence by (1.2) we get

(1.3) Dolbeault Isomorphism Theorem (1953). *For every holomorphic vector bundle F on X, there is a canonical isomorphism*

$$H^{p,q}(X, F) \simeq H^q(X, \Omega^p_X \otimes \mathcal{O}(F)). \qquad \square$$

If X is projective algebraic and F is an algebraic vector bundle, Serre's GAGA theorem [Ser56] shows that the algebraic sheaf cohomology group $H^q(X, \Omega^p_X \otimes \mathcal{O}(F))$ computed with algebraic sections over Zariski open sets is actually isomorphic to the analytic cohomology group. These results are the most basic tools to attack algebraic problems via analytic methods. Another important tool is the theory of plurisubharmonic functions and positive currents originated by K. Oka and P. Lelong in the decades 1940-1960.

1.B. Plurisubharmonic Functions

Plurisubharmonic functions have been introduced independently by Lelong and Oka in the study of holomorphic convexity. We refer to [Lel67, 69] for more details.

(1.4) Definition. *A function $u : \Omega \longrightarrow [-\infty, +\infty[$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic (psh for short) if*

a) *u is upper semicontinuous ;*

b) *for every complex line $L \subset \mathbb{C}^n$, $u_{\restriction \Omega \cap L}$ is subharmonic on $\Omega \cap L$, that is, for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$ with $|\xi| < d(a, \complement\Omega)$, the function u satisfies the mean value inequality*

$$u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{\mathrm{i}\,\theta}\,\xi)\,d\theta.$$

The set of psh functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

We list below the most basic properties of psh functions. They all follow easily from the definition.

(1.5) Basic properties.

a) Every function $u \in \mathrm{Psh}(\Omega)$ is subharmonic, namely it satisfies the mean value inequality on euclidean balls or spheres:

$$u(a) \le \frac{1}{\pi^n r^{2n}/n!} \int_{B(a,r)} u(z)\,d\lambda(z)$$

for every $a \in \Omega$ and $r < d(a, \complement\Omega)$. Either $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}$ on every connected component of Ω.

b) For any decreasing sequence of psh functions $u_k \in \mathrm{Psh}(\Omega)$, the limit $u = \lim u_k$ is psh on Ω.

c) Let $u \in \mathrm{Psh}(\Omega)$ be such that $u \not\equiv -\infty$ on every connected component of Ω. If (ρ_ε) is a family of smoothing kernels, then $u \star \rho_\varepsilon$ is C^∞ and psh on

$$\Omega_\varepsilon = \big\{x \in \Omega\,;\, d(x, \complement\Omega) > \varepsilon\big\},$$

the family $(u \star \rho_\varepsilon)$ is increasing in ε and $\lim_{\varepsilon\to 0} u \star \rho_\varepsilon = u$.

d) Let $u_1, \ldots, u_p \in \mathrm{Psh}(\Omega)$ and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_p)$ is increasing in each t_j. Then $\chi(u_1, \ldots, u_p)$ is psh on Ω. In particular $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(e^{u_1} + \cdots + e^{u_p})$ are psh on Ω. □

(1.6) Lemma. *A function $u \in C^2(\Omega, \mathbb{R})$ is psh on Ω if and only if the hermitian form $Hu(a)(\xi) = \sum_{1\le j,k\le n} \partial^2 u/\partial z_j \partial\overline{z}_k(a)\, \xi_j \overline{\xi}_k$ is semipositive at every point $a \in \Omega$.*

Proof. This is an easy consequence of the following standard formula

$$\frac{1}{2\pi}\int_0^{2\pi} u(a + e^{\mathrm{i}\,\theta}\,\xi)\, d\theta - u(a) = \frac{2}{\pi}\int_0^1 \frac{dt}{t}\int_{|\zeta|<t} Hu(a + \zeta\xi)(\xi)\, d\lambda(\zeta),$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}$. Lemma 1.6 is a strong evidence that plurisubharmonicity is the natural complex analogue of linear convexity. □

For non smooth functions, a similar characterization of plurisubharmonicity can be obtained by means of a regularization process.

(1.7) Theorem. *If $u \in \mathrm{Psh}(\Omega)$, $u \not\equiv -\infty$ on every connected component of Ω, then for all $\xi \in \mathbb{C}^n$*

$$Hu(\xi) = \sum_{1\le j,k\le n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\, \xi_j \overline{\xi}_k \in \mathcal{D}'(\Omega)$$

is a positive measure. Conversely, if $v \in \mathcal{D}'(\Omega)$ is such that $Hv(\xi)$ is a positive measure for every $\xi \in \mathbb{C}^n$, there exists a unique function $u \in \mathrm{Psh}(\Omega)$ which is locally integrable on Ω and such that v is the distribution associated to u. □

In order to get a better geometric insight of this notion, we assume more generally that u is a function on a complex n-dimensional manifold X. If $\Phi : X \to Y$ is a holomorphic mapping and if $v \in C^2(Y, \mathbb{R})$, we have $d'd''(v \circ \Phi) = \Phi^\star d'd''v$, hence

$$H(v \circ \Phi)(a, \xi) = Hv\big(\Phi(a), \Phi'(a).\xi\big).$$

In particular Hu, viewed as a hermitian form on T_X, does not depend on the choice of coordinates $(z_1, \ldots, z_n)$. Therefore, the notion of psh function makes sense on any complex manifold. More generally, we have

(1.8) Proposition. *If $\Phi : X \longrightarrow Y$ is a holomorphic map and $v \in \mathrm{Psh}(Y)$, then $v \circ \Phi \in \mathrm{Psh}(X)$.* □

(1.9) Example. It is a standard fact that $\log|z|$ is psh (i.e. subharmonic) on $\mathbb{C}$. Thus $\log|f| \in \mathrm{Psh}(X)$ for every holomorphic function $f \in H^0(X, \mathcal{O}_X)$. More generally

$$\log\big(|f_1|^{\alpha_1} + \cdots + |f_q|^{\alpha_q}\big) \in \mathrm{Psh}(X)$$

for every $f_j \in H^0(X, \mathcal{O}_X)$ and $\alpha_j \geq 0$ (apply Property 1.5 d with $u_j = \alpha_j \log|f_j|$). We will be especially interested in the singularities obtained at points of the zero variety $f_1 = \ldots = f_q = 0$, when the α_j are rational numbers. □

(1.10) Definition. *A psh function $u \in \mathrm{Psh}(X)$ will be said to have analytic singularities if u can be written locally as*

$$u = \frac{\alpha}{2} \log\big(|f_1|^2 + \cdots + |f_N|^2\big) + v,$$

where $\alpha \in \mathbb{R}_+$, v is a locally bounded function and the f_j are holomorphic functions. If X is algebraic, we say that u has algebraic singularities if u can be written as above on sufficiently small Zariski open sets, with $\alpha \in \mathbb{Q}_+$ and f_j algebraic.

We then introduce the ideal $\mathcal{J} = \mathcal{J}(u/\alpha)$ of germs of holomorphic functions h such that $|h| \leq Ce^{u/\alpha}$ for some constant C, i.e.

$$|h| \leq C\big(|f_1| + \cdots + |f_N|\big).$$

This is a globally defined ideal sheaf on X, locally equal to the integral closure $\overline{\mathcal{I}}$ of the ideal sheaf $\mathcal{I} = (f_1, \ldots, f_N)$, thus $\mathcal{J}$ is coherent on X. If $(g_1, \ldots, g_{N'})$ are local generators of $\mathcal{J}$, we still have

$$u = \frac{\alpha}{2} \log\big(|g_1|^2 + \cdots + |g_{N'}|^2\big) + O(1).$$

If X is projective algebraic and u has analytic singularities with $\alpha \in \mathbb{Q}_+$, then u automatically has algebraic singularities. From an algebraic point of view, the singularities of u are in 1:1 correspondence with the "algebraic data" $(\mathcal{J}, \alpha)$. Later on, we will see another important method for associating an ideal sheaf to a psh function.

(1.11) Exercise. Show that the above definition of the integral closure of an ideal $\mathcal{I}$ is equivalent to the following more algebraic definition: $\overline{\mathcal{I}}$ consists of all germs h satisfying an integral equation

$$h^d + a_1 h^{d-1} + \ldots + a_{d-1} h + a_d = 0, \qquad a_k \in \mathcal{I}^k.$$

Hint. One inclusion is clear. To prove the other inclusion, consider the normalization of the blow-up of X along the (non necessarily reduced) zero variety $V(\mathcal{I})$. □

1.C. Positive Currents

The reader can consult [Fed69] for a more thorough treatment of current theory. Let us first recall a few basic definitions. A *current* of degree q on an oriented differentiable manifold M is simply a differential q-form Θ with distribution coefficients. The space of currents of degree q over M will be denoted by $\mathcal{D}'^q(M)$. Alternatively, a current of degree q can be seen as an element Θ in the dual space $\mathcal{D}'_p(M) := \big(\mathcal{D}^p(M)\big)'$ of the space $\mathcal{D}^p(M)$ of smooth differential forms of degree $p = \dim M - q$ with compact support; the duality pairing is given by

$$(1.12)\qquad \langle \Theta, \alpha\rangle = \int_M \Theta\wedge\alpha, \qquad \alpha\in\mathcal{D}^p(M).$$

A basic example is the *current of integration* $[S]$ over a compact oriented submanifold S of M:

$$(1.13)\qquad \langle [S], \alpha\rangle = \int_S \alpha, \qquad \deg\alpha = p = \dim_{\mathbb{R}} S.$$

Then $[S]$ is a current with measure coefficients, and Stokes' formula shows that $d[S] = (-1)^{q-1}[\partial S]$, in particular $d[S]=0$ if S has no boundary. Because of this example, the integer p is said to be the dimension of Θ when $\Theta\in\mathcal{D}'_p(M)$. The current Θ is said to be *closed* if $d\Theta=0$.

On a complex manifold X, we have similar notions of bidegree and bidimension; as in the real case, we denote by

$$\mathcal{D}'^{p,q}(X) = \mathcal{D}'_{n-p,n-q}(X), \qquad n=\dim X,$$

the space of currents of bidegree (p,q) and bidimension $(n-p,n-q)$ on X. According to [Lel57], a current Θ of bidimension (p,p) is said to be (*weakly*) *positive* if for every choice of smooth $(1,0)$-forms $\alpha_1,\ldots,\alpha_p$ on X the distribution

$$(1.14)\qquad \Theta\wedge \mathrm{i}\,\alpha_1\wedge\overline{\alpha}_1\wedge\ldots\wedge\mathrm{i}\,\alpha_p\wedge\overline{\alpha}_p \quad \text{is a positive measure.}$$

(1.15) Exercise. If Θ is positive, show that the coefficients $\Theta_{I,J}$ of Θ are complex measures, and that, up to constants, they are dominated by the trace measure

$$\sigma_\Theta = \Theta\wedge\frac{1}{p!}\beta^p = 2^{-p}\sum\Theta_{I,I}, \qquad \beta = \frac{\mathrm{i}}{2}d'd''|z|^2 = \frac{\mathrm{i}}{2}\sum_{1\le j\le n} dz_j\wedge d\overline{z}_j,$$

which is a positive measure.
Hint. Observe that $\sum\Theta_{I,I}$ is invariant by unitary changes of coordinates and that the (p,p)-forms $\mathrm{i}\,\alpha_1\wedge\overline{\alpha}_1\wedge\ldots\wedge\mathrm{i}\,\alpha_p\wedge\overline{\alpha}_p$ generate $\Lambda^{p,p}T^\star_{\mathbb{C}^n}$ as a $\mathbb{C}$-vector space. □

A current $\Theta = \mathrm{i}\sum_{1\le j,k\le n}\Theta_{jk}dz_j\wedge dz_k$ of bidegree $(1,1)$ is easily seen to be positive if and only if the complex measure $\sum\lambda_j\overline{\lambda}_k\Theta_{jk}$ is a positive measure for every n-tuple $(\lambda_1,\ldots,\lambda_n)\in\mathbb{C}^n$.

(1.16) Example. If u is a (not identically $-\infty$) psh function on X, we can associate with u a (closed) positive current $\Theta = \mathrm{i}\,\partial\overline{\partial}u$ of bidegree $(1,1)$. Conversely, every closed positive current of bidegree $(1,1)$ can be written under this form on any open subset $\Omega\subset X$ such that $H^2_{DR}(\Omega,\mathbb{R}) = H^1(\Omega,\mathcal{O}) = 0$, e.g. on small coordinate balls (exercise to the reader). □

It is not difficult to show that a product $\Theta_1\wedge\ldots\wedge\Theta_q$ of positive currents of bidegree $(1,1)$ is positive whenever the product is well defined (this is certainly the case if all Θ_j but one at most are smooth; much finer conditions will be discussed in Section 2).

We now discuss another very important example of closed positive current. In fact, with every closed analytic set $A\subset X$ of pure dimension p is associated a current of integration

$$\langle [A], \alpha \rangle = \int_{A_{\rm reg}} \alpha, \qquad \alpha \in \mathcal{D}^{p,p}(X), \tag{1.17}$$

obtained by integrating over the regular points of A. In order to show that (1.17) is a correct definition of a current on X, one must show that $A_{\rm reg}$ has locally finite area in a neighborhood of $A_{\rm sing}$. This result, due to [Lel57] is shown as follows. Suppose that 0 is a singular point of A. By the local parametrization theorem for analytic sets, there is a linear change of coordinates on $\mathbb{C}^n$ such that all projections

$$\pi_I : (z_1, \ldots, z_n) \mapsto (z_{i_1}, \ldots, z_{i_p})$$

define a finite ramified covering of the intersection $A \cap \Delta$ with a small polydisk Δ in $\mathbb{C}^n$ onto a small polydisk Δ_I in $\mathbb{C}^p$. Let n_I be the sheet number. Then the p-dimensional area of $A \cap \Delta$ is bounded above by the sum of the areas of its projections counted with multiplicities, i.e.

$$\mathrm{Area}(A \cap \Delta) \le \sum n_I \mathrm{Vol}(\Delta_I).$$

The fact that $[A]$ is positive is also easy. In fact

$$\mathrm{i}\,\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\,\alpha_p \wedge \overline{\alpha}_p = |\det(\alpha_{jk})|^2\,\mathrm{i}\,w_1 \wedge \overline{w}_1 \wedge \ldots \wedge \mathrm{i}\,w_p \wedge \overline{w}_p$$

if $\alpha_j = \sum \alpha_{jk} dw_k$ in terms of local coordinates $(w_1, \ldots, w_p)$ on $A_{\rm reg}$. This shows that all such forms are ≥ 0 in the canonical orientation defined by $\mathrm{i}\,w_1 \wedge \overline{w}_1 \wedge \ldots \wedge \mathrm{i}\,w_p \wedge \overline{w}_p$. More importantly, Lelong [Lel57] has shown that $[A]$ is d-closed in X, even at points of $A_{\rm sing}$. This last result can be seen today as a consequence of the Skoda-El Mir extension theorem. For this we need the following definition: a *complete pluripolar* set is a set E such that there is an open covering (Ω_j) of X and psh functions u_j on Ω_j with $E \cap \Omega_j = u_j^{-1}(-\infty)$. Any (closed) analytic set is of course complete pluripolar (take u_j as in Example 1.9).

(1.18) Theorem (Skoda [Sko82], El Mir [EM84], Sibony [Sib85])**.** *Let E be a closed complete pluripolar set in X, and let Θ be a closed positive current on $X \smallsetminus E$ such that the coefficients $\Theta_{I,J}$ of Θ are measures with locally finite mass near E. Then the trivial extension $\widetilde{\Theta}$ obtained by extending the measures $\Theta_{I,J}$ by 0 on E is still closed on X.*

Lelong's result $d[A] = 0$ is obtained by applying the Skoda-El Mir theorem to $\Theta = [A_{\rm reg}]$ on $X \smallsetminus A_{\rm sing}$.

Proof of Theorem 1.18. The statement is local on X, so we may work on a small open set Ω such that $E \cap \Omega = v^{-1}(-\infty)$, $v \in \mathrm{Psh}(\Omega)$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a convex increasing function such that $\chi(t) = 0$ for $t \le -1$ and $\chi(0) = 1$. By shrinking Ω and putting $v_k = \chi(k^{-1} v \star \rho_{\varepsilon_k})$ with $\varepsilon_k \to 0$ fast, we get a sequence of functions $v_k \in \mathrm{Psh}(\Omega) \cap C^\infty(\Omega)$ such that $0 \le v_k \le 1$, $v_k = 0$ in a neighborhood of $E \cap \Omega$ and $\lim v_k(x) = 1$ at every point of $\Omega \smallsetminus E$. Let $\theta \in C^\infty([0,1])$ be a function such that $\theta = 0$ on $[0, 1/3]$, $\theta = 1$ on $[2/3, 1]$ and $0 \le \theta \le 1$. Then $\theta \circ v_k = 0$ near $E \cap \Omega$ and $\theta \circ v_k \to 1$ on $\Omega \smallsetminus E$. Therefore $\tilde{\Theta} = \lim_{k \to +\infty} (\theta \circ v_k)\Theta$ and

$$d'\tilde{\Theta} = \lim_{k \to +\infty} \Theta \wedge d'(\theta \circ v_k)$$

in the weak topology of currents. It is therefore sufficient to verify that $\Theta \wedge d'(\theta \circ v_k)$ converges weakly to 0 (note that $d''\tilde{\Theta}$ is conjugate to $d'\tilde{\Theta}$, thus $d''\tilde{\Theta}$ will also vanish).

Assume first that $\Theta \in \mathcal{D}'^{n-1,n-1}(X)$. Then $\Theta \wedge d'(\theta \circ v_k) \in \mathcal{D}'^{n,n-1}(\Omega)$, and we have to show that

$$\langle \Theta \wedge d'(\theta \circ v_k), \overline{\alpha} \rangle = \langle \Theta, \theta'(v_k) d'v_k \wedge \overline{\alpha} \rangle \underset{k \to +\infty}{\longrightarrow} 0, \qquad \forall \alpha \in \mathcal{D}^{1,0}(\Omega).$$

As $\gamma \mapsto \langle \Theta, \mathrm{i}\,\gamma \wedge \overline{\gamma} \rangle$ is a non-negative hermitian form on $\mathcal{D}^{1,0}(\Omega)$, the Cauchy-Schwarz inequality yields

$$\big|\langle \Theta, \mathrm{i}\,\beta \wedge \overline{\gamma} \rangle\big|^2 \le \langle \Theta, \mathrm{i}\,\beta \wedge \overline{\beta} \rangle\ \langle \Theta, \mathrm{i}\,\gamma \wedge \overline{\gamma} \rangle, \qquad \forall \beta, \gamma \in \mathcal{D}^{1,0}(\Omega).$$

Let $\psi \in \mathcal{D}(\Omega)$, $0 \le \psi \le 1$, be equal to 1 in a neighborhood of $\operatorname{Supp} \alpha$. We find

$$\big|\langle \Theta, \theta'(v_k) d'v_k \wedge \overline{\alpha} \rangle\big|^2 \le \langle \Theta, \psi \mathrm{i}\, d'v_k \wedge d''v_k \rangle\ \langle \Theta, \theta'(v_k)^2 \mathrm{i}\,\alpha \wedge \overline{\alpha} \rangle.$$

By hypothesis $\int_{\Omega \smallsetminus E} \Theta \wedge \mathrm{i}\,\alpha \wedge \overline{\alpha} < +\infty$ and $\theta'(v_k)$ converges everywhere to 0 on Ω, thus $\langle \Theta, \theta'(v_k)^2 \mathrm{i}\,\alpha \wedge \overline{\alpha} \rangle$ converges to 0 by Lebesgue's dominated convergence theorem. On the other hand

$$\begin{aligned} &\mathrm{i}\, d'd''v_k^2 = 2v_k\, \mathrm{i}\, d'd''v_k + 2\mathrm{i}\, d'v_k \wedge d''v_k \ge 2\mathrm{i}\, d'v_k \wedge d''v_k, \\ &2\langle \Theta, \psi \mathrm{i}\, d'v_k \wedge d''v_k \rangle \le \langle \Theta, \psi \mathrm{i}\, d'd''v_k^2 \rangle. \end{aligned}$$

As $\psi \in \mathcal{D}(\Omega)$, $v_k = 0$ near E and $d\Theta = 0$ on $\Omega \smallsetminus E$, an integration by parts yields

$$\langle \Theta, \psi \mathrm{i}\, d'd''v_k^2 \rangle = \langle \Theta, v_k^2 \mathrm{i}\, d'd''\psi \rangle \le C \int_{\Omega \smallsetminus E} \|\Theta\| < +\infty$$

where C is a bound for the coefficients of $\mathrm{i}\, d'd''\psi$. Thus $\langle \Theta, \psi \mathrm{i}\, d'v_k \wedge d''v_k \rangle$ is bounded, and the proof is complete when $\Theta \in \mathcal{D}'^{n-1,n-1}$.

In the general case $\Theta \in \mathcal{D}'^{p,p}$, $p < n$, we simply apply the result already proved to all positive currents $\Theta \wedge \gamma \in \mathcal{D}'^{n-1,n-1}$ where $\gamma = \mathrm{i}\,\gamma_1 \wedge \overline{\gamma}_1 \wedge \ldots \wedge \mathrm{i}\,\gamma_{n-p-1,} \wedge \overline{\gamma}_{n-p-1}$ runs over a basis of forms of $\Lambda^{n-p-1,n-p-1} T^\star_\Omega$ with constant coefficients. Then we get $d(\tilde{\Theta} \wedge \gamma) = d\tilde{\Theta} \wedge \gamma = 0$ for all such γ, hence $d\tilde{\Theta} = 0$. □

(1.19) Corollary. *Let Θ be a closed positive current on X and let E be a complete pluripolar set. Then $\mathbf{1}_E \Theta$ and $\mathbf{1}_{X \smallsetminus E} \Theta$ are closed positive currents. In fact, $\widetilde{\Theta} = \mathbf{1}_{X \smallsetminus E} \Theta$ is the trivial extension of $\Theta_{\restriction X \smallsetminus E}$ to X, and $\mathbf{1}_E \Theta = \Theta - \widetilde{\Theta}$.* □

As mentioned above, any current $\Theta = \mathrm{i}\, d'd''u$ associated with a psh function u is a closed positive $(1,1)$-current. In the special case $u = \log|f|$ where $f \in H^0(X, \mathcal{O}_X)$ is a non zero holomorphic function, we have the important

(1.20) Lelong-Poincaré equation. *Let $f \in H^0(X, \mathcal{O}_X)$ be a non zero holomorphic function, $Z_f = \sum m_j Z_j$, $m_j \in \mathbb{N}$, the zero divisor of f and $[Z_f] = \sum m_j [Z_j]$ the associated current of integration. Then*

$$\frac{\mathrm{i}}{\pi} \partial\overline{\partial} \log|f| = [Z_f].$$

Proof (sketch). It is clear that $\mathrm{i}\,d'd''\log|f| = 0$ in a neighborhood of every point $x \notin \mathrm{Supp}(Z_f) = \bigcup Z_j$, so it is enough to check the equation in a neighborhood of every point of $\mathrm{Supp}(Z_f)$. Let A be the set of singular points of $\mathrm{Supp}(Z_f)$, i.e. the union of the pairwise intersections $Z_j \cap Z_k$ and of the singular loci $Z_{j,\mathrm{sing}}$; we thus have $\dim A \le n-2$. In a neighborhood of any point $x \in \mathrm{Supp}(Z_f) \smallsetminus A$ there are local coordinates $(z_1, \ldots, z_n)$ such that $f(z) = z_1^{m_j}$ where m_j is the multiplicity of f along the component Z_j which contains x and $z_1 = 0$ is an equation for Z_j near x. Hence

$$\frac{\mathrm{i}}{\pi} d'd'' \log|f| = m_j \frac{\mathrm{i}}{\pi} d'd'' \log|z_1| = m_j [Z_j]$$

in a neighborhood of x, as desired (the identity comes from the standard formula $\frac{\mathrm{i}}{\pi} d'd'' \log|z| =$ Dirac measure δ_0 in $\mathbb{C}$). This shows that the equation holds on $X \smallsetminus A$. Hence the difference $\frac{\mathrm{i}}{\pi} d'd'' \log|f| - [Z_f]$ is a closed current of degree 2 with measure coefficients, whose support is contained in A. By Exercise 1.21, this current must be 0, for A has too small dimension to carry its support (A is stratified by submanifolds of real codimension ≥ 4). $\square$

(1.21) Exercise. Let Θ be a current of degree q on a real manifold M, such that both Θ and $d\Theta$ have measure coefficients ("normal current"). Suppose that $\mathrm{Supp}\,\Theta$ is contained in a real submanifold A with $\mathrm{codim}_{\mathbb{R}} A > q$. Show that $\Theta = 0$.
Hint: Let $m = \dim_{\mathbb{R}} M$ and let $(x_1, \ldots, x_m)$ be a coordinate system in a neighborhood Ω of a point $a \in A$ such that $A \cap \Omega = \{x_1 = \ldots = x_k = 0\}$, $k > q$. Observe that $x_j \Theta = x_j d\Theta = 0$ for $1 \le j \le k$, thanks to the hypothesis on supports and on the normality of Θ, hence $dx_j \wedge \Theta = d(x_j \Theta) - x_j d\Theta = 0$, $1 \le j \le k$. Infer from this that all coefficients in $\Theta = \sum_{|I|=q} \Theta_I dx_I$ vanish. $\square$

We now recall a few basic facts of slicing theory (the reader will profitably consult [Fed69] and [Siu74] for further developments). Let $\sigma : M \to M'$ be a submersion of smooth differentiable manifolds and let Θ be a *locally flat* current on M, that is, a current which can be written locally as $\Theta = U + dV$ where U, V have L^1_{loc} coefficients. It is a standard fact (see Federer) that every current Θ such that both Θ and $d\Theta$ have measure coefficients is locally flat; in particular, closed positive currents are locally flat. Then, for almost every $x' \in M'$, there is a well defined *slice* $\Theta_{x'}$, which is the current on the fiber $\sigma^{-1}(x')$ defined by

$$\Theta_{x'} = U_{\restriction \sigma^{-1}(x')} + dV_{\restriction \sigma^{-1}(x')}.$$

The restrictions of U, V to the fibers exist for almost all x' by the Fubini theorem. The slices $\Theta_{x'}$ are currents on the fibers with the same degree as Θ (thus of dimension $\dim\Theta - \dim(\text{fibers})$). Of course, every slice $\Theta_{x'}$ coincides with the usual restriction of Θ to the fiber if Θ has smooth coefficients. By using a regularization $\Theta_\varepsilon = \Theta \star \rho_\varepsilon$, it is easy to show that the slices of a closed positive current are again closed and positive: in fact $U_{\varepsilon,x'}$ and $V_{\varepsilon,x'}$ converge to $U_{x'}$ and $V_{x'}$ in $L^1_{\mathrm{loc}}(\sigma^{-1}(x'))$, thus $\Theta_{\varepsilon,x'}$ converges weakly to $\Theta_{x'}$ for almost every x'. Now, the basic slicing formula is

$$(1.22) \qquad \int_M \Theta \wedge \alpha \wedge \sigma^\star \beta = \int_{x' \in M'} \Big(\int_{x'' \in \sigma^{-1}(x')} \Theta_{x'}(x'') \wedge \alpha_{\restriction \sigma^{-1}(x')}(x'') \Big) \beta(x')$$

for every smooth form α on M and β on M', such that α has compact support and $\deg\alpha = \dim M - \dim M' - \deg\Theta$, $\deg\beta = \dim M'$. This is an easy consequence of the usual Fubini theorem applied to U and V in the decomposition $\Theta = U + dV$, if we identify locally σ with a projection map $M = M' \times M'' \to M'$, $x = (x', x'') \mapsto x'$, and use a partitition of unity on the support of α.

To conclude this section, we discuss De Rham and Dolbeault cohomology theory in the context of currents. A basic observation is that the Poincaré and Dolbeault-Grothendieck lemmas still hold for currents. Namely, if $(\mathcal{D}'^q, d)$ and $(\mathcal{D}'(F)^{p,q}, d'')$ denote the complex of sheaves of degree q currents (resp. of (p,q)-currents with values in a holomorphic vector bundle F), we still have De Rham and Dolbeault sheaf resolutions

$$0 \to \mathbb{R} \to \mathcal{D}'^\bullet, \qquad 0 \to \Omega^p_X \otimes \mathcal{O}(F) \to \mathcal{D}'(F)^{p,\bullet}.$$

Hence we get canonical isomorphisms

$$\begin{aligned} H^q_{\mathrm{DR}}(M,\mathbb{R}) &= H^q\big((\Gamma(M,\mathcal{D}'^\bullet), d)\big), \\ H^{p,q}(X,F) &= H^q\big((\Gamma(X,\mathcal{D}'(F)^{p,\bullet}), d'')\big). \end{aligned} \tag{1.23}$$

In other words, we can attach a cohomology class $\{\Theta\} \in H^q_{\mathrm{DR}}(M,\mathbb{R})$ to any closed current Θ of degree q, resp. a cohomology class $\{\Theta\} \in H^{p,q}(X,F)$ to any d''-closed current of bidegree (p,q). Replacing if necessary every current by a smooth representative in the same cohomology class, we see that there is a well defined cup product given by the wedge product of differential forms

$$\begin{aligned} H^{q_1}(M,\mathbb{R}) \times \ldots \times H^{q_m}(M,\mathbb{R}) &\longrightarrow H^{q_1+\ldots+q_m}(M,\mathbb{R}), \\ (\{\Theta_1\}, \ldots, \{\Theta_1\}) &\longmapsto \{\Theta_1\} \wedge \ldots \wedge \{\Theta_m\}. \end{aligned}$$

In particular, if M is a compact oriented variety and $q_1 + \ldots + q_m = \dim M$, there is a well defined intersection number

$$\{\Theta_1\} \cdot \{\Theta_2\} \cdot \cdots \cdot \{\Theta_m\} = \int_M \{\Theta_1\} \wedge \ldots \wedge \{\Theta_m\}.$$

However, as we will see in the next section, the pointwise product $\Theta_1 \wedge \ldots \wedge \Theta_m$ need not exist in general.

2. Lelong Numbers and Intersection Theory

2.A. Multiplication of Currents and Monge-Ampère Operators

Let X be a n-dimensional complex manifold. We set

$$d^c = \frac{1}{2\mathrm{i}\,\pi}(d' - d'').$$

It follows in particular that d^c is a real operator, i.e. $\overline{d^c u} = d^c \overline{u}$, and that $dd^c = \frac{\mathrm{i}}{\pi} d'd''$. Although not quite standard, the $1/2\mathrm{i}\,\pi$ normalization is very convenient for many purposes, since we may then forget the factor π or 2π almost everywhere (e.g. in the Lelong-Poincaré equation (1.20)).

Let u be a psh function and let Θ be a closed positive current on X. Our desire is to define the wedge product $dd^c u \wedge \Theta$ even when neither u nor Θ are smooth. In general, this product does not make sense because $dd^c u$ and Θ have measure coefficients and measures cannot be multiplied; see Kiselman [Kis84] for interesting counterexamples. Even in the algebraic setting considered here, multiplication of currents is not always possible: suppose e.g. that $\Theta = [D]$ is the exceptional divisor of a blow-up in a surface; then $D\cdot D = -1$ cannot be the cohomology class of a closed positive current $[D]^2$. Assume however that u is a *locally bounded* psh function. Then the current $u\Theta$ is well defined since u is a locally bounded Borel function and Θ has measure coefficients. According to Bedford-Taylor [BT82] we define

$$dd^c u \wedge \Theta = dd^c(u\Theta)$$

where $dd^c(\ \ )$ is taken in the sense of distribution theory.

(2.1) Proposition. *If u is a locally bounded psh function, the wedge product $dd^c u\wedge\Theta$ is again a closed positive current.*

Proof. The result is local. Use a convolution $u_\nu = u \star \rho_{1/\nu}$ to get a decreasing sequence of smooth psh functions converging to u. Then write

$$dd^c(u\Theta) = \lim_{\nu\to+\infty} dd^c(u_\nu\Theta) = dd^c u_\nu \wedge \Theta$$

as a weak limit of closed positive currents. Observe that $u_\nu\Theta$ converges weakly to $u\Theta$ by Lebesgue's monotone convergence theorem. □

More generally, if $u_1, \ldots, u_m$ are locally bounded psh functions, we can define

$$dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \Theta = dd^c\big(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_m \wedge \Theta\big)$$

by induction on m. Chern, Levine and Nirenberg [CLN69] noticed the following useful inequality. Define the *mass* of a current Θ on a compact set K to be

$$||\Theta||_K = \int_K \sum_{I,J} |\Theta_{I,J}|$$

whenever K is contained in a coordinate patch and $\Theta = \sum \Theta_{I,J} dz_I \wedge d\overline{z}_J$. Up to seminorm equivalence, this does not depend on the choice of coordinates. If K is not contained in a coordinate patch, we use a partition of unity to define a suitable seminorm $||\Theta||_K$. If $\Theta \geq 0$, Exercise 1.15 shows that the mass is controlled by the trace measure, i.e. $||\Theta||_K \leq C\int_K \Theta\wedge\beta^p$.

(2.2) Chern-Levine-Nirenberg inequality. *For all compact subsets K, L of X with $L \subset K^\circ$, there exists a constant $C_{K,L} \geq 0$ such that*

$$||dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \Theta||_L \leq C_{K,L}\ ||u_1||_{L^\infty(K)} \ldots ||u_m||_{L^\infty(K)}\, ||\Theta||_K$$

Proof. By induction, it is sufficient to prove the result for $m = 1$ and $u_1 = u$. There is a covering of L by a family of open balls $B'_j \subset\subset B_j \subset K$ contained in coordinate patches of X. Let (p,p) be the bidimension of Θ, let $\beta = \frac{\mathrm{i}}{2} d'd''|z|^2$, and let $\chi \in \mathcal{D}(B_j)$ be equal to 1 on $\overline{B}'_j$. Then

$$||dd^c u\wedge\Theta||_{L\cap\overline{B}'_j} \leq C\int_{\overline{B}'_j} dd^c u\wedge\Theta\wedge\beta^{p-1} \leq C\int_{B_j} \chi\, dd^c u\wedge\Theta\wedge\beta^{p-1}.$$

As Θ and β are closed, an integration by parts yields

$$||dd^c u \wedge \Theta||_{L\cap \overline{B}'_j} \le C \int_{B_j} u\,\Theta \wedge dd^c\chi \wedge \beta^{p-1} \le C'||u||_{L^\infty(K)}||\Theta||_K$$

where C' is equal to C multiplied by a bound for the coefficients of the smooth form $dd^c\chi \wedge \beta^{p-1}$. □

Various examples (cf. [Kis84]) show however that products of $(1,1)$-currents $dd^c u_j$ cannot be defined in a reasonable way for arbitrary psh functions u_j. However, functions u_j with $-\infty$ poles can be admitted if the polar sets are sufficiently small.

(2.3) Proposition. *Let u be a psh function on X, and let Θ be a closed positive current of bidimension (p,p). Suppose that u is locally bounded on $X \smallsetminus A$, where A is an analytic subset of X of dimension $< p$ at each point. Then $dd^c u \wedge \Theta$ can be defined in such a way that $dd^c u \wedge \Theta = \lim_{\nu \to +\infty} dd^c u_\nu \wedge \Theta$ in the weak topology of currents, for any decreasing sequence $(u_\nu)_{\nu \ge 0}$ of psh functions converging to u.*

Proof. When u is locally bounded everywhere, we have $\lim u_\nu \Theta = u\Theta$ by the monotone convergence theorem and the result follows from the continuity of dd^c with respect to the weak topology.

First assume that A is discrete. Since our results are local, we may suppose that X is a ball $B(0,R) \subset \mathbb{C}^n$ and that $A = \{0\}$. For every $s \le 0$, the function $u^{\ge s} = \max(u,s)$ is locally bounded on X, so the product $\Theta \wedge dd^c u^{\ge s}$ is well defined. For $|s|$ large, the function $u^{\ge s}$ differs from u only in a small neighborhood of the origin, at which u may have a $-\infty$ pole. Let γ be a $(p-1,p-1)$-form with constant coefficients and set $s(r) = \liminf_{|z| \to r-0} u(z)$. By Stokes' formula, we see that the integral

$$I(s) := \int_{B(0,r)} dd^c u^{\ge s} \wedge \Theta \wedge \gamma \tag{2.4}$$

does not depend on s when $s < s(r)$, for the difference $I(s) - I(s')$ of two such integrals involves the dd^c of a current $(u^{\ge s} - u^{\ge s'}) \wedge \Theta \wedge \gamma$ with compact support in $B(0,r)$. Taking $\gamma = (dd^c|z|^2)^{p-1}$, we see that the current $dd^c u \wedge \Theta$ has finite mass on $B(0,r) \smallsetminus \{0\}$ and we can define $\langle \mathbf{1}_{\{0\}}(dd^c u \wedge \Theta), \gamma \rangle$ to be the limit of the integrals (2.4) as r tends to zero and $s < s(r)$. In this case, the weak convergence statement is easily deduced from the locally bounded case discussed above.

In the case where $0 < \dim A < p$, we use a slicing technique to reduce the situation to the discrete case. Set $q = p-1$. There are linear coordinates $(z_1, \ldots, z_n)$ centered at any point of A, such that 0 is an isolated point of $A \cap (\{0\} \times \mathbb{C}^{n-q})$. Then there are small balls $B' = B(0,r')$ in $\mathbb{C}^q$, $B'' = B(0,r'')$ in $\mathbb{C}^{n-q}$ such that $A \cap (\overline{B}' \times \partial B'') = \emptyset$, and the projection map

$$\pi : \mathbb{C}^n \to \mathbb{C}^q, \qquad z = (z_1, \ldots, z_n) \mapsto z' = (z_1, \ldots, z_q)$$

defines a finite proper mapping $A \cap (B' \times B'') \to B'$. These properties are preserved if we slightly change the direction of projection. Take sufficiently many projections π_m associated to coordinate systems $(z_1^m, \ldots, z_n^m)$, $1 \le m \le N$, in such a way that the family of (q,q)-forms

$$i\,dz_1^m \wedge d\overline{z}_1^m \wedge \ldots \wedge i\,dz_q^m \wedge d\overline{z}_q^m$$

defines a basis of the space of (q,q)-forms. Expressing any compactly supported smooth (q,q)-form in such a basis, we see that we need only define

$$(2.5)\qquad \int_{B'\times B''} dd^c u \wedge \Theta \wedge f(z',z'')\,\mathrm{i}\,dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}\,dz_q \wedge d\overline{z}_q = \int_{B'} \Big\{ \int_{B''} f(z',\bullet)\, dd^c u(z',\bullet) \wedge \Theta(z',\bullet) \Big\} \mathrm{i}\,dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}\,dz_q \wedge d\overline{z}_q$$

where f is a test function with compact support in $B' \times B''$, and $\Theta(z',\bullet)$ denotes the slice of Θ on the fiber $\{z'\} \times B''$ of the projection $\pi : \mathbb{C}^n \to \mathbb{C}^q$. Each integral $\int_{B''}$ in the right hand side of (2.5) makes sense since the slices $(\{z'\} \times B'') \cap A$ are discrete. Moreover, the double integral $\int_{B'}\int_{B''}$ is convergent. Indeed, observe that u is bounded on any compact cylinder

$$K_{\delta,\varepsilon} = \overline{B}\big((1-\delta)r'\big) \times \Big(\overline{B}(r'') \smallsetminus \overline{B}\big((1-\varepsilon)r''\big)\Big)$$

disjoint from A. Take $\varepsilon \ll \delta \ll 1$ so small that

$$\operatorname{Supp} f \subset \overline{B}\big((1-\delta)r'\big) \times \overline{B}\big((1-\varepsilon)r''\big).$$

For all $z' \in \overline{B}((1-\delta)r')$, the proof of the Chern-Levine-Nirenberg inequality (2.2) with a cut-off function $\chi(z'')$ equal to 1 on $B((1-\varepsilon)r'')$ and with support in $B((1-\varepsilon/2)r'')$ shows that

$$\int_{B((1-\varepsilon)r'')} dd^c u(z',\bullet) \wedge \Theta(z',\bullet) \le C_\varepsilon ||u||_{L^\infty(K_{\delta,\varepsilon})} \int_{z''\in B((1-\varepsilon/2)r'')} \Theta(z',z'') \wedge dd^c|z''|^2.$$

This implies that the double integral is convergent. Now replace u everywhere by u_ν and observe that $\lim_{\nu\to+\infty}\int_{B''}$ is the expected integral for every z' such that $\Theta(z',\bullet)$ exists (apply the discrete case already proven). Moreover, the Chern-Levine-Nirenberg inequality yields uniform bounds for all functions u_ν, hence Lebesgue's dominated convergence theorem can be applied to $\int_{B'}$. We conclude from this that the sequence of integrals (2.5) converges when $u_\nu \downarrow u$, as expected. □

(2.6) Remark. In the above proof, the fact that A is an analytic set does not play an essential role. The main point is just that the slices $(\{z'\} \times B'') \cap A$ consist of isolated points for generic choices of coordinates (z',z''). In fact, the proof even works if the slices are totally discontinuous, in particular if they are of zero Hausdorff measure $\mathcal{H}_1$. It follows that Proposition 2.3 still holds whenever A is a closed set such that $\mathcal{H}_{2p-1}(A) = 0$. □

2.B. Lelong Numbers

The concept of Lelong number is an analytic analogue of the algebraic notion of multiplicity. It is a very useful technique to extend results of the intersection theory of algebraic cycles to currents. Lelong numbers have been introduced for the first time by Lelong in [Lel57]. See also [Lel69], [Siu74], [Dem82a, 85a, 87] for further developments.

Let us first recall a few definitions. Let Θ be a closed positive current of bidimension (p,p) on a coordinate open set $\Omega \subset \mathbb{C}^n$ of a complex manifold X. The Lelong number of Θ at a point $x \in \Omega$ is defined to be the limit

$$\nu(\Theta,x) = \lim_{r\to 0+} \nu(\Theta,x,r), \qquad \text{where } \nu(\Theta,x,r) = \frac{\sigma_\Theta(B(x,r))}{\pi^p r^{2p}/p!}$$

measures the ratio of the area of Θ in the ball $B(x,r)$ to the area of the ball of radius r in $\mathbb{C}^p$. As $\sigma_\Theta = \Theta \wedge \frac{1}{p!}(\pi dd^c|z|^2)^p$ by 1.15, we also get

$$\nu(\Theta,x,r) = \frac{1}{r^{2p}} \int_{B(x,r)} \Theta(z) \wedge (dd^c|z|^2)^p. \tag{2.7}$$

The main results concerning Lelong numbers are summarized in the following theorems, due respectively to Lelong, Thie and Siu.

(2.8) Theorem ([Lel57]).

a) *For every positive current Θ, the ratio $\nu(\Theta,x,r)$ is a nonnegative increasing function of r, in particular the limit $\nu(\Theta,x)$ as $r \to 0+$ always exists.*

b) *If $\Theta = dd^c u$ is the bidegree $(1,1)$-current associated with a psh function u, then*

$$\nu(\Theta,x) = \sup\big\{\gamma \geq 0\,;\ u(z) \leq \gamma \log|z-x| + O(1) \ \ \text{at } x\big\}.$$

In particular, if $u = \log|f|$ with $f \in H^0(X,\mathcal{O}_X)$ and $\Theta = dd^c u = [Z_f]$, we have

$$\nu([Z_f],x) = \mathrm{ord}_x(f) = \max\{m \in \mathbb{N}\,;\ D^\alpha f(x) = 0,\ |\alpha| < m\}.$$

(2.9) Theorem ([Thi67]). *In the case where Θ is a current of integration $[A]$ over an analytic subvariety A, the Lelong number $\nu([A],x)$ coincides with the multiplicity of A at x (defined e.g. as the sheet number in the ramified covering obtained by taking a generic linear projection of the germ (A,x) onto a p-dimensional linear subspace through x in any coordinate patch Ω).*

(2.10) Theorem ([Siu74]). *Let Θ be a closed positive current of bidimension (p,p) on the complex manifold X.*

a) *The Lelong number $\nu(\Theta,x)$ is invariant by holomorphic changes of local coordinates.*

b) *For every $c > 0$, the set $E_c(\Theta) = \big\{x \in X\,;\ \nu(\Theta,x) \geq c\big\}$ is a closed analytic subset of X of dimension $\leq p$.*

The most important result is 2.10 b), which is a deep application of Hörmander L^2 estimates (see Section 5). The earlier proofs of all other results were rather intricate in spite of their rather simple nature. We reproduce below a sketch of elementary arguments based on the use of a more general and more flexible notion of Lelong number introduced in [Dem87]. Let φ be a continuous psh function with an isolated $-\infty$ pole at x, e.g. a function of the form $\varphi(z) = \log \sum_{1\leq j\leq N} |g_j(z)|^{\gamma_j}$, $\gamma_j > 0$, where $(g_1,\ldots,g_N)$ is an ideal of germs of holomorphic functions in $\mathcal{O}_x$

with $g^{-1}(0) = \{x\}$. The *generalized Lelong number* $\nu(\Theta, \varphi)$ of Θ with respect to the weight φ is simply defined to be the mass of the measure $\Theta \wedge (dd^c\varphi)^p$ carried by the point x (the measure $\Theta \wedge (dd^c\varphi)^p$ is always well defined thanks to Proposition 2.3). This number can also be seen as the limit $\nu(\Theta, \varphi) = \lim_{t\to-\infty} \nu(\Theta, \varphi, t)$, where

$$\nu(\Theta, \varphi, t) = \int_{\varphi(z)<t} \Theta \wedge (dd^c\varphi)^p. \tag{2.11}$$

The relation with our earlier definition of Lelong numbers (as well as part a) of Theorem 2.8) comes from the identity

$$\nu(\Theta, x, r) = \nu(\Theta, \varphi, \log r), \qquad \varphi(z) = \log|z - x|, \tag{2.12}$$

in particular $\nu(\Theta, x) = \nu(\Theta, \log|\bullet - x|)$. This equality is in turn a consequence of the following general formula, applied to $\chi(t) = e^{2t}$ and $t = \log r$:

$$\int_{\varphi(z)<t} \Theta \wedge (dd^c\chi\circ\varphi)^p = \chi'(t-0)^p \int_{\varphi(z)<t} \Theta \wedge (dd^c\varphi)^p, \tag{2.13}$$

where χ is an arbitrary convex increasing function. To prove the formula, we use a regularization and thus suppose that Θ, φ and χ are smooth, and that t is a non critical value of φ. Then Stokes' formula shows that the integrals on the left and right hand side of (2.13) are equal respectively to

$$\int_{\varphi(z)=t} \Theta \wedge (dd^c\chi\circ\varphi)^{p-1} \wedge d^c(\chi\circ\varphi), \qquad \int_{\varphi(z)=t} \Theta \wedge \big(dd^c\varphi\big)^{p-1} \wedge d^c\varphi,$$

and the differential form of bidegree $(p-1, p)$ appearing in the integrand of the first integral is equal to $(\chi'\circ\varphi)^p\,(dd^c\varphi)^{p-1} \wedge d^c\varphi$. The expected formula follows. Part b) of Theorem 2.8 is a consequence of the Jensen-Lelong formula, whose proof is left as an exercise to the reader.

(2.14) Jensen-Lelong formula. *Let u be any psh function on X. Then u is integrable with respect to the measure $\mu_r = (dd^c\varphi)^{n-1} \wedge d^c\varphi$ supported by the pseudo-sphere $\{\varphi(z) = r\}$ and*

$$\mu_r(u) = \int_{\{\varphi<r\}} u(dd^c\varphi)^n + \int_{-\infty}^r \nu(dd^cu, \varphi, t)\,dt. \qquad \square$$

In our case, we set $\varphi(z) = \log|z - x|$. Then $(dd^c\varphi)^n = \delta_x$ and μ_r is just the unitary invariant mean value measure on the sphere $S(x, e^r)$. For $r < r_0$, Formula 2.14 implies

$$\mu_r(u) - \mu_{r_0}(u) = \int_{r_0}^r \nu(dd^cu, x, t) \sim (r - r_0)\nu(dd^cu, x) \qquad \text{as } r \to -\infty.$$

From this, using the Harnack inequality for subharmonic functions, we get

$$\liminf_{z\to x} \frac{u(z)}{\log|z-x|} = \lim_{r\to-\infty} \frac{\mu_r(u)}{r} = \nu(dd^cu, x).$$

These equalities imply statement 2.8 b).

Next, we show that the Lelong numbers $\nu(T,\varphi)$ only depend on the asymptotic behaviour of φ near the polar set $\varphi^{-1}(-\infty)$. In a precise way:

(2.15) Comparison theorem. *Let Θ be a closed positive current on X, and let $\varphi,\psi : X \to [-\infty,+\infty[$ be continuous psh functions with isolated poles at some point $x\in X$. Assume that*

$$\ell := \limsup_{z\to x} \frac{\psi(z)}{\varphi(z)} < +\infty.$$

Then $\nu(\Theta,\psi) \le \ell^p \nu(\Theta,\varphi)$, and the equality holds if $\ell = \lim \psi/\varphi$.

Proof. (2.12) shows that $\nu(\Theta,\lambda\varphi) = \lambda^p\nu(\Theta,\varphi)$ for every positive constant λ. It is thus sufficient to verify the inequality $\nu(\Theta,\psi)\le\nu(\Theta,\varphi)$ under the hypothesis $\limsup\psi/\varphi<1$. For any $c>0$, consider the psh function

$$u_c = \max(\psi - c, \varphi).$$

Fix $r \ll 0$. For $c>0$ large enough, we have $u_c=\varphi$ on a neighborhood of $\varphi^{-1}(r)$ and Stokes' formula gives

$$\nu(\Theta,\varphi,r) = \nu(\Theta,u_c,r) \ge \nu(\Theta,u_c).$$

On the other hand, the hypothesis $\limsup\psi/\varphi<1$ implies that there exists $t_0<0$ such that $u_c=\psi-c$ on $\{u_c<t_0\}$. We thus get

$$\nu(\Theta,u_c) = \nu(\Theta,\psi-c) = \nu(\Theta,\psi),$$

hence $\nu(\Theta,\psi)\le\nu(\Theta,\varphi)$. The equality case is obtained by reversing the roles of φ and ψ and observing that $\lim\varphi/\psi = 1/l$. □

Part a) of Theorem 2.10 follows immediately from 2.15 by considering the weights $\varphi(z)=\log|\tau(z)-\tau(x)|$, $\psi(z)=\log|\tau'(z)-\tau'(x)|$ associated to coordinates systems $\tau(z)=(z_1,\ldots,z_n)$, $\tau'(z)=(z'_1,\ldots,z'_n)$ in a neighborhood of x. Another application is a direct simple proof of Thie's Theorem 2.9 when $\Theta=[A]$ is the current of integration over an analytic set $A\subset X$ of pure dimension p. For this, we have to observe that Theorem 2.15 still holds provided that x is an isolated point in $\mathrm{Supp}(\Theta)\cap\varphi^{-1}(-\infty)$ and $\mathrm{Supp}(\Theta)\cap\psi^{-1}(-\infty)$ (even though x is not isolated in $\varphi^{-1}(-\infty)$ or $\psi^{-1}(-\infty)$), under the weaker assumption that $\limsup_{\mathrm{Supp}(\Theta)\ni z\to x}\psi(z)/\varphi(z)=\ell$. The reason for this is that all integrals involve currents supported on $\mathrm{Supp}(\Theta)$. Now, by a generic choice of local coordinates $z'=(z_1,\ldots,z_p)$ and $z''=(z_{p+1},\ldots,z_n)$ on (X,x), the germ (A,x) is contained in a cone $|z''|\le C|z'|$. If $B'\subset\mathbb{C}^p$ is a ball of center 0 and radius r' small, and $B''\subset\mathbb{C}^{n-p}$ is the ball of center 0 and radius $r''=Cr'$, the projection

$$\mathrm{pr} : A\cap(B'\times B'')\longrightarrow B'$$

is a ramified covering with finite sheet number m. When $z\in A$ tends to $x=0$, the functions

$$\varphi(z)=\log|z| = \log(|z'|^2+|z''|^2)^{1/2}, \qquad \psi(z)=\log|z'|.$$

satisfy $\lim_{z\to x}\psi(z)/\varphi(z)=1$. Hence Theorem 2.15 implies

$$\nu([A],x)=\nu([A],\varphi)=\nu([A],\psi).$$

Now, Formula 2.13 with $\chi(t)=e^{2t}$ yields

$$\begin{aligned}\nu([A],\psi,\log t) &= t^{-2p}\int_{\{\psi<\log t\}}[A]\wedge\Big(\frac{1}{2}dd^c e^{2\psi}\Big)^p\\ &= t^{-2p}\int_{A\cap\{|z'|<t\}}\Big(\frac{1}{2}\,\mathrm{pr}^\star\, dd^c|z'|^2\Big)^p\\ &= m\,t^{-2p}\int_{\mathbb{C}^p\cap\{|z'|<t\}}\Big(\frac{1}{2}dd^c|z'|^2\Big)^p = m,\end{aligned}$$

hence $\nu([A],\psi)=m$. Here, we have used the fact that pr is an étale covering with m sheets over the complement of the ramification locus $S\subset B'$, and the fact that S is of zero Lebesgue measure in B'.

(2.16) Proposition. *Under the assumptions of Proposition 2.3, we have*

$$\nu(dd^c u\wedge\Theta,x)\ge\nu(u,x)\,\nu(\Theta,x)$$

at every point $x\in X$.

Proof. Assume that $X=B(0,r)$ and $x=0$. By definition

$$\nu(dd^c u\wedge\Theta,x)=\lim_{r\to 0}\int_{|z|\le r}dd^c u\wedge\Theta\wedge(dd^c\log|z|)^{p-1}.$$

Set $\gamma=\nu(u,x)$ and

$$u_\nu(z)=\max\big(u(z),(\gamma-\varepsilon)\log|z|-\nu\big)$$

with $0<\varepsilon<\gamma$ (if $\gamma=0$, there is nothing to prove). Then u_ν decreases to u and

$$\int_{|z|\le r}dd^c u\wedge\Theta\wedge(dd^c\log|z|)^{p-1}\ge\limsup_{\nu\to+\infty}\int_{|z|\le r}dd^c u_\nu\wedge\Theta\wedge(dd^c\log|z|)^{p-1}$$

by the weak convergence of $dd^c u_\nu\wedge\Theta$; here $(dd^c\log|z|)^{p-1}$ is not smooth on $\overline{B}(0,r)$, but the integrals remain unchanged if we replace $\log|z|$ by $\chi(\log|z|/r)$ with a smooth convex function χ such that $\chi(t)=t$ for $t\ge -1$ and $\chi(t)=0$ for $t\le -2$. Now, we have $u(z)\le\gamma\log|z|+C$ near 0, so $u_\nu(z)$ coincides with $(\gamma-\varepsilon)\log|z|-\nu$ on a small ball $B(0,r_\nu)\subset B(0,r)$ and we infer

$$\begin{aligned}\int_{|z|\le r}dd^c u_\nu\wedge\Theta\wedge(dd^c\log|z|)^{p-1}&\ge(\gamma-\varepsilon)\int_{|z|\le r_\nu}\Theta\wedge(dd^c\log|z|)^p\\ &\ge(\gamma-\varepsilon)\nu(\Theta,x).\end{aligned}$$

As $r\in\,]0,R[$ and $\varepsilon\in\,]0,\gamma[$ were arbitrary, the desired inequality follows. □

We will later need an important decomposition formula of [Siu74]. We start with the following lemma.

(2.17) Lemma. *If Θ is a closed positive current of bidimension (p,p) and Z is an irreducible analytic set in X, we set*

$$m_Z=\inf\{x\in Z\,;\,\nu(\Theta,x)\}.$$

a) *There is a countable family of proper analytic subsets* (Z'_j) *of* Z *such that* $\nu(\Theta,x) = m_Z$ *for all* $x \in Z \smallsetminus \bigcup Z'_j$. *We say that* m_Z *is the generic Lelong number of* Θ *along* Z.

b) *If* $\dim Z = p$, *then* $\Theta \geq m_Z[Z]$ *and* $\mathbf{1}_Z\Theta = m_Z[Z]$.

Proof. a) By definition of m_Z and $E_c(\Theta)$, we have $\nu(\Theta,x) \geq m_Z$ for every $x \in Z$ and

$$\nu(\Theta,x) = m_Z \quad \text{on} \ \ Z \smallsetminus \bigcup_{c\in\mathbb{Q},\, c>m_Z} Z \cap E_c(\Theta).$$

However, for $c > m_Z$, the intersection $Z \cap E_c(\Theta)$ is a proper analytic subset of A.

b) Left as an exercise to the reader. It is enough to prove that $\Theta \geq m_Z[Z_{\mathrm{reg}}]$ at regular points of Z, so one may assume that Z is a p-dimensional linear subspace in $\mathbb{C}^n$. Show that the measure $(\Theta - m_Z[Z]) \wedge (dd^c|z|^2)^p$ has nonnegative mass on every ball $|z-a| < r$ with center $a \in Z$. Conclude by using arbitrary affine changes of coordinates that $\Theta - m_Z[Z] \geq 0$. □

(2.18) Decomposition formula ([Siu74]). *Let* Θ *be a closed positive current of bidimension* (p,p). *Then* Θ *can be written as a convergent series of closed positive currents*

$$\Theta = \sum_{k=1}^{+\infty} \lambda_k\,[Z_k] + R,$$

where $[Z_k]$ *is a current of integration over an irreducible analytic set of dimension* p, *and* R *is a residual current with the property that* $\dim E_c(R) < p$ *for every* $c > 0$. *This decomposition is locally and globally unique: the sets* Z_k *are precisely the* p*-dimensional components occurring in the sublevel sets* $E_c(\Theta)$, *and* $\lambda_k = \min_{x\in Z_k} \nu(\Theta,x)$ *is the generic Lelong number of* Θ *along* Z_k.

Proof of uniqueness. If Θ has such a decomposition, the p-dimensional components of $E_c(\Theta)$ are $(Z_j)_{\lambda_j\geq c}$, for $\nu(\Theta,x) = \sum \lambda_j\nu([Z_j],x) + \nu(R,x)$ is non zero only on $\bigcup Z_j \cup \bigcup E_c(R)$, and is equal to λ_j generically on Z_j $\big($more precisely, $\nu(\Theta,x) = \lambda_j$ at every regular point of Z_j which does not belong to any intersection $Z_j \cup Z_k$, $k \neq j$ or to $\bigcup E_c(R)\big)$. In particular Z_j and λ_j are unique.

Proof of existence. Let $(Z_j)_{j\geq 1}$ be the countable collection of p-dimensional components occurring in one of the sets $E_c(\Theta)$, $c \in \mathbb{Q}_+^\star$, and let $\lambda_j > 0$ be the generic Lelong number of Θ along Z_j. Then Lemma 2.17 shows by induction on N that $R_N = \Theta - \sum_{1\leq j\leq N} \lambda_j[Z_j]$ is positive. As R_N is a decreasing sequence, there must be a limit $R = \lim_{N\to+\infty} R_N$ in the weak topology. Thus we have the asserted decomposition. By construction, R has zero generic Lelong number along Z_j, so $\dim E_c(R) < p$ for every $c > 0$. □

It is very important to note that some components of lower dimension can actually occur in $E_c(R)$, but they cannot be subtracted because R has bidimension (p,p). A typical case is the case of a bidimension $(n-1,n-1)$ current $\Theta = dd^cu$ with $u = \log(|f_j|^{\gamma_1} + \ldots |f_N|^{\gamma_N})$ and $f_j \in H^0(X,\mathcal{O}_X)$. In general $\bigcup E_c(\Theta) = \bigcap f_j^{-1}(0)$ has dimension $< n-1$.

Corollary 2.19. *Let $\Theta_j = dd^c u_j$, $1 \le j \le p$, be closed positive $(1,1)$-currents on a complex manifold X. Suppose that there are analytic sets $A_2 \supset \ldots \supset A_p$ in X with* $\operatorname{codim} A_j \ge j$ *at every point such that each u_j, $j \ge 2$, is locally bounded on $X \smallsetminus A_j$. Let $\{A_{p,k}\}_{k\ge 1}$ be the irreducible components of A_p of codimension p exactly and let $\nu_{j,k} = \min_{x\in A_{p,k}} \nu(\Theta_j, x)$ be the generic Lelong number of Θ_j along $A_{p,k}$. Then $\Theta_1 \wedge \ldots \wedge \Theta_p$ is well-defined and*

$$\Theta_1 \wedge \ldots \wedge \Theta_p \ge \sum_{k=1}^{+\infty} \nu_{1,k} \ldots \nu_{p,k}\, [A_{p,k}].$$

Proof. By induction on p, Proposition 2.3 shows that $\Theta_1 \wedge \ldots \wedge \Theta_p$ is well defined. Moreover, Proposition 2.16 implies

$$\nu(\Theta_1 \wedge \ldots \wedge \Theta_p, x) \ge \nu(\Theta_1, x) \ldots \nu(\Theta_p, x) \ge \nu_{1,k} \ldots \nu_{p,k}$$

at every point $x \in A_{p,k}$. The desired inequality is then a consequence of Siu's decomposition theorem. □

3. Hermitian Vector Bundles, Connections and Curvature

The goal of this section is to recall the most basic definitions of hemitian differential geometry related to the concepts of connection, curvature and first Chern class of a line bundle.

Let F be a complex vector bundle of rank r over a smooth differentiable manifold M. A *connection* D on F is a linear differential operator of order 1

$$D : C^\infty(M, \Lambda^q T^\star_M \otimes F) \to C^\infty(M, \Lambda^{q+1} T^\star_M \otimes F)$$

such that

$$D(f \wedge u) = df \wedge u + (-1)^{\deg f} f \wedge Du \tag{3.1}$$

for all forms $f \in C^\infty(M, \Lambda^p T^\star_M)$, $u \in C^\infty(X, \Lambda^q T^\star_M \otimes F)$. On an open set $\Omega \subset M$ where F admits a trivialization $\theta : F_{|\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}^r$, a connection D can be written

$$Du \simeq_\theta du + \Gamma \wedge u$$

where $\Gamma \in C^\infty(\Omega, \Lambda^1 T^\star_M \otimes \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ is an arbitrary matrix of 1-forms and d acts componentwise. It is then easy to check that

$$D^2 u \simeq_\theta (d\Gamma + \Gamma \wedge \Gamma) \wedge u \quad \text{on } \Omega.$$

Since D^2 is a globally defined operator, there is a global 2-form

$$\Theta(D) \in C^\infty(M, \Lambda^2 T^\star_M \otimes \mathrm{Hom}_{\mathbb{C}}(F, F)) \tag{3.2}$$

such that $D^2 u = \Theta(D) \wedge u$ for every form u with values in F.

Assume now that F is endowed with a C^∞ hermitian metric h along the fibers and that the isomorphism $F_{|\Omega} \simeq \Omega \times \mathbb{C}^r$ is given by a C^∞ frame (e_λ). We then have a canonical sesquilinear pairing $\{\bullet, \bullet\} = \{\bullet, \bullet\}_h$

$$(3.3)\quad C^\infty(M,\Lambda^pT^\star_M\otimes F)\times C^\infty(M,\Lambda^qT^\star_M\otimes F)\longrightarrow C^\infty(M,\Lambda^{p+q}T^\star_M\otimes\mathbb{C})$$
$$(u,v)\longmapsto\{u,v\}_h$$

given by

$$\{u,v\}_h=\sum_{\lambda,\mu}u_\lambda\wedge\overline{v}_\mu\langle e_\lambda,e_\mu\rangle_h,\qquad u=\sum u_\lambda\otimes e_\lambda,\quad v=\sum v_\mu\otimes e_\mu.$$

We will frequently omit the subscript h when no confusion can arise. The connection D is said to be *hermitian* (with respect to h) if it satisfies the additional property

$$d\{u,v\}=\{Du,v\}+(-1)^{\deg u}\{u,Dv\}.$$

Assuming that (e_λ) is orthonormal, one easily checks that D is hermitian if and only if $\Gamma^\star=-\Gamma$. In this case $\Theta(D)^\star=-\Theta(D)$ [observe that $(\Gamma\wedge\Gamma)^*=-\Gamma^*\wedge\Gamma^*$ and more generally $(A\wedge B)^*=-B^*\wedge A^*$ for products of matrices of 1-forms, since reversing the order of the product of 1-forms changes the sign]. Therefore the 2-form $\mathrm{i}\,\Theta(D)$ takes values in hermitian symmetric tensors $\mathrm{Herm}(F,F)$, i.e.

$$\mathrm{i}\,\Theta(D)\in C^\infty(M,\Lambda^2T^\star_M\otimes_{\mathbb{R}}\mathrm{Herm}(F,F))$$

where $\mathrm{Herm}(F,F)\subset\mathrm{Hom}(F,F)$ is the real subspace of hermitian endomorphisms.

(3.4) Special case. For a bundle F of rank 1, the connection form Γ of a hermitian connection D can be seen as a 1-form with purely imaginary coefficients (i.e. $\Gamma=\mathrm{i}\,A$, A real). Then we have $\Theta(D)=d\Gamma=\mathrm{i}\,dA$. In particular $\mathrm{i}\,\Theta(F)$ is a closed 2-form. The *first Chern class* of F is defined to be the cohomology class

$$c_1(F)_{\mathbb{R}}=\Big\{\frac{\mathrm{i}}{2\pi}\Theta(D)\Big\}\in H^2_{\mathrm{DR}}(M,\mathbb{R}).$$

The cohomology class is actually independent of the connection, since any other connection D_1 differs by a global 1-form, $D_1u=Du+B\wedge u$, so that $\Theta(D_1)=\Theta(D)+dB$. It is well-known that $c_1(F)_{\mathbb{R}}$ is the image in $H^2(M,\mathbb{R})$ of an integral class $c_1(F)\in H^2(M,\mathbb{Z})$; by using the exponential exact sequence

$$0\to\mathbb{Z}\to\mathcal{E}\to\mathcal{E}^\star\to 0,$$

$c_1(F)$ can be defined in Čech cohomology theory as the image by the coboundary map $H^1(M,\mathcal{E}^\star)\to H^2(M,\mathbb{Z})$ of the cocycle $\{g_{jk}\}\in H^1(M,\mathcal{E}^\star)$ defining F; see e.g. [GrH78] for details. □

We now concentrate ourselves on the complex analytic case. If $M=X$ is a complex manifold X, every connection D on a complex C^∞ vector bundle F can be splitted in a unique way as a sum of a $(1,0)$ and of a $(0,1)$-connection, $D=D'+D''$. In a local trivialization θ given by a C^∞ frame, one can write

$$(3.5')\qquad D'u\simeq_\theta d'u+\Gamma'\wedge u,$$
$$(3.5'')\qquad D''u\simeq_\theta d''u+\Gamma''\wedge u,$$

with $\Gamma=\Gamma'+\Gamma''$. The connection is hermitian if and only if $\Gamma'=-(\Gamma'')^\star$ in any orthonormal frame. Thus there exists a unique hermitian connection D corresponding to a prescribed $(0,1)$ part D''.

Assume now that the hermitian bundle (F,h) itself has a *holomorphic* structure. The unique hermitian connection D_h for which D''_h is the d'' operator defined in § 1 is called the *Chern connection* of F. In a local holomorphic frame (e_λ) of $E_{|\Omega}$, the metric h is then given by a hermitian matrix $H=(h_{\lambda\mu})$, $h_{\lambda\mu}=\langle e_\lambda,e_\mu\rangle$. We have

$$\{u,v\}=\sum_{\lambda,\mu}h_{\lambda\mu}u_\lambda\wedge\overline{v}_\mu=u^\dagger\wedge H\overline{v},$$

where $u^\dagger$ is the transposed matrix of u. Easy computations yield

$$\begin{aligned}d\{u,v\}&=(du)^\dagger\wedge H\overline{v}+(-1)^{\deg u}u^\dagger\wedge(dH\wedge\overline{v}+H\overline{dv})\\&=\big(du+\overline{H}^{\,-1}d'\overline{H}\wedge u\big)^\dagger\wedge H\overline{v}+(-1)^{\deg u}u^\dagger\wedge\overline{(dv+\overline{H}^{\,-1}d'\overline{H}\wedge v)}\end{aligned}$$

using the fact that $dH=d'H+\overline{d'\overline{H}}$ and $\overline{H}^\dagger=H$. Therefore the Chern connection D_h coincides with the hermitian connection defined by

$$\text{(3.6)}\qquad\begin{cases}D_hu\simeq_\theta du+\overline{H}^{\,-1}d'\overline{H}\wedge u,\\ D'_h\simeq_\theta d'+\overline{H}^{\,-1}d'\overline{H}\wedge\bullet=\overline{H}^{\,-1}d'(\overline{H}\bullet),\qquad D''_h=d''.\end{cases}$$

It is clear from the above relations (3.6) that $D'^2_h=D''^2_h=0$. Consequently D^2_h is given by to $D^2_h=D'_hD''_h+D''_hD'_h$, and the curvature tensor $\Theta(D_h)$ is of type $(1,1)$. Since $d'd''+d''d'=0$, we get

$$\begin{aligned}(D'_hD''_h+D''_hD'_h)u&\simeq_\theta\overline{H}^{\,-1}d'\overline{H}\wedge d''u+d''(\overline{H}^{\,-1}d'\overline{H}\wedge u)\\&=d''(\overline{H}^{\,-1}d'\overline{H})\wedge u.\end{aligned}$$

By the above calculation $\Theta(D_h)$ is given by the matrix of $(1,1)$-forms

$$\Theta(D_h)\simeq_\theta d''(\overline{H}^{\,-1}d'\overline{H})=\overline{H}^{\,-1}d''d'\overline{H}-\overline{H}^{\,-1}d''\overline{H}\wedge\overline{H}^{\,-1}d'\overline{H}$$

Since $H=\overline{H}^\dagger$ is hermitian symmetric and transposition reverses products, we find again in this setting that $\Theta(D_h)$ is hermitian skew symmetric

$$\Theta(D_h)^*\simeq_\theta\overline{H}^{\,-1}\overline{\Theta(F)}^\dagger\overline{H}=-\Theta(D_h).$$

(3.7) Definition and proposition. *The Chern curvature tensor of (F,h) is defined to be $\Theta_h(F):=\Theta(D_h)$. It is such that*

$$\mathrm{i}\,\Theta_h(F)\in C^\infty\big(X,\Lambda^{1,1}_{\mathbb{R}}T^\star_X\otimes_{\mathbb{R}}\mathrm{Herm}(F,F)\big)\subset C^\infty\big(X,\Lambda^{1,1}T^\star_X\otimes_{\mathbb{C}}\mathrm{Hom}(F,F)\big).$$

If $\theta:F_{\restriction\Omega}\to\Omega\times\mathbb{C}^r$ is a holomorphic trivialization and if H is the hermitian matrix representing the metric along the fibers of $F_{\restriction\Omega}$, then

$$\mathrm{i}\,\Theta_h(F)\simeq_\theta\mathrm{i}\,d''(\overline{H}^{\,-1}d'\overline{H})\quad\text{on }\Omega.$$

□

[*We will frequently omit the subscript h and write simply $D_h=D$, $\Theta_h(F)=\Theta(F)$ when no confusion can arise*].

The next proposition shows that the Chern curvature tensor is the obstruction to the existence of orthonormal holomorphic frames: a holomorphic frame can be made "almost orthonormal" only up to curvature terms of order 2 in a neighborhood of any point.

(3.8) Proposition. *For every point $x_0 \in X$ and every holomorphic coordinate system $(z_j)_{1\le j\le n}$ at x_0, there exists a holomorphic frame $(e_\lambda)_{1\le\lambda\le r}$ of F in a neighborhood of x_0 such that*

$$\langle e_\lambda(z), e_\mu(z)\rangle = \delta_{\lambda\mu} - \sum_{1\le j,k\le n} c_{jk\lambda\mu}\, z_j\overline{z}_k + O(|z|^3)$$

where $(c_{jk\lambda\mu})$ are the coefficients of the Chern curvature tensor $\Theta(F)_{x_0}$, namely

$$\mathrm{i}\,\Theta(F) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu} dz_j\wedge d\overline{z}_k\otimes e_\lambda^\star\otimes e_\mu.$$

Such a frame (e_λ) is called a normal coordinate frame at x_0.

Proof. Let (h_λ) be a holomorphic frame of F. After replacing (h_λ) by suitable linear combinations with constant coefficients, we may assume that $\big(h_\lambda(x_0)\big)$ is an orthonormal basis of F_{x_0}. Then the inner products $\langle h_\lambda, h_\mu\rangle$ have an expansion

$$\langle h_\lambda(z), h_\mu(z)\rangle = \delta_{\lambda\mu} + \sum_j (a_{j\lambda\mu}\, z_j + a'_{j\lambda\mu}\,\overline{z}_j) + O(|z|^2)$$

for some complex coefficients $a_{j\lambda\mu}$, $a'_{j\lambda\mu}$ such that $a'_{j\lambda\mu} = \overline{a}_{j\mu\lambda}$. Set first

$$g_\lambda(z) = h_\lambda(z) - \sum_{j,\mu} a_{j\lambda\mu}\, z_j\, h_\mu(z).$$

Then there are coefficients $a_{jk\lambda\mu}$, $a'_{jk\lambda\mu}$, $a''_{jk\lambda\mu}$ such that

$$\begin{aligned}\langle g_\lambda(z), g_\mu(z)\rangle &= \delta_{\lambda\mu} + O(|z|^2)\\ &= \delta_{\lambda\mu} + \sum_{j,k}\big(a_{jk\lambda\mu}\, z_j\overline{z}_k + a'_{jk\lambda\mu}\, z_j z_k + a''_{jk\lambda\mu}\overline{z}_j\overline{z}_k\big) + O(|z|^3).\end{aligned}$$

The holomorphic frame (e_λ) we are looking for is

$$e_\lambda(z) = g_\lambda(z) - \sum_{j,k,\mu} a'_{jk\lambda\mu}\, z_j z_k\, g_\mu(z).$$

Since $a''_{jk\lambda\mu} = \overline{a}'_{jk\mu\lambda}$, we easily find

$$\begin{aligned}\langle e_\lambda(z), e_\mu(z)\rangle &= \delta_{\lambda\mu} + \sum_{j,k} a_{jk\lambda\mu}\, z_j\overline{z}_k + O(|z|^3),\\ d'\langle e_\lambda, e_\mu\rangle = \{D'e_\lambda, e_\mu\} &= \sum_{j,k} a_{jk\lambda\mu}\,\overline{z}_k\, dz_j + O(|z|^2),\\ \Theta(F)\cdot e_\lambda = D''(D'e_\lambda) &= \sum_{j,k,\mu} a_{jk\lambda\mu}\, d\overline{z}_k\wedge dz_j\otimes e_\mu + O(|z|),\end{aligned}$$

therefore $c_{jk\lambda\mu} = -a_{jk\lambda\mu}$. □

According to (3.8) and (3.9), one can identify the curvature tensor of F to a hermitian form

$$\widetilde{\Theta}(F)(\xi\otimes v) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda\overline{v}_\mu \tag{3.9}$$

on $T_X \otimes F$. This leads in a natural way to positivity concepts, following definitions introduced by Kodaira [Kod53], Nakano [Nak55] and Griffiths [Gri69].

(3.10) Definition. *The hermitian vector bundle F is said to be*

a) *positive in the sense of Nakano if $\widetilde{\Theta}(F)(\tau) > 0$ for all non zero tensors $\tau = \sum \tau_{j\lambda}\partial/\partial z_j \otimes e_\lambda \in T_X \otimes F$.*

b) *positive in the sense of Griffiths if $\widetilde{\Theta}(F)(\xi \otimes v) > 0$ for all non zero decomposable tensors $\xi \otimes v \in T_X \otimes F$;*

Corresponding semipositivity concepts are defined by relaxing the strict inequalities.

(3.11) Special case of rank 1 bundles. Assume that F is a line bundle. The hermitian matrix $H = (h_{11})$ associated to a trivialization $\theta : F_{\restriction\Omega} \simeq \Omega \times \mathbb{C}$ is simply a positive function which we find convenient to denote by $e^{-2\varphi}$, $\varphi \in C^\infty(\Omega, \mathbb{R})$. In this case the curvature form $\Theta_h(F)$ can be identified to the $(1,1)$-form $2d'd''\varphi$, and

$$\frac{\mathrm{i}}{2\pi}\Theta_h(F) = \frac{\mathrm{i}}{\pi}d'd''\varphi = dd^c\varphi$$

is a real $(1,1)$-form. Hence F is semipositive (in either the Nakano or Griffiths sense) if and only if φ is psh, resp. positive if and only if φ is *strictly psh*. In this setting, the Lelong-Poincaré equation can be generalized as follows: let $\sigma \in H^0(X, F)$ be a non zero holomorphic section. Then

$$dd^c \log \|\sigma\|_h = [Z_\sigma] - \frac{\mathrm{i}}{2\pi}\Theta_h(F). \tag{3.12}$$

Formula (3.12) is immediate if we write $\|\sigma\|_h = |\theta(\sigma)|e^{-\varphi}$ and if we apply (1.20) to the holomorphic function $f = \theta(\sigma)$. As we shall see later, it is very important for the applications to consider also singular hermitian metrics.

(3.13) Definition. *A singular (hermitian) metric h on a line bundle F is a metric h which is given in any trivialization $\theta : F_{\restriction\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}$ by*

$$\|\xi\|_h = |\theta(\xi)|\, e^{-\varphi(x)}, \qquad x \in \Omega,\ \xi \in F_x$$

where φ is an arbitrary measurable function in $L^1_{\mathrm{loc}}(\Omega)$, called the weight of the metric with respect to the trivialization θ.

If $\theta' : F_{\restriction\Omega'} \longrightarrow \Omega' \times \mathbb{C}$ is another trivialization, φ' the associated weight and $g \in \mathcal{O}^\star(\Omega \cap \Omega')$ the transition function, then $\theta'(\xi) = g(x)\,\theta(\xi)$ for $\xi \in F_x$, and so $\varphi' = \varphi + \log|g|$ on $\Omega \cap \Omega'$. The curvature form of F is then given formally by the closed $(1,1)$-current $\frac{\mathrm{i}}{2\pi}\Theta_h(F) = dd^c\varphi$ on Ω ; our assumption $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ guarantees that $\Theta_h(F)$ exists in the sense of distribution theory. As in the smooth case, $\frac{\mathrm{i}}{2\pi}\Theta_h(F)$ is globally defined on X and independent of the choice of trivializations, and its De Rham cohomology class is the image of the first Chern class $c_1(F) \in H^2(X, \mathbb{Z})$ in $H^2_{DR}(X, \mathbb{R})$. Before going further, we discuss two basic examples.

(3.14) Example. Let $D = \sum \alpha_j D_j$ be a divisor with coefficients $\alpha_j \in \mathbb{Z}$ and let $F = \mathcal{O}(D)$ be the associated invertible sheaf of meromorphic functions u such that $\mathrm{div}(u) + D \geq 0$; the corresponding line bundle can be equipped with the singular metric defined by $\|u\| = |u|$. If g_j is a generator of the ideal of D_j on an open set $\Omega \subset X$ then $\theta(u) = u \prod g_j^{\alpha_j}$ defines a trivialization of $\mathcal{O}(D)$ over Ω, thus our singular metric is associated to the weight $\varphi = \sum \alpha_j \log |g_j|$. By the Lelong-Poincaré equation, we find

$$\frac{\mathrm{i}}{2\pi}\Theta\big(\mathcal{O}(D)\big) = dd^c\varphi = [D],$$

where $[D] = \sum \alpha_j [D_j]$ denotes the current of integration over D. □

(3.15) Example. Assume that $\sigma_1, \ldots, \sigma_N$ are non zero holomorphic sections of F. Then we can define a natural (possibly singular) hermitian metric h^* on $F^\star$ by

$$\|\xi^\star\|^2_{h^*} = \sum_{1\le j\le n} \big|\xi^\star.\sigma_j(x)\big|^2 \quad \text{for } \xi^\star \in F^\star_x.$$

The dual metric h on F is given by

$$\text{(3.15 a)} \qquad \|\xi\|^2_h = \frac{|\theta(\xi)|^2}{|\theta(\sigma_1(x))|^2 + \ldots + |\theta(\sigma_N(x))|^2}$$

with respect to any trivialization θ. The associated weight function is thus given by $\varphi(x) = \log\big(\sum_{1\le j\le N} |\theta(\sigma_j(x))|^2\big)^{1/2}$. In this case φ is a psh function, thus $\mathrm{i}\,\Theta_h(F)$ is a closed positive current, given explicity by

$$\text{(3.15 b)} \qquad \frac{\mathrm{i}}{2\pi}\Theta_h(F) = dd^c\varphi = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\log\Big(\sum_{1\le j\le N} |\theta(\sigma_j(x))|^2\Big).$$

Let us denote by Σ the linear system defined by $\sigma_1, \ldots, \sigma_N$ and by $B_\Sigma = \bigcap \sigma_j^{-1}(0)$ its base locus. We have a meromorphic map

$$\Phi_\Sigma : X \smallsetminus B_\Sigma \to \mathbb{P}^{N-1}, \qquad x \mapsto (\sigma_1(x) : \sigma_2(x) : \ldots : \sigma_N(x)).$$

Then $\frac{\mathrm{i}}{2\pi}\Theta(F)$ is equal to the pull-back by Φ_Σ over $X \smallsetminus B_\Sigma$ of the so called *Fubini-Study metric* on $\mathbb{P}^{N-1}$ (see also (4.15) in the next section) :

$$\text{(3.15 c)} \qquad \omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\log(|z_1|^2 + \ldots + |z_N|^2)$$ □

(3.16) Ample and very ample line bundles. *A holomorphic line bundle F over a compact complex manifold X is said to be*

a) *very ample if the map $\Phi_{|F|} : X \to \mathbb{P}^{N-1}$ associated to the complete linear system $|F| = P(H^0(X, F))$ is a regular embedding (by this we mean in particular that the base locus is empty, i.e. $B_{|F|} = \emptyset$).*

b) *ample if some multiple mF, $m > 0$, is very ample.*

Here we use an additive notation for $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^\star)$, hence the symbol mF denotes the line bundle $F^{\otimes m}$. By Example 3.15, every ample line bundle F has a smooth hermitian metric with positive definite curvature form; indeed, if the linear system $|mF|$ gives an embedding in projective space, then we get a smooth hermitian metric on $F^{\otimes m}$, and the m-th root yields a metric on F such that $\frac{\mathrm{i}}{2\pi}\Theta(F) = \frac{1}{m}\Phi^\star_{|mF|}\omega_{\mathrm{FS}}$. Conversely, the Kodaira embedding theorem [Kod54] tells us that every positive line bundle F is ample (see Exercise 6.14 for a straightforward analytic proof of the Kodaira embedding theorem).

4. Kähler identities and Hodge Theory

We briefly explain here the most basic facts of L^2 Hodge theory. Assume for the moment that M is a differentiable manifold equipped with a Riemannian metric $g = \sum g_{ij} dx_i \otimes dx_j$. Given a q-form u on M with values in a hermitian vector bundle (F,h), we consider the global L^2 norm

$$\|u\|^2 = \int_M |u(x)|^2 dV_g(x) \tag{4.1}$$

where $|u| = |u|_{g,h}$ is the pointwise hermitian norm on $\Lambda^q T^*_M \otimes F$ associated with g, h and dV_g is the Riemannian volume form. Let D be a hermitian connection on (F,h). The Laplace-Beltrami operator associated with the connection D is defined to be

$$\Delta = DD^\star + D^\star D \tag{4.2}$$

where

$$D^\star : C^\infty(M, \Lambda^q T^\star_M \otimes F) \to C^\infty(M, \Lambda^{q-1} T^\star_M \otimes F)$$

is the (formal) adjoint of D with respect to the L^2 inner product. More generally, recall that if $P : C^\infty(M,E) \to C^\infty(M,F)$ is a differential operator and both E, F are euclidean or hermitian vector bundles, there exists a unique differential operator

$$P^\star : C^\infty(M,F) \to C^\infty(M,E),$$

called the *formal adjoint* of P, such that for all sections $u \in C^\infty(M,E)$ and $v \in C^\infty(M,F)$ there is an identity

$$\langle\!\langle Pu, v\rangle\!\rangle = \langle\!\langle u, P^\star v\rangle\!\rangle, \qquad \text{whenever } \operatorname{Supp} u \cap \operatorname{Supp} v \subset\subset M. \tag{4.3}$$

In fact, let

$$Pu(x) = \sum_{|\alpha|\le\delta} a_\alpha(x) D^\alpha u(x) \tag{4.4}$$

be the expansion of P with respect to trivializations of E, F given by orthonormal frames over some coordinate open set $\Omega \subset M$ and let $dV_g(x) = \gamma(x)\, dx_1 \ldots dx_n$ be the volume element. When $\operatorname{Supp} u \cap \operatorname{Supp} v \subset\subset \Omega$, an integration by parts yields

$$\begin{aligned}
\langle\!\langle Pu, v\rangle\!\rangle &= \int_\Omega \sum_{|\alpha|\le\delta,\lambda,\mu} a_{\alpha\lambda\mu} D^\alpha u_\mu(x) \overline{v}_\lambda(x)\, \gamma(x)\, dx_1, \ldots, dx_m \\
&= \int_\Omega \sum_{|\alpha|\le\delta,\lambda,\mu} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha(\gamma(x)\, \overline{a}_{\alpha\lambda\mu} v_\lambda(x)}\, dx_1, \ldots, dx_m \\
&= \int_\Omega \langle u, \sum_{|\alpha|\le\delta} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha\big(\gamma(x)\, {}^t\overline{a}_\alpha v(x)\big)\rangle\, dV(x).
\end{aligned}$$

Hence we see that $P^\star$ exists and is uniquely defined by

(4.5) $$P^\star v(x) = \sum_{|\alpha|\le\delta} (-1)^{|\alpha|}\gamma(x)^{-1}D^\alpha\big(\gamma(x)\,{}^t\overline{a}_\alpha v(x)\big).$$

Now, assume that M is *compact*. It can be checked that

$$\Delta : C^\infty(M,\Lambda^q T_M^\star\otimes F)\to C^\infty(M,\Lambda^q T_M^\star\otimes F)$$

is a self-adjoint elliptic operator (for arbitrary M, F and arbitrary metrics on them). Standard results of PDE theory show that there is an orthogonal decomposition

(4.6) $$C^\infty(M,\Lambda^q T_M^\star\otimes F) = \mathcal{H}^q(M,F)\oplus \operatorname{Im}\Delta$$

where $\mathcal{H}^q(M,F) = \operatorname{Ker}\Delta$ is by definition the *space of harmonic forms of degree q in F* which, under these circumstances, must be a *finite dimensional space* by the ellipticity of P (This is the analogue of the theorem of finite dimensional linear algebra which says that for a hermitian symmetric operator, the kernel and the image are orthogonal and form a direct sum decomposition of the space.) Notice also that by definition of $\Delta = DD^* + D^*D$, we get

(4.7) $$\langle\!\langle \Delta u, u\rangle\!\rangle = \|Du\|^2 + \|D^*u\|^2$$

by means of an integration by parts. Therefore the space $\mathcal{H}^q(M,F) = \operatorname{Ker}\Delta$ consists of all forms h such that $Dh = D^*h = 0$ (This is not true if M is non compact : for instance, on $M=\mathbb{C}$, there are harmonic functions which are non constant ...).

(4.8) Basic L^2 decompositions. *Assume that the hermitian connection D on H is integrable, i.e. that $D^2=0$. Then we have orthogonal direct sums*

(4.8 a) $$\operatorname{Im}\Delta = \operatorname{Im}D\oplus\operatorname{Im}D^\star,$$

(4.8 b) $$C^\infty(M,\Lambda^q T_M^\star\otimes F) = \mathcal{H}^q(M,F)\oplus\operatorname{Im}D\oplus\operatorname{Im}D^\star,$$

(4.8 c) $$\operatorname{Ker}D = \mathcal{H}^q(M,F)\oplus\operatorname{Im}D.$$

Proof. Indeed $\langle\!\langle Du, D^\star v\rangle\!\rangle = \langle\!\langle D^2u, v\rangle\!\rangle = 0$ for all u, v, hence $\operatorname{Im}D\perp\operatorname{Im}D^*$. Likewise, we have $\langle\!\langle h, Du\rangle\!\rangle = \langle\!\langle D^*h, u\rangle\!\rangle = 0$ and $\langle\!\langle h, D^*v\rangle\!\rangle = \langle\!\langle Dh, v\rangle\!\rangle = 0$ for every harmonic form h, hence $\mathcal{H}^q(M,F)\perp\operatorname{Im}D$ and $\mathcal{H}^q(M,F)\perp\operatorname{Im}D^*$. By the above and (4.6), it follows that

$$\operatorname{Im}D\oplus\operatorname{Im}D^*\subset\mathcal{H}^q(M,F)^\perp = \operatorname{Im}\Delta,$$

and clearly we also have $\operatorname{Im}\Delta\subset\operatorname{Im}D+\operatorname{Im}D^*$, since $\Delta = DD^*+D^*D$. Decompositions (4.8 a) and (4.8 b) follow. Now, Let $w = h + Du + D^*v$ be an arbitrary element in $C^\infty(M,\Lambda^q T_M^\star\otimes F)$ written according to decomposition (4.8 b). We then find $Dw = DD^*v$, and since $\langle\!\langle Dw, v\rangle\!\rangle = \langle\!\langle DD^*v, v\rangle\!\rangle = \|D^*v\|^2$, we have $Dw=0$ if and only if $D^*v=0$. This proves (4.8 c). □

Notice that if $D^2=0$, one can define De Rham cohomology groups

$$H^q_{\mathrm{DR}}(M,F) = \operatorname{Ker}D/\operatorname{Im}D$$

which just generalize the usual De Rham cohomology groups associated with the exterior derivative d acting on the trivial bundle $M\times\mathbb{C}$.

(4.9) Corollary (Hodge Fundamental Theorem). *If M is a compact Riemannian manifold and F a hermitian line bundle over M with an integrable hermitian connection D, there is an isomorphism*

$$H^q_{\mathrm{DR}}(M,F) \simeq \mathcal{H}^q(M,F)$$

from De Rham cohomology groups onto spaces of harmonic forms. This applies in particular to the case of the trivial bundle $F = M \times \mathbb{C}$. □

A rather important consequence of the Hodge fundamental theorem is a proof of the *Poincaré duality theorem.* Assume that the Riemannian manifold (M, g) is oriented. Then there is a (conjugate linear) Hodge star operator

$$\star : \Lambda^q T^\star_M \otimes \mathbb{C} \to \Lambda^{m-q} T^\star_M \otimes \mathbb{C}, \qquad m = \dim_{\mathbb{R}} M$$

defined by $u \wedge \star v = \langle u, v\rangle \, dV_g$ for any two complex valued q-forms u, v. With respect to an orthonormal frame $(\xi_j)_{1\le j\le n}$ of T_M, standard computations left to the reader show that

$$u = \sum_{|J|=q} u_J \, \xi^*_J \quad \longmapsto \quad * \, u = \sum_{|J|=q} \mathrm{sign}(J, \complement J) \, \overline{u}_J \, \xi^*_{\complement J},$$

and as a consequence $** = \pm \,\mathrm{Id}$ and $D^* = \pm * D *$. It follows that $\star$ commutes with Δ and that $\star u$ is harmonic if and only if u itself is harmonic. This implies

(4.10) Poincaré duality theorem. *If M is a compact oriented differential manifold, the natural pairing*

$$H^q_{\mathrm{DR}}(M,\mathbb{C}) \times H^{m-q}_{\mathrm{DR}}(M,\mathbb{C}), \qquad (\{u\},\{v\}) \mapsto \int_M u \wedge v$$

is a nondegenerate duality.

Proof. The statement follows from the following observations :

- the dual of a class $\{u\}$ represented by a harmonic form is $\{\star u\}$,
- $(u, * \, u)$ is paired to $\int_M u \wedge * \, u = \|u\|^2 > 0$ if $u \ne 0$. □

Let us now suppose that X is a compact complex manifold equipped with a hermitian metric $g = \sum \omega_{jk} dz_j \otimes d\overline{z}_k$. Let F be a holomorphic vector bundle on X equipped with a hermitian metric h, and let $D = D' + D''$ be its Chern curvature form. All that we said above for the Laplace-Beltrami operator Δ still applies to the complex Laplace operators

(4.11) $$\Delta' = D'D'^\star + D'^\star D', \qquad \Delta'' = D''D''^\star + D''^\star D'',$$

with the great advantage that we always have $D'^2 = D''^2 = 0$ without making any special assumption on the connection. Especially, if X is a compact complex manifold, there is an isomorphism

(4.12) $$H^{p,q}(X,F) \simeq \mathcal{H}^{p,q}(X,F)$$

between the Dolbeault cohomology group $H^{p,q}(X, F)$ and the space $\mathcal{H}^{p,q}(X, F)$ of Δ''-harmonic forms of bidegree (p, q) with values in F. The isomorphism is derived as above from the L^2 direct sum decomposition

$$C^\infty(M,\Lambda^{p,q}T^\star_M\otimes F)=\mathcal{H}^{p,q}(M,F)\oplus \operatorname{Im} D''\oplus \operatorname{Im} D''^\star,$$

Now, there is a generalized (conjugate linear) Hodge star operator such that

$$u\wedge \star v=\langle u,v\rangle dV_g,$$

for any two F-valued (p,q)-forms u, v, when the wedge product $u\wedge\star v$ is combined with the natural obvious pairing $F\times F^\star\to\mathbb{C}$. It is given by

$$\begin{aligned}&\star:\Lambda^{p,q}T^\star_X\otimes F\to\Lambda^{n-p,n-q}T^\star_X\otimes F^\star,\qquad n=\dim_{\mathbb{C}}X,\\ &u=\sum_{I,J,\lambda}u_{I,J,\lambda}\,\xi^*_I\wedge\overline{\xi}^*_J\otimes e_\lambda\quad\longmapsto *u=\sum_{I,J,\lambda}\varepsilon_{I,J}\,\overline{u}_{I,J,\lambda}\,\xi^*_{\complement I}\wedge\overline{\xi}^*_{\complement J}\otimes e^*_\lambda\,,\\ &\varepsilon_{I,J}:=(-1)^{q(n-p)}\operatorname{sign}(I,\complement I)\operatorname{sign}(J,\complement J),\qquad \text{for } |I|=p,\ |J|=q,\ 1\le\lambda\le r,\end{aligned}$$

with respect to orthonormal frames $(\xi_j)_{1\le j\le n}$ of (T_X,g) and $(e_\lambda)_{1\le\lambda\le r}$ of (F,h). This leads to the *Serre duality theorem* [Ser55]:

(4.13) Serre duality theorem. *If X is a compact complex manifold, the bilinear pairing*

$$H^{p,q}(X,F)\times H^{n-p,n-q}(X,F^\star),\qquad (\{u\},\{v\})\mapsto\int_X u\wedge v$$

is a nondegenerate duality. □

Combining this with the Dolbeault isomorphism (1.3), we may restate the result in the form of the duality formula

$$H^q(X,\Omega^p_X\otimes\mathcal{O}(F))^\star\simeq H^{n-q}(X,\Omega^{n-p}_X\otimes\mathcal{O}(F^\star)). \tag{4.13$'$}$$

We now proceed to explain the basic commutation relations satisfied by the operators of hermitian geometry. Great simplifications occur in the computations if the hermitian metric on X is supposed to be *Kähler*.

(4.14) Definition. *A hermitian metric $g=\sum_{j,k}\omega_{jk}dz_j\otimes d\overline{z}_k$ on T_X is said to be Kähler if the associated fundamental $(1,1)$-form*

$$\omega=\mathrm{i}\sum_{j,k}\omega_{jk}dz_j\wedge d\overline{z}_k$$

satisfies $d\omega=0$, i.e. is also a symplectic 2-form.

(4.15) Examples.

- $\mathbb{C}^n$ equipped with its canonical metric $\omega=\mathrm{i}\sum_{1\le j\le n}dz_j\wedge d\overline{z}_j$ (or any other hermitian metric $\omega=\mathrm{i}\sum_{1\le j,k\le n}\omega_{jk}dz_j\wedge d\overline{z}_k$ with constant coefficients) is Kähler.
- A *complex torus* is a quotient $X=\mathbb{C}^n/\Gamma$ by a lattice (closed discrete subgroup) Γ of rank $2n$. Then X is a compact complex manifold. Any positive definite hermitian form $\omega=\mathrm{i}\sum\omega_{jk}dz_j\wedge d\overline{z}_k$ with constant coefficients defines a Kähler metric on X.
- The complex projective space $\mathbb{P}^n$ is Kähler. A natural Kähler metric ω_{FS} on $\mathbb{P}^n$ is the *Fubini-Study metric* defined by

$$p^\star\omega_{\mathrm{FS}}=\frac{\mathrm{i}}{2\pi}d'd''\log\big(|\zeta_0|^2+|\zeta_1|^2+\cdots+|\zeta_n|^2\big)$$

where $\zeta_0, \zeta_1, \ldots, \zeta_n$ are coordinates of $\mathbb{C}^{n+1}$ and where $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the projection. Let $z = (\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0)$ be non homogeneous coordinates on $\mathbb{C}^n \subset \mathbb{P}^n$. A calculation shows that

$$\omega_{\rm FS} = \frac{\mathrm{i}}{2\pi} d'd'' \log(1+|z|^2), \qquad \int_{\mathbb{P}^n} \omega_{\rm FS}^n = 1.$$

It follows from this that $\{\omega_{\rm FS}\} = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^n, \mathbb{Z})$ is a generator of the cohomology algebra $H^\bullet(\mathbb{P}^n, \mathbb{Z})$.

• Every (complex) submanifold Y of a Kähler manifold (X, ω) is Kähler with metric $\omega' = \omega_{\restriction Y}$. Especially, all *projective manifolds*, i.e. all submanifolds $X \subset \mathbb{P}^N$, are Kähler with Kähler metric $\omega = \omega_{{\rm FS}\restriction X}$. Since $\omega_{\rm FS}$ is in $H^2(\mathbb{P}, \mathbb{Z})$, the restriction ω is an integral class in $H^2(X, \mathbb{Z})$. Conversely, the Kodaira embedding theorem [Kod54] states that every compact Kähler manifold X possessing a Kähler metric ω with an integral cohomology class $\{\omega\} \in H^2(X, \mathbb{Z})$ can be embedded in projective space as a projective algebraic subvariety. We will be able to prove this in section 6.

(4.16) Lemma. *Let ω be a C^∞ positive definite $(1,1)$-form on X. In order that ω be Kähler, it is necessary and sufficient that to every point $x_0 \in X$ corresponds a holomorphic coordinate system $(z_1, \ldots, z_n)$ centered at x_0 such that*

$$\text{(4.16 a)} \qquad \omega = \mathrm{i} \sum_{1 \le l, m \le n} \omega_{lm}\, dz_l \wedge d\overline{z}_m, \qquad \omega_{lm} = \delta_{lm} + O(|z|^2).$$

If ω is Kähler, the coordinates $(z_j)_{1 \le j \le n}$ can be chosen such that

$$\text{(4.16 b)} \qquad \omega_{lm} = \langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \rangle = \delta_{lm} - \sum_{1 \le j,k \le n} c_{jklm}\, z_j \overline{z}_k + O(|z|^3),$$

where (c_{jklm}) are the coefficients of the Chern curvature tensor

$$\text{(4.16 c)} \qquad \Theta(T_X)_{x_0} = \sum_{j,k,l,m} c_{jklm}\, dz_j \wedge d\overline{z}_k \otimes \Big(\frac{\partial}{\partial z_l}\Big)^\star \otimes \frac{\partial}{\partial z_m}$$

associated to (T_X, ω) at x_0. Such a system (z_j) will be called a geodesic coordinate system at x_0.

Proof. It is clear that (4.16 a) implies $d_{x_0}\omega = 0$, so the condition is sufficient. Assume now that ω is Kähler and show that (4.16 b) can be achieved. The calculations are somewhat similar to those already made for the proof of (3.8). One can choose local coordinates $(x_1, \ldots, x_n)$ such that $(dx_1, \ldots, dx_n)$ is an ω-orthonormal basis of $T^\star_{x_0}X$. Therefore

$$\omega = \mathrm{i} \sum_{1 \le l,m \le n} \widetilde{\omega}_{lm}\, dx_l \wedge d\overline{x}_m, \qquad \text{where}$$

$$\text{(4.16 d)} \qquad \widetilde{\omega}_{lm} = \delta_{lm} + O(|x|) = \delta_{lm} + \sum_{1 \le j \le n} (a_{jlm} x_j + a'_{jlm} \overline{x}_j) + O(|x|^2).$$

Since ω is real, we have $a'_{jlm} = \overline{a}_{jml}$; on the other hand the Kähler condition $\partial\omega_{lm}/\partial x_j = \partial\omega_{jm}/\partial x_l$ at x_0 implies $a_{jlm} = a_{ljm}$. Set now

$$z_m = x_m + \frac{1}{2}\sum_{j,l} a_{jlm} x_j x_l, \qquad 1 \le m \le n.$$

Then (z_m) is a coordinate system at x_0, and

$$\begin{aligned} dz_m &= dx_m + \sum_{j,l} a_{jlm} x_j dx_l, \\ \mathrm{i}\sum_m dz_m \wedge d\overline{z}_m &= \mathrm{i}\sum_m dx_m \wedge d\overline{x}_m + \mathrm{i}\sum_{j,l,m} a_{jlm} x_j\, dx_l \wedge d\overline{x}_m \\ &\qquad + \mathrm{i}\sum_{j,l,m} \overline{a}_{jlm}\overline{x}_j\, dx_m \wedge d\overline{x}_l + O(|x|^2) \\ &= \mathrm{i}\sum_{l,m} \widetilde{\omega}_{lm}\, dx_l \wedge \overline{d}x_m + O(|x|^2) = \omega + O(|z|^2). \end{aligned}$$

Condition (4.16 a) is proved. Suppose the coordinates (x_m) chosen from the beginning so that (4.9) holds with respect to (x_m). Then the Taylor expansion (4.16 d) can be refined into

$$\begin{aligned} (4.16\,\mathrm{e}) \qquad \widetilde{\omega}_{lm} &= \delta_{lm} + O(|x|^2) \\ &= \delta_{lm} + \sum_{j,k}\big(a_{jklm} x_j \overline{x}_k + a'_{jklm} x_j x_k + a''_{jklm}\overline{x}_j\overline{x}_k\big) + O(|x|^3). \end{aligned}$$

These new coefficients satisfy the relations

$$a'_{jklm} = a'_{kjlm}, \qquad a''_{jklm} = \overline{a}'_{jkml}, \qquad \overline{a}_{jklm} = a_{kjml}.$$

The Kähler condition $\partial\omega_{lm}/\partial x_j = \partial\omega_{jm}/\partial x_l$ at $x=0$ gives the equality $a'_{jklm} = a'_{lkjm}$; in particular a'_{jklm} is invariant under all permutations of j,k,l. If we set

$$z_m = x_m + \frac{1}{3}\sum_{j,k,l} a'_{jklm}\, x_j x_k x_l, \qquad 1 \le m \le n,$$

then by (4.16 e) we find

$$\begin{aligned} dz_m &= dx_m + \sum_{j,k,l} a'_{jklm}\, x_j x_k\, dx_l, \qquad 1 \le m \le n, \\ \omega &= \mathrm{i}\sum_{1\le m\le n} dz_m \wedge d\overline{z}_m + \mathrm{i}\sum_{j,k,l,m} a_{jklm}\, x_j\overline{x}_k\, dx_l \wedge d\overline{x}_m + O(|x|^3), \\ (4.16\,\mathrm{f}) \qquad \omega &= \mathrm{i}\sum_{1\le m\le n} dz_m \wedge d\overline{z}_m + \mathrm{i}\sum_{j,k,l,m} a_{jklm}\, z_j\overline{z}_k\, dz_l \wedge d\overline{z}_m + O(|z|^3). \end{aligned}$$

It is now easy to compute the Chern curvature tensor $\Theta(T_X)_{x_0}$ in terms of the coefficients a_{jklm}. Indeed

$$\begin{aligned} \langle\frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m}\rangle &= \delta_{lm} + \sum_{j,k} a_{jklm}\, z_j\overline{z}_k + O(|z|^3), \\ d'\langle\frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m}\rangle &= \Big\{D'\frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m}\Big\} = \sum_{j,k} a_{jklm}\,\overline{z}_k\, dz_j + O(|z|^2), \\ \Theta(T_X)\cdot\frac{\partial}{\partial z_l} &= D''D'\Big(\frac{\partial}{\partial z_l}\Big) = -\sum_{j,k,m} a_{jklm}\, dz_j \wedge d\overline{z}_k \otimes \frac{\partial}{\partial z_m} + O(|z|), \end{aligned}$$

therefore $c_{jklm} = -a_{jklm}$ and the expansion (4.16 b) follows from (4.16 f). □

The next step is to compute the basic operators on $X = \mathbb{C}^n$, in the case of a trivial line bundle $F = X \times \mathbb{C}$ (with the trivial constant metric!). For any form $u \in C^\infty(\Omega, \Lambda^{p,q}T^\star_\Omega)$ we have

$$d'u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J = \sum_{1\le k\le n} dz_k \wedge \Big(\frac{\partial u}{\partial z_k}\Big),$$
$$d''u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J = \sum_{1\le k\le n} d\overline{z}_k \wedge \Big(\frac{\partial u}{\partial \overline{z}_k}\Big),$$

where $\partial u/\partial z_k$ and $\partial u/\partial \overline{z}_k$ are the differentiations of u in z_k, $\overline{z}_k$, taken componentwise on each coefficient u_{IJ}. From this we easily get

(4.17) Lemma. *On any open subset $\omega \subset \mathbb{C}^n$ equipped with the constant hermitian metric $\omega = \mathrm{i}\sum_{1\le j\le n} dz_j \wedge d\overline{z}_j$, we have*

$$d'^\star u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k} \frac{\partial}{\partial z_k} \,\lrcorner\, (dz_I \wedge d\overline{z}_J),$$
$$d''^\star u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} \frac{\partial}{\partial \overline{z}_k} \,\lrcorner\, (dz_I \wedge d\overline{z}_J),$$

where $\lrcorner$ denotes the interior product of a form by a vector field. These formulas can be written more briefly as

$$d'^\star u = -\sum_{1\le k\le n} \frac{\partial}{\partial z_k} \,\lrcorner\, \Big(\frac{\partial u}{\partial \overline{z}_k}\Big), \qquad d''^\star u = -\sum_{1\le k\le n} \frac{\partial}{\partial \overline{z}_k} \,\lrcorner\, \Big(\frac{\partial u}{\partial z_k}\Big).$$

Proof. The adjoint of $dz_j \wedge \bullet$ is $\frac{\partial}{\partial z_j} \lrcorner \bullet$. In the case of $d'^\star$, for instance, we get

$$\langle\!\langle d'u, v\rangle\!\rangle = \int_\Omega \langle \sum_{1\le k\le n} dz_k \wedge \Big(\frac{\partial u}{\partial z_k}\Big), v\rangle\, dV = \int_\Omega \sum_{1\le k\le n} \langle \frac{\partial u}{\partial z_k}, \frac{\partial}{\partial z_k} \,\lrcorner\, v\rangle\, dV$$
$$= \int_\Omega \langle u, -\sum_{1\le k\le n} \frac{\partial}{\partial \overline{z}_k}\Big(\frac{\partial}{\partial z_k} \,\lrcorner\, v\Big)\rangle\, dV = \int_\Omega \langle u, -\sum_{1\le k\le n} \frac{\partial}{\partial z_k} \,\lrcorner\, \Big(\frac{\partial v}{\partial \overline{z}_k}\Big)\rangle\, dV$$

whenever u (resp. v) is a $(p-1, q)$-form (resp. (p, q)-form), with $\mathrm{Supp}\, u \cap \mathrm{Supp}\, v \subset\subset \Omega$. The third equality is simply obtained through an integration par parts, and amounts to observe that the formal adjoint of $\partial/\partial z_k$ is $-\partial/\partial \overline{z}_k$. □

We now prove a useful lemma due to Akizuki and Nakano [AN54]. If (X, ω) is a Kähler manifold, we define the operators L and Λ by

(4, 18) $\quad u \mapsto Lu = \omega \wedge u, \quad L : C^\infty(X, \Lambda^{p,q}T^*_X \otimes F) \longrightarrow C^\infty(X, \Lambda^{p+1,q+1}T^*_X \otimes F),$

(4.18*) $\quad \Lambda = L^* : C^\infty(X, \Lambda^{p,q}T^*_X \otimes F) \longrightarrow C^\infty(X, \Lambda^{p-1,q-1}T^*_X \otimes F).$

Again, in the flat hermitian complex space $(\mathbb{C}^n, \omega)$ we find :

(4.19) Lemma. *In* $\mathbb{C}^n$*, we have* $[d''^\star, L] = \mathrm{i}\, d'$*.*

Proof. Using Lemma 4.17, we get

$$\begin{aligned}[d''^\star, L]u &= d''^\star(\omega \wedge u) - \omega \wedge d''^\star u \\ &= -\sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner \Big(\frac{\partial}{\partial z_k}(\omega \wedge u)\Big) + \omega \wedge \sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner \Big(\frac{\partial u}{\partial z_k}\Big).\end{aligned}$$

Since ω has constant coefficients, we have $\frac{\partial}{\partial z_k}(\omega \wedge u) = \omega \wedge \frac{\partial u}{\partial z_k}$ and therefore

$$\begin{aligned}[d''^\star, L]\, u &= -\sum_k \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner \Big(\omega \wedge \frac{\partial u}{\partial z_k}\Big) - \omega \wedge \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner \frac{\partial u}{\partial z_k}\Big)\Big) \\ &= -\sum_k \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner\, \omega\Big) \wedge \frac{\partial u}{\partial z_k}\end{aligned}$$

by the derivation property of $\lrcorner$. Clearly

$$\frac{\partial}{\partial \overline{z}_k} \lrcorner\, \omega = \frac{\partial}{\partial \overline{z}_k} \lrcorner\, \mathrm{i} \sum_{1\le j\le n} dz_j \wedge d\overline{z}_j = -\mathrm{i}\, dz_k,$$

hence

$$[d''^\star, L]\, u = \mathrm{i} \sum_k dz_k \wedge \frac{\partial u}{\partial z_k} = \mathrm{i}\, d'u. \qquad \square$$

The final step is to extend these results to an arbitrary Kähler manifold (X, ω) and an arbitrary hermitian vector bundle (F, h). When we use a geodesic coordinate system $(z_1, \ldots, z_n)$ for ω in combination with a normal coordinate frame for (F, h), all norms expressed in a neighborhood of x_0 differ from constant norms by $O(|z|^2)$ terms only. It follows that all order 1 operators D, D', D'' and their adjoints $D^\star$, $D'^\star$, $D''^\star$ admit at x_0 the same expansion as the analogous operators obtained when the hermitian metrics on X or F are freezed at their values at x_0 (first order derivatives applied to $O(|z|^2)$ terms yield 0). This can be used to derive the basic commutation relations of Kähler geometry from the simpler case of constant metrics in an open subset of $\mathbb{C}^n$.

If A, B are differential operators acting on the algebra $C^\infty(X, \Lambda^{\bullet,\bullet} T^\star_X \otimes F)$, their graded commutator (or graded Lie bracket) is defined by

$$[A, B] = AB - (-1)^{ab} BA \tag{4.20}$$

where a, b are the degrees of A and B respectively. If C is another endomorphism of degree c, the following purely formal *Jacobi identity* holds:

$$(-1)^{ca}\big[A, [B, C]\big] + (-1)^{ab}\big[B, [C, A]\big] + (-1)^{bc}\big[C, [A, B]\big] = 0. \tag{4.21}$$

(4.22) Basic commutation relations. *Let* (X, ω) *be a Kähler manifold and let* L *be the operators defined by* $Lu = \omega \wedge u$ *and* $\Lambda = L^\star$*. Then*

$$\begin{aligned}[D''^\star, L] &= \mathrm{i}\, D', & [D'^\star, L] &= -\mathrm{i}\, D'', \\ [\Lambda, D''] &= -\mathrm{i}\, D'^\star, & [\Lambda, D'] &= \mathrm{i}\, D''^\star.\end{aligned}$$

Proof. The first formula follows from Lemma 4.19, by looking at Taylor expansions of the coefficients of the operators up to order 1. The three other identities are obtained by taking conjugates or adjoints in the first one. □

(4.23) Bochner-Kodaira-Nakano identity. *If (X,ω) is Kähler, the complex Laplace operators Δ' and Δ'' acting on F-valued forms satisfy the identity*

$$\Delta'' = \Delta' + [\mathrm{i}\,\Theta(F), \Lambda].$$

Proof. The last equality in (4.22) yields $D''^\star = -\mathrm{i}\,[\Lambda, D']$, hence

$$\Delta'' = [D'', \delta''] = -\mathrm{i}\,\big[D'', [\Lambda, D']\big].$$

By the Jacobi identity (4.21) we get

$$\big[D'', [\Lambda, D']\big] = \big[\Lambda, [D', D'']\big] + \big[D', [D'', \Lambda]\big] = [\Lambda, \Theta(F)] + \mathrm{i}\,[D', D'^\star],$$

taking into account that $[D', D''] = D^2 = \Theta(F)$. The formula follows. □

(4.24) Corollary (Hodge decomposition). *If (X,ω) is a compact Kähler manifold, there is a canonical decomposition*

$$H^k_{\mathrm{DR}}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C})$$

of the De Rham cohomology groups in terms of the Dolbeault cohomology groups.

Proof. If we apply the Bochner-Kodaira-Nakano identity to the trivial bundle $F = X \times \mathbb{C}$, we find $\Delta'' = \Delta'$. Morever

$$\Delta = [d' + d'', d'^\star + d''^\star] = \Delta' + \Delta'' + [d', d''^\star] + [d'', d'^\star].$$

We claim that $[d', d''^\star] = [d'', d'^\star] = 0$. Indeed, we have $[d', d''^\star] = -\mathrm{i}\,\big[d', [\Lambda, d']\big]$ by (4.22), and the Jacobi identity (4.21) implies

$$-\big[d', [\Lambda, d']\big] + \big[\Lambda, [d', d']\big] + \big[d', [d', \Lambda]\big] = 0,$$

hence $-2\big[d', [\Lambda, d']\big] = 0$ and $[d', d''^\star] = 0$. The second identity is similar. As a consequence

$$\Delta' = \Delta'' = \frac{1}{2}\Delta.$$

We infer that Δ preserves the bidegree of forms and operates "separately" on each term $C^\infty(X, \Lambda^{p,q}T^*_X)$. Hence, on the level of harmonic forms we have

$$\mathcal{H}^k(X,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X,\mathbb{C}).$$

The decomposition theorem (4.24) now follows from the Hodge isomorphisms for De Rham and Dolbeault groups. The decomposition is canonical since $H^{p,q}(X)$ coincides with the set of classes in $H^k(X,\mathbb{C})$ which can be represented by d-closed (p,q)-forms. □

(4.25) Remark. The decomposition formula shows that the so called *Betti numbers* $b_k := \dim H^k_{\mathrm{DR}}(X, \mathbb{C})$ satisfy the relation

$$b_k = \sum_{p+q=k} h^{p,q}$$

in terms of the *Hodge numbers* $h^{p,q} := \dim H^{p,q}(X, \mathbb{C})$. In addition to this, the complex conjugation $u \mapsto \overline{u}$ takes (p, q)-harmonic forms to (q, p)-harmonic forms, hence there is a canonical conjugate linear isomorphism

$$H^{q,p}(X, \mathbb{C}) \simeq \overline{H^{p,q}(X, \mathbb{C})}$$

and $h^{q,p} = h^{p,q}$. In particular, if X is compact Kähler, the Betti numbers b_k of odd index are even. The Serre duality theorem gives the further relation

$$H^{n-p,n-q}(X, \mathbb{C}) \simeq H^{p,q}(X, \mathbb{C})^*$$

(actually, this is even true for an arbitrary compact complex manifold X).

This observation implies the existence of compact complex manifolds X which are non Kähler (hence non projective) : for instance the *Hopf surface* defined by $X = \mathbb{C}^2 \smallsetminus \{0\}/\Gamma$, where $\Gamma \simeq \mathbb{Z}$ is the discrete group generated by a contraction $z \mapsto \lambda z$, $0 < \lambda < 1$, is easily seen to be diffeomorphic to $S^1 \times S^3$. Hence $b_1 = 1$ by the Künneth formula, and X cannot be Kähler.

5. Bochner Technique and Vanishing Theorems

Assume that X is compact and that $u \in C^\infty(X, \Lambda^{p,q}T^\star X \otimes F)$ is an arbitrary (p, q)-form. An integration by parts yields

$$\langle \Delta' u, u\rangle = \|D'u\|^2 + \|D'^\star u\|^2 \geq 0$$

and similarly for Δ'', hence we get the basic a priori inequality

$$\|D''u\|^2 + \|D''^\star u\|^2 \geq \int_X \langle [\mathrm{i}\,\Theta(F), \Lambda]u, u\rangle dV_\omega. \tag{5.1}$$

This inequality is known as the *Bochner-Kodaira-Nakano* inequality (see [Boc48], [Kod53], [Nak55]). When u is Δ''-harmonic, we get

$$\int_X \big(\langle [\mathrm{i}\,\Theta(F), \Lambda]u, u\rangle + \langle T_\omega u, u\rangle\big) dV \leq 0.$$

If the hermitian operator $[\mathrm{i}\,\Theta(F), \Lambda]$ acting on $\Lambda^{p,q}T^\star_X \otimes F$ is positive on each fiber, we infer that u must be zero, hence

$$H^{p,q}(X, F) = \mathcal{H}^{p,q}(X, F) = 0$$

by Hodge theory. The main point is thus to compute the curvature form $\Theta(F)$ and find sufficient conditions under which the operator $[\mathrm{i}\,\Theta(F), \Lambda]$ is positive definite. Elementary (but somewhat tedious) calculations yield the following formulae: if the curvature of F is written as in (3.9) and $u = \sum u_{J,K,\lambda} dz_I \wedge d\overline{z}_J \otimes e_\lambda$, $|J| = p$, $|K| = q$, $1 \leq \lambda \leq r$ is a (p, q)-form with values in F, then

$$\langle[\mathrm{i}\,\Theta(F),\Lambda]u,u\rangle = \sum_{j,k,\lambda,\mu,J,S} c_{jk\lambda\mu}\,u_{J,jS,\lambda}\,\overline{u_{J,kS,\mu}} + \sum_{j,k,\lambda,\mu,R,K} c_{jk\lambda\mu}\,u_{kR,K,\lambda}\,\overline{u_{jR,K,\mu}} - \sum_{j,\lambda,\mu,J,K} c_{jj\lambda\mu}\,u_{J,K,\lambda}\,\overline{u_{J,K,\mu}}, \tag{5.2}$$

where the sum is extended to all indices $1 \le j,k \le n$, $1 \le \lambda,\mu \le r$ and multiindices $|R| = p-1$, $|S| = q-1$ (here the notation $u_{JK\lambda}$ is extended to non necessarily increasing multiindices by making it alternate with respect to permutations). It is usually hard to decide the sign of the curvature term (5.2), except in some special cases.

The easiest case is when $p = n$. Then all terms in the second summation of (5.2) must have $j = k$ and $R = \{1,\ldots,n\} \smallsetminus \{j\}$, therefore the second and third summations are equal. It follows that $[\mathrm{i}\,\Theta(F),\Lambda]$ is positive on (n,q)-forms under the assumption that F is positive in the sense of Nakano. In this case X is automatically Kähler since

$$\omega = \mathrm{Tr}_F(\mathrm{i}\,\Theta(F)) = \mathrm{i}\sum_{j,k,\lambda} c_{jk\lambda\lambda} dz_j \wedge d\overline{z}_k = \mathrm{i}\,\Theta(\det F)$$

is a Kähler metric.

(5.3) Nakano vanishing theorem ([Nak55]). *Let X be a compact complex manifold and let F be a Nakano positive vector bundle on X. Then*

$$H^{n,q}(X,F) = H^q(X, K_X \otimes F) = 0 \qquad \text{for every } q \ge 1. \qquad \square$$

Another tractable case is the case where F is a line bundle ($r = 1$). Indeed, at each point $x \in X$, we may then choose a coordinate system which diagonalizes simultaneously the hermitians forms $\omega(x)$ and $\mathrm{i}\,\Theta(F)(x)$, in such a way that

$$\omega(x) = \mathrm{i}\sum_{1\le j\le n} dz_j \wedge d\overline{z}_j, \qquad \mathrm{i}\,\Theta(F)(x) = \mathrm{i}\sum_{1\le j\le n} \gamma_j dz_j \wedge d\overline{z}_j$$

with $\gamma_1 \le \ldots \le \gamma_n$. The curvature eigenvalues $\gamma_j = \gamma_j(x)$ are then uniquely defined and depend continuously on x. With our previous notation, we have $\gamma_j = c_{jj11}$ and all other coefficients $c_{jk\lambda\mu}$ are zero. For any (p,q)-form $u = \sum u_{JK} dz_J \wedge d\overline{z}_K \otimes e_1$, this gives

$$\begin{aligned}\langle[\mathrm{i}\,\Theta(F),\Lambda]u,u\rangle &= \sum_{|J|=p,\,|K|=q} \Big(\sum_{j\in J}\gamma_j + \sum_{j\in K}\gamma_j - \sum_{1\le j\le n}\gamma_j\Big)|u_{JK}|^2 \\ &\ge (\gamma_1 + \ldots + \gamma_q - \gamma_{n-p+1} - \ldots - \gamma_n)|u|^2.\end{aligned} \tag{5.4}$$

Assume that $\mathrm{i}\,\Theta(F)$ is positive. It is then natural to make the special choice $\omega = \mathrm{i}\,\Theta(F)$ for the Kähler metric. Then $\gamma_j = 1$ for $j = 1,2,\ldots,n$ and we obtain $\langle[\mathrm{i}\,\Theta(F),\Lambda]u,u\rangle = (p+q-n)|u|^2$. As a consequence:

(5.5) Akizuki-Kodaira-Nakano vanishing theorem ([AN54]). *If F is a positive line bundle on a compact complex manifold X, then*

$$H^{p,q}(X,F) = H^q(X,\Omega_X^p \otimes F) = 0 \qquad \textit{for} \;\; p+q \geq n+1. \qquad \square$$

More generally, if F is a Griffiths positive (or ample) vector bundle of rank $r \geq 1$, Le Potier [LP75] proved that $H^{p,q}(X,F) = 0$ for $p+q \geq n+r$. The proof is not a direct consequence of the Bochner technique. A rather easy proof has been found by M. Schneider [Sch74], using the Leray spectral sequence associated to the projectivized bundle projection $\mathbb{P}(F) \to X$.

(5.7) Exercise. It is important for various applications to obtain vanishing theorems which are also valid in the case of semipositive line bundles. The easiest case is the following result of Girbau [Gir76]: let (X,ω) be compact Kähler; assume that F is a line bundle and that $\mathrm{i}\,\Theta(F) \geq 0$ has at least $n-k$ positive eigenvalues at each point, for some integer $k \geq 0$; show that $H^{p,q}(X,F) = 0$ for $p+q \geq n+k+1$.
Hint: use the Kähler metric $\omega_\varepsilon = \mathrm{i}\,\Theta(F) + \varepsilon\omega$ with $\varepsilon > 0$ small.

A stronger and more natural "algebraic version" of this result has been obtained by Sommese [Som78]: define F to be k-ample if some multiple mF is such that the canonical map

$$\Phi_{|mF|} : X \smallsetminus B_{|mF|} \to \mathbb{P}^{N-1}$$

has at most k-dimensional fibers and $\dim B_{|mF|} \leq k$. If X is projective and F is k-ample, show that $H^{p,q}(X,F) = 0$ for $p+q \geq n+k+1$.
Hint: prove the dual result $H^{p,q}(X,F^{-1}) = 0$ for $p+q \leq n-k-1$ by induction on k. First show that F 0-ample $\Rightarrow$ F positive; then use hyperplane sections $Y \subset X$ to prove the induction step, thanks to the exact sequences

$$0 \longrightarrow \Omega_X^p \otimes F^{-1} \otimes \mathcal{O}(-Y) \longrightarrow \Omega_X^p \otimes F^{-1} \longrightarrow \big(\Omega_X^p \otimes F^{-1}\big)_{\restriction Y} \longrightarrow 0,$$

$$0 \longrightarrow \Omega_Y^{p-1} \otimes F^{-1}_{\restriction Y} \otimes \mathcal{O}(-Y)_{\restriction Y} \longrightarrow \big(\Omega_X^p \otimes F^{-1}\big)_{\restriction Y} \longrightarrow \Omega_Y^p \otimes F^{-1}_{\restriction Y} \longrightarrow 0. \qquad \square$$

6. L^2 Estimates and Existence Theorems

The starting point is the following L^2 existence theorem, which is essentially due to Hörmander [Hör65, 66], and Andreotti-Vesentini [AV65]. We will only outline the main ideas, referring e.g. to [Dem82b] for a detailed exposition of the technical situation considered here.

(6.1) Theorem. *Let (X,ω) be a Kähler manifold. Here X is not necessarily compact, but we assume that the geodesic distance δ_ω is complete on X. Let F be a hermitian vector bundle of rank r over X, and assume that the curvature operator $A = A^{p,q}_{F,\omega} = [\mathrm{i}\,\Theta(F),\Lambda_\omega]$ is positive definite everywhere on $\Lambda^{p,q}T^\star_X \otimes F$, $q \geq 1$. Then for any form $g \in L^2(X,\Lambda^{p,q}T^\star_X \otimes F)$ satisfying $D''g = 0$ and $\int_X \langle A^{-1}g,g\rangle\, dV_\omega < +\infty$, there exists $f \in L^2(X,\Lambda^{p,q-1}T^\star_X \otimes F)$ such that $D''f = g$ and*

$$\int_X |f|^2\, dV_\omega \leq \int_X \langle A^{-1}g,g\rangle\, dV_\omega.$$

Proof. A well-known result of Riemannian geometry (Hopf-Rinow lemma) asserts that the geodesic distance δ_ω is complete if and only if all closed geodesic balls

$\overline{B}_\omega(x_0,r)$, $r>0$, are compact (none can "approach the boundary", i.e. the boundary of X is at infinity from the point of view of distance). This assumption implies (is in fact equivalent to) the existence of cut-off functions ψ_ν with arbitrarily large compact support such that $|d\psi_\nu|\le 2^{-\nu}$ (take ψ_ν to be a function of the rescaled distance $x\mapsto 2^{-\nu}\delta_\omega(x_0,x)$ (the distance function is an almost everywhere differentiable 1-Lipschitz function, so one might have to regularize it beforehand).

From this, it follows that very form $u\in L^2(X,\Lambda^{p,q}T^\star_X\otimes F)$ such that $D''u\in L^2$ and $D''^\star u\in L^2$ in the sense of distribution theory is a limit of a sequence of smooth forms u_ν with compact support, in such a way that $u_\nu\to u$, $D''u_\nu\to D''u$ and $D''^\star u_\nu\to D''^\star u$ in L^2 (just take u_ν to be a regularization of $\psi_\nu u$ and notice that the terms coming from the differentiation of ψ_ν tend to 0 in L^2 norm).

As a consequence, the basic a priori inequality (5.1) extends to arbitrary forms u such that u, $D''u, D''^\star u\in L^2$ (observe that the proof of (5.1) requires integration by parts; hence, if X is non compact, the inequality is a priori guaranteed only for compactly supported forms). Now, consider the Hilbert space orthogonal decomposition

$$L^2(X,\Lambda^{p,q}T^\star_X\otimes F)=\operatorname{Ker}D''\oplus(\operatorname{Ker}D'')^\perp,$$

observing that $\operatorname{Ker}D''$ is weakly (hence strongly) closed. Let $v=v_1+v_2$ be the decomposition of a smooth form $v\in\mathcal{D}^{p,q}(X,F)$ with compact support according to this decomposition (v_1, v_2 do not have compact support in general!). Since $(\operatorname{Ker}D'')^\perp\subset\operatorname{Ker}D''^\star$ by duality and $g,v_1\in\operatorname{Ker}D''$ by hypothesis, we get $D''^\star v_2=0$ and

$$|\langle g,v\rangle|^2=|\langle g,v_1\rangle|^2\le\int_X\langle A^{-1}g,g\rangle\,dV_\omega\int_X\langle Av_1,v_1\rangle\,dV_\omega$$

thanks to the Cauchy-Schwarz inequality. The a priori inequality (5.1) applied to $u=v_1$ yields

$$\int_X\langle Av_1,v_1\rangle\,dV_\omega\le\|D''v_1\|^2+\|D''^\star v_1\|^2=\|D''^\star v_1\|^2=\|D''^\star v\|^2.$$

Combining both inequalities, we find

$$|\langle g,v\rangle|^2\le\Big(\int_X\langle A^{-1}g,g\rangle\,dV_\omega\Big)\|D''^\star v\|^2$$

for every smooth (p,q)-form v with compact support. This shows that we have a well defined linear form

$$w=D''^\star v\longmapsto\langle v,g\rangle,\qquad L^2(X,\Lambda^{p,q-1}T^\star_X\otimes F)\supset D''^\star(\mathcal{D}^{p,q}(F))\longrightarrow\mathbb{C}$$

on the range of $D''^\star$. This linear form is continuous in L^2 norm and has norm $\le C$ with

$$C=\Big(\int_X\langle A^{-1}g,g\rangle\,dV_\omega\Big)^{1/2}.$$

By the Hahn-Banach theorem, there is an element $f\in L^2(X,\Lambda^{p,q-1}T^\star_X\otimes F)$ with $||f||\le C$, such that $\langle v,g\rangle=\langle D''^\star v,f\rangle$ for every v, hence $D''f=g$ in the sense of distributions. The inequality $||f||\le C$ is equivalent to the last estimate in the theorem. □

The above L^2 existence theorem can be applied in the fairly general context of *weakly pseudoconvex* manifolds. By this, we mean a complex manifold X such that there exists a smooth psh exhaustion function ψ on X (ψ is said to be an exhaustion if for every $c > 0$ the sublevel set $X_c = \psi^{-1}(c)$ is relatively compact, i.e. $\psi(z)$ tends to $+\infty$ when z is taken outside larger and larger compact subsets of X). In particular, every compact complex manifold X is weakly pseudoconvex (take $\psi = 0$), as well as every Stein manifold, e.g. affine algebraic submanifolds of $\mathbb{C}^N$ (take $\psi(z) = |z|^2$), open balls $X = B(z_0, r)$ $\big($take $\psi(z) = 1/(r - |z - z_0|^2)\big)$, convex open subsets, etc. Now, a basic observation is that every weakly pseudoconvex Kähler manifold (X, ω) carries a *complete* Kähler metric: let $\psi \geq 0$ be a psh exhaustion function and set

$$\omega_\varepsilon = \omega + \varepsilon\,\mathrm{i}\,d'd''\psi^2 = \omega + 2\varepsilon(2\mathrm{i}\,\psi d'd''\psi + \mathrm{i}\,d'\psi \wedge d''\psi).$$

Then $|d\psi|_{\omega_\varepsilon} \leq 1/\varepsilon$ and $|\psi(x) - \psi(y)| \leq \varepsilon^{-1}\delta_{\omega_\varepsilon}(x, y)$. It follows easily from this estimate that the geodesic balls are relatively compact, hence $\delta_{\omega_\varepsilon}$ is complete for every $\varepsilon > 0$. Therefore, the L^2 existence theorem can be applied to each Kähler metric ω_ε, and by passing to the limit it can even be applied to the non necessarily complete metric ω. An important special case is the following

(6.2) Theorem. *Let* (X, ω) *be a Kähler manifold,* $\dim X = n$. *Assume that* X *is weakly pseudoconvex. Let* F *be a hermitian line bundle and let*

$$\gamma_1(x) \leq \ldots \leq \gamma_n(x)$$

be the curvature eigenvalues (i.e. the eigenvalues of $\mathrm{i}\,\Theta(F)$ *with respect to the metric* ω*) at every point. Assume that the curvature is positive, i.e.* $\gamma_1 > 0$ *everywhere. Then for any form* $g \in L^2(X, \Lambda^{n,q}T^\star_X \otimes F)$ *satisfying* $D''g = 0$ *and* $\int_X \langle(\gamma_1 + \ldots + \gamma_q)^{-1}|g|^2\, dV_\omega < +\infty$*, there exists* $f \in L^2(X, \Lambda^{p,q-1}T^\star_X \otimes F)$ *such that* $D''f = g$ *and*

$$\int_X |f|^2\, dV_\omega \leq \int_X (\gamma_1 + \ldots + \gamma_q)^{-1}|g|^2\, dV_\omega.$$

Proof. Indeed, for $p = n$, Formula 5.4 shows that

$$\langle Au, u\rangle \geq (\gamma_1 + \ldots + \gamma_q)|u|^2,$$

hence $\langle A^{-1}u, u\rangle \geq (\gamma_1 + \ldots + \gamma_q)^{-1}|u|^2$. □

An important observation is that the above theorem still applies when the hermitian metric on F is a singular metric with positive curvature in the sense of currents. In fact, by standard regularization techniques (convolution of psh functions by smoothing kernels), the metric can be made smooth and the solutions obtained by (6.1) or (6.2) for the smooth metrics have limits satisfying the desired estimates (see [Dem82b, 92]. Especially, we get the following

(6.3) Corollary. *Let* (X, ω) *be a Kähler manifold,* $\dim X = n$. *Assume that* X *is weakly pseudoconvex. Let* F *be a holomorphic line bundle equipped with a singular metric whose local weights are denoted* $\varphi \in L^1_{\mathrm{loc}}$. *Suppose that*

$$\mathrm{i}\,\Theta(F) = 2\mathrm{i}\,d'd''\varphi \geq \varepsilon\omega$$

for some $\varepsilon > 0$*. Then for any form* $g \in L^2(X, \Lambda^{n,q}T^\star_X \otimes F)$ *satisfying* $D''g = 0$*, there exists* $f \in L^2(X, \Lambda^{p,q-1}T^\star_X \otimes F)$ *such that* $D''f = g$ *and*

$$\int_X |f|^2 e^{-2\varphi}\, dV_\omega \le \frac{1}{q\varepsilon}\int_X |g|^2 e^{-2\varphi}\, dV_\omega. \qquad \square$$

Here we denoted somewhat incorrectly the metric by $|f|^2 e^{-2\varphi}$, as if the weight φ was globally defined on X (of course, this is so only if F is globally trivial). We will use this notation anyway, because it clearly describes the dependence of the L^2 norm on the psh weights.

We now introduce the concept of *multiplier ideal sheaf*, following A. Nadel [Nad89]. The main idea actually goes back to the fundamental works of Bombieri [Bom70] and H. Skoda [Sko72a].

(6.4) Definition. *Let* φ *be a psh function on an open subset* $\Omega \subset X$ *; to* φ *is associated the ideal subsheaf* $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$ *of germs of holomorphic functions* $f \in \mathcal{O}_{\Omega,x}$ *such that* $|f|^2 e^{-2\varphi}$ *is integrable with respect to the Lebesgue measure in some local coordinates near* x*.*

The zero variety $V(\mathcal{I}(\varphi))$ is thus the set of points in a neighborhood of which $e^{-2\varphi}$ is non integrable. Of course, such points occur only if φ has logarithmic poles. This is made precise as follows.

(6.5) Definition. *A psh function* φ *is said to have a logarithmic pole of coefficient* γ *at a point* $x \in X$ *if the Lelong number*

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{\log|z - x|}$$

is non zero and if $\nu(\varphi, x) = \gamma$*.*

(6.6) Lemma (Skoda [Sko72a])**.** *Let* φ *be a psh function on an open set* Ω *and let* $x \in \Omega$*.*

a) *If* $\nu(\varphi, x) < 1$*, then* $e^{-2\varphi}$ *is integrable in a neighborhood of* x*, in particular* $\mathcal{I}(\varphi)_x = \mathcal{O}_{\Omega,x}$*.*

b) *If* $\nu(\varphi, x) \ge n + s$ *for some integer* $s \ge 0$*, then* $e^{-2\varphi} \ge C|z - x|^{-2n-2s}$ *in a neighborhood of* x *and* $\mathcal{I}(\varphi)_x \subset \mathfrak{m}^{s+1}_{\Omega,x}$*, where* $\mathfrak{m}_{\Omega,x}$ *is the maximal ideal of* $\mathcal{O}_{\Omega,x}$*.*

c) *The zero variety* $V(\mathcal{I}(\varphi))$ *of* $\mathcal{I}(\varphi)$ *satisfies*

$$E_n(\varphi) \subset V(\mathcal{I}(\varphi)) \subset E_1(\varphi)$$

where $E_c(\varphi) = \{x \in X\,;\, \nu(\varphi, x) \ge c\}$ *is the c-sublevel set of Lelong numbers of* φ*.*

Proof. a) Set $\Theta = dd^c\varphi$ and $\gamma = \nu(\Theta, x) = \nu(\varphi, x)$. Let χ be a cut-off function with support in a small ball $B(x, r)$, equal to 1 in $B(x, r/2)$. As $(dd^c \log|z|)^n = \delta_0$, we get

$$\begin{aligned}\varphi(z) &= \int_{B(x,r)} \chi(\zeta)\varphi(\zeta)(dd^c \log|\zeta - z|)^n \\ &= \int_{B(x,r)} dd^c(\chi(\zeta)\varphi(\zeta)) \wedge \log|\zeta - z|(dd^c \log|\zeta - z|)^{n-1}\end{aligned}$$

for $z \in B(x, r/2)$. Expanding $dd^c(\chi\varphi)$ and observing that $d\chi = dd^c\chi = 0$ on $B(x, r/2)$, we find

$$\varphi(z) = \int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge \log|\zeta - z|(dd^c \log|\zeta - z|)^{n-1} + \text{smooth terms}$$

on $B(x, r/2)$. Fix r so small that

$$\int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - x|)^{n-1} \le \nu(\Theta, x, r) < 1.$$

By continuity, there exists $\delta, \varepsilon > 0$ such that

$$I(z) := \int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - z|)^{n-1} \le 1 - \delta$$

for all $z \in B(x, \varepsilon)$. Applying Jensen's convexity inequality to the probability measure

$$d\mu_z(\zeta) = I(z)^{-1}\chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - z|)^{n-1},$$

we find

$$\begin{aligned}-\varphi(z) &= \int_{B(x,r)} I(z) \log|\zeta - z|^{-1}\, d\mu_z(\zeta) + O(1) \quad \Longrightarrow \\ e^{-2\varphi(z)} &\le C \int_{B(x,r)} |\zeta - z|^{-2I(z)}\, d\mu_z(\zeta).\end{aligned}$$

As

$$d\mu_z(\zeta) \le C_1|\zeta - z|^{-(2n-2)}\Theta(\zeta) \wedge (dd^c|\zeta|^2)^{n-1} = C_2|\zeta - z|^{-(2n-2)} d\sigma_\Theta(\zeta),$$

we get

$$e^{-2\varphi(z)} \le C_3 \int_{B(x,r)} |\zeta - z|^{-2(1-\delta)-(2n-2)} d\sigma_\Theta(\zeta),$$

and the Fubini theorem implies that $e^{-2\varphi(z)}$ is integrable on a neighborhood of x.

b) If $\nu(\varphi, x) = \gamma$, the convexity properties of psh functions, namely, the convexity of $\log r \mapsto \sup_{|z-x|=r} \varphi(z)$ implies that

$$\varphi(z) \le \gamma \log|z - x|/r_0 + M,$$

where M is the supremum on $B(x, r_0)$. Hence there exists a constant $C > 0$ such that $e^{-2\varphi(z)} \ge C|z - x|^{-2\gamma}$ in a neighborhood of x. The desired result follows from the identity

$$\int_{B(0,r_0)} \frac{\left|\sum a_\alpha z^\alpha\right|^2}{|z|^{2\gamma}} dV(z) = \text{Const} \int_0^{r_0} \Big(\sum |a_\alpha|^2 r^{2|\alpha|}\Big) r^{2n-1-2\gamma}\, dr,$$

which is an easy consequence of Parseval's formula. In fact, if γ has integral part $[\gamma] = n + s$, the integral converges if and only if $a_\alpha = 0$ for $|\alpha| \le s$.

c) is just a simple formal consequence of a) and b). □

(6.7) Proposition ([Nad89]). *For any psh function φ on $\Omega \subset X$, the sheaf $\mathcal{I}(\varphi)$ is a coherent sheaf of ideals over Ω. Moreover, if Ω is a bounded Stein open set, the sheaf $\mathcal{I}(\varphi)$ is generated by any Hilbert basis of the L^2 space $\mathcal{H}^2(\Omega,\varphi)$ of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2\varphi}\,d\lambda < +\infty$.*

Proof. Since the result is local, we may assume that Ω is a bounded pseudoconvex open set in $\mathbb{C}^n$. By the strong noetherian property of coherent sheaves, the family of sheaves generated by finite subsets of $\mathcal{H}^2(\Omega,\varphi)$ has a maximal element on each compact subset of Ω, hence $\mathcal{H}^2(\Omega,\varphi)$ generates a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_\Omega$. It is clear that $\mathcal{J} \subset \mathcal{I}(\varphi)$; in order to prove the equality, we need only check that $\mathcal{J}_x + \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1} = \mathcal{I}(\varphi)_x$ for every integer s, in view of the Krull lemma. Let $f \in \mathcal{I}(\varphi)_x$ be defined in a neighborhood V of x and let θ be a cut-off function with support in V such that $\theta = 1$ in a neighborhood of x. We solve the equation $d''u = g := d''(\theta f)$ by means of Hörmander's L^2 estimates 6.3, where F is the trivial line bundle $\Omega \times \mathbb{C}$ equipped with the strictly psh weight

$$\widetilde{\varphi}(z) = \varphi(z) + (n+s)\log|z-x| + |z|^2.$$

We get a solution u such that $\int_\Omega |u|^2 e^{-2\varphi}|z-x|^{-2(n+s)}d\lambda < \infty$, thus $F = \theta f - u$ is holomorphic, $F \in \mathcal{H}^2(\Omega,\varphi)$ and $f_x - F_x = u_x \in \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1}$. This proves the coherence. Now, $\mathcal{J}$ is generated by any Hilbert basis of $\mathcal{H}^2(\Omega,\varphi)$, because it is well-known that the space of sections of any coherent sheaf is a Fréchet space, therefore closed under local L^2 convergence. □

The multiplier ideal sheaves satisfy the following basic functoriality property with respect to direct images of sheaves by modifications.

(6.8) Proposition. *Let $\mu : X' \to X$ be a modification of non singular complex manifolds (i.e. a proper generically 1:1 holomorphic map), and let φ be a psh function on X. Then*

$$\mu_\star\big(\mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi\circ\mu)\big) = \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi).$$

Proof. Let $n = \dim X = \dim X'$ and let $S \subset X$ be an analytic set such that $\mu : X' \smallsetminus S' \to X \smallsetminus S$ is a biholomorphism. By definition of multiplier ideal sheaves, $\mathcal{O}(K_X) \otimes \mathcal{I}(\varphi)$ is just the sheaf of holomorphic n-forms f on open sets $U \subset X$ such that $\mathrm{i}^{n^2} f \wedge \overline{f}\, e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)$. Since φ is locally bounded from above, we may even consider forms f which are a priori defined only on $U \smallsetminus S$, because f will be in $L^2_{\mathrm{loc}}(U)$ and therefore will automatically extend through S. The change of variable formula yields

$$\int_U \mathrm{i}^{n^2} f \wedge \overline{f}\, e^{-2\varphi} = \int_{\mu^{-1}(U)} \mathrm{i}^{n^2} \mu^\star f \wedge \overline{\mu^\star f}\, e^{-2\varphi\circ\mu},$$

hence $f \in \Gamma(U, \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi))$ iff $\mu^\star f \in \Gamma(\mu^{-1}(U), \mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi\circ\mu))$. Proposition 6.8 is proved. □

(6.9) Remark. If φ has analytic singularities (according to Definition 1.10), the computation of $\mathcal{I}(\varphi)$ can be reduced to a purely algebraic problem.

The first observation is that $\mathcal{I}(\varphi)$ can be computed easily if φ has the form $\varphi = \sum \alpha_j \log|g_j|$ where $D_j = g_j^{-1}(0)$ are nonsingular irreducible divisors with normal crossings. Then $\mathcal{I}(\varphi)$ is the sheaf of functions h on open sets $U \subset X$ such that

$$\int_U |h|^2 \prod |g_j|^{-2\alpha_j} dV < +\infty.$$

Since locally the g_j can be taken to be coordinate functions from a local coordinate system $(z_1, \ldots, z_n)$, the condition is that h is divisible by $\prod g_j^{m_j}$ where $m_j - \alpha_j > -1$ for each j, i.e. $m_j \geq \lfloor \alpha_j \rfloor$ (integer part). Hence

$$\mathcal{I}(\varphi) = \mathcal{O}(-\lfloor D \rfloor) = \mathcal{O}(-\sum \lfloor \alpha_j \rfloor D_j)$$

where $\lfloor D \rfloor$ denotes the integral part of the $\mathbb{Q}$-divisor $D = \sum \alpha_j D_j$.

Now, consider the general case of analytic singularities and suppose that $\varphi \sim \frac{\alpha}{2} \log\big(|f_1|^2 + \cdots + |f_N|^2\big)$ near the poles. By the remarks after Definition 1.10, we may assume that the (f_j) are generators of the integrally closed ideal sheaf $\mathcal{J} = \mathcal{J}(\varphi/\alpha)$, defined as the sheaf of holomorphic functions h such that $|h| \leq C \exp(\varphi/\alpha)$. In this case, the computation is made as follows (see also L. Bonavero's work [Bon93], where similar ideas are used in connection with "singular" holomorphic Morse inequalities).

First, one computes a smooth modification $\mu : \widetilde{X} \to X$ of X such that $\mu^\star \mathcal{J}$ is an invertible sheaf $\mathcal{O}(-D)$ associated with a normal crossing divisor $D = \sum \lambda_j D_j$, where (D_j) are the components of the exceptional divisor of $\widetilde{X}$ (take the blow-up X' of X with respect to the ideal $\mathcal{J}$ so that the pull-back of $\mathcal{J}$ to X' becomes an invertible sheaf $\mathcal{O}(-D')$, then blow up again by Hironaka [Hir64] to make X' smooth and D' have normal crossings). Now, we have $K_{\widetilde{X}} = \mu^\star K_X + R$ where $R = \sum \rho_j D_j$ is the zero divisor of the Jacobian function J_μ of the blow-up map. By the direct image formula 6.8, we get

$$\mathcal{I}(\varphi) = \mu_\star\big(\mathcal{O}(K_{\widetilde{X}} - \mu^\star K_X) \otimes \mathcal{I}(\varphi \circ \mu)\big) = \mu_\star\big(\mathcal{O}(R) \otimes \mathcal{I}(\varphi \circ \mu)\big).$$

Now, $(f_j \circ \mu)$ are generators of the ideal $\mathcal{O}(-D)$, hence

$$\varphi \circ \mu \sim \alpha \sum \lambda_j \log|g_j|$$

where g_j are local generators of $\mathcal{O}(-D_j)$. We are thus reduced to computing multiplier ideal sheaves in the case where the poles are given by a $\mathbb{Q}$-divisor with normal crossings $\sum \alpha\lambda_j D_j$. We obtain $\mathcal{I}(\varphi \circ \mu) = \mathcal{O}(-\sum \lfloor \alpha\lambda_j \rfloor D_j)$, hence

$$\mathcal{I}(\varphi) = \mu_\star \mathcal{O}_{\widetilde{X}}\big(\sum (\rho_j - \lfloor \alpha\lambda_j \rfloor) D_j\big). \qquad \square$$

(6.10) Exercise. Compute the multiplier ideal sheaf $\mathcal{I}(\varphi)$ associated with $\varphi = \log(|z_1|^{\alpha_1} + \ldots + |z_p|^{\alpha_p})$ for arbitrary real numbers $\alpha_j > 0$.
Hint: using Parseval's formula and polar coordinates $z_j = r_j e^{\mathrm{i}\theta_j}$, show that the problem is equivalent to determining for which p-tuples $(\beta_1, \ldots, \beta_p) \in \mathbb{N}^p$ the integral

$$\int_{[0,1]^p} \frac{r_1^{2\beta_1} \ldots r_p^{2\beta_p}\, r_1 dr_1 \ldots r_p dr_p}{r_1^{2\alpha_1} + \ldots + r_p^{2\alpha_p}} = \int_{[0,1]^p} \frac{t_1^{(\beta_1+1)/\alpha_1} \ldots t_p^{(\beta_p+1)/\alpha_p}}{t_1 + \ldots + t_p} \prod_{j=1}^{p} \frac{1}{2\alpha_j} \frac{dt_j}{t_j}$$

is convergent. Conclude from this that $\mathcal{I}(\varphi)$ is generated by the monomials $z_1^{\beta_1} \ldots z_p^{\beta_p}$ such that $\sum(\beta_p + 1)/\alpha_p > 1$. (This exercise shows that the analytic definition of $\mathcal{I}(\varphi)$ is sometimes also quite convenient for computations). □

Let F be a line bundle over X with a singular metric h of curvature current $\Theta_h(F)$. If φ is the weight representing the metric in an open set $\Omega \subset X$, the ideal sheaf $\mathcal{I}(\varphi)$ is independent of the choice of the trivialization and so it is the restriction to Ω of a global coherent sheaf $\mathcal{I}(h)$ on X. We will sometimes still write $\mathcal{I}(h) = \mathcal{I}(\varphi)$ by abuse of notation. In this context, we have the following fundamental vanishing theorem, which is probably one of the most central results of analytic and algebraic geometry (as we will see later, it contains the Kawamata-Viehweg vanishing theorem as a special case).

(6.11) Nadel vanishing theorem ([Nad89], [Dem93b])**.** *Let (X, ω) be a Kähler weakly pseudoconvex manifold, and let F be a holomorphic line bundle over X equipped with a singular hermitian metric h of weight φ. Assume that $\mathrm{i}\,\Theta_h(F) \geq \varepsilon\omega$ for some continuous positive function ε on X. Then*

$$H^q\big(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(h)\big) = 0 \qquad \text{for all } q \geq 1.$$

Proof. Let $\mathcal{L}^q$ be the sheaf of germs of (n, q)-forms u with values in F and with measurable coefficients, such that both $|u|^2 e^{-2\varphi}$ and $|d''u|^2 e^{-2\varphi}$ are locally integrable. The d'' operator defines a complex of sheaves $(\mathcal{L}^\bullet, d'')$ which is a resolution of the sheaf $\mathcal{O}(K_X + F) \otimes \mathcal{I}(\varphi)$: indeed, the kernel of d'' in degree 0 consists of all germs of holomorphic n-forms with values in F which satisfy the integrability condition; hence the coefficient function lies in $\mathcal{I}(\varphi)$; the exactness in degree $q \geq 1$ follows from Corollary 6.3 applied on arbitrary small balls. Each sheaf $\mathcal{L}^q$ is a $\mathcal{C}^\infty$-module, so $\mathcal{L}^\bullet$ is a resolution by acyclic sheaves. Let ψ be a smooth psh exhaustion function on X. Let us apply Corollary 6.3 globally on X, with the original metric of F multiplied by the factor $e^{-\chi\circ\psi}$, where χ is a convex increasing function of arbitrary fast growth at infinity. This factor can be used to ensure the convergence of integrals at infinity. By Corollary 6.3, we conclude that $H^q\big(\Gamma(X, \mathcal{L}^\bullet)\big) = 0$ for $q \geq 1$. The theorem follows. □

(6.12) Corollary. *Let (X, ω), F and φ be as in Theorem 6.11 and let $x_1, \ldots, x_N$ be isolated points in the zero variety $V(\mathcal{I}(\varphi))$. Then there is a surjective map*

$$H^0(X, K_X + F) \longrightarrow\!\!\!\!\!\rightarrow \bigoplus_{1 \leq j \leq N} \mathcal{O}(K_X + L)_{x_j} \otimes \big(\mathcal{O}_X/\mathcal{I}(\varphi)\big)_{x_j}.$$

Proof. Consider the long exact sequence of cohomology associated to the short exact sequence $0 \to \mathcal{I}(\varphi) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}(\varphi) \to 0$ twisted by $\mathcal{O}(K_X + F)$, and apply Theorem 6.11 to obtain the vanishing of the first H^1 group. The asserted surjectivity property follows. □

(6.13) Corollary. *Let (X, ω), F and φ be as in Theorem 6.11 and suppose that the weight function φ is such that $\nu(\varphi, x) \geq n + s$ at some point $x \in X$ which is an isolated point of $E_1(\varphi)$. Then $H^0(X, K_X + F)$ generates all s-jets at x.*

Proof. The assumption is that $\nu(\varphi, y) < 1$ for y near x, $y \neq x$. By Skoda's lemma 6.6 b), we conclude that $e^{-2\varphi}$ is integrable at all such points y, hence $\mathcal{I}(\varphi)_y = \mathcal{O}_{X,y}$, whilst $\mathcal{I}(\varphi)_x \subset \mathfrak{m}_{X,x}^{s+1}$ by 6.6 a). Corollary 6.13 is thus a special case of 6.12. □

The philosophy of these results (which can be seen as generalizations of the Hörmander-Bombieri-Skoda theorem [Bom70], [Sko72a, 75]) is that the problem of constructing holomorphic sections of $K_X + F$ can be solved by constructing suitable hermitian metrics on F such that the weight φ has isolated poles at given points x_j. The following result gives a somewhat general result in this direction.

(6.14) Theorem. *Let X be a compact complex manifold, E a holomorphic vector bundle and (F, h_F) a hermitian line bundle with a smooth metric h such that*

$$\omega = \mathrm{i}\,\Theta_{h_F}(F) > 0.$$

Let φ be a quasi-psh function on X, i.e. a function φ such that $\mathrm{i}\,d'd''\varphi \geq -C\omega$ for some constant $C > 0$. Then

a) *There exists an integer m_0 such that*

$$H^q(X, E \otimes F^{\otimes m} \otimes \mathcal{I}(\varphi)) = 0$$

for $q \geq 1$ and $m \geq m_0$.

b) *The restriction map*

$$H^0(X, E \otimes F^{\otimes m} \otimes \mathcal{I}(\varphi)) \longrightarrow H^0(X, E \otimes F^{\otimes m} \otimes \mathcal{O}_X/\mathcal{I}(\varphi))$$

is surjective for $m \geq m_0$.

b) *The vector bundle $E \otimes F^{\otimes m}$ generates its sections (or jets of any order s) for $m \geq m_0(s)$ large enough.*

Proof. a) Put an arbitrary smooth hermitian metric h_E en E and consider the singular hermitian metric $h_E \cdot h_F^m \cdot e^{-\varphi}$ on $E \otimes F^{\otimes m}$. The L^2 holomorphic sections of $E \otimes F^{\otimes m}$ are exactly the sections of the sheaf $E \otimes F^{\otimes m} \otimes \mathcal{I}(\varphi)$. On the other hand, the curvature of the metric is

$$\mathrm{i}\,\Theta_{h_E}(E) + (m\,\mathrm{i}\,\Theta_{h_F}(F) + \mathrm{i}\,d'd''\varphi) \otimes \mathrm{Id}_E$$

and therefore the curvature is Nakano > 0 for m large. This implies the vanishing of $H^q(X, E \otimes F^{\otimes m} \otimes \mathcal{I}(\varphi))$.

b) The vanishing of H^1 itself implies the surjevtivity statement on the H^0 groups by the same argument as in Corollary (6.12).

c) Clearly, one can construct a quasi-psh function φ with a single logarithmic pole at a point $x \in X$ by taking

$$\varphi(z) = \theta(z)(n + s - 1)\log \sum |z_j - x_j|^2$$

in some local coordinates near x, where θ is a cut-off function with support in the coordinate open set. Then $\mathcal{I}(\varphi) = \mathfrak{m}_x^s$ and we conclude by b). □

(6.15) Remark. Assume that X is compact and that F is a positive line bundle on X. Let $\{x_1, \ldots, x_N\}$ be a finite set. Show that there are constants $C_1, C_2 \geq 0$

depending only on X and E, F such that $H^0(X, E\otimes F^{\otimes m})$ interpolates given jets of order s_j at x_j for $m \ge C_1\sum s_j + C_2$. To see this, we take a quasi-psh weight

$$\varphi(z) = \sum \theta_j(z)(n+s_j-1)\log|w^{(j)}(z)|$$

with respect to coordinate systems $(w^{(j)}_k(z))_{1\le k\le n}$ centered at x_j. The cut-off functions can be taken of a fixed radius (bounded away from 0) with respect to a finite collection of coordinate patches covering X. It is then easy to see that $\mathrm{i}\,d'd''\varphi \ge -C(\sum s_j+1)\omega$. □

(6.16) Theorem (Kodaira [Kod54]). *Let X be a compact complex manifold. A line bundle L on X is ample if and only if L is positive. In particular, a manifold X posessing a positive line bundle is projective, and can be embedded in projective space via the canonical map $\Phi_{|mL|}: X \to \mathbb{P}^N$ for N large.*

Proof. If the line bundle L is ample, then by definition the canonical map $\Phi_{|mL|}: X\to\mathbb{P}^N$ is an embedding for m large, and $mL = \Phi_{|mL|}^{-1}\mathcal{O}(1)$. This implies that L can be equipped with a metric of positive curvature, as we saw in (1.15). Conversely, if L possesses a metric with positive curvature, then by Theorem 6.14 c), there exists an integer m_0 such that for $m \ge m_0$ sections in $H^0(X, mL)$ separate any pair of points $\{x,y\}\subset X$ and generate 1-jets of sections at every point $x\in X$. However, as is easily seen, separation of points is equivalent to the injectivity of the map $\Phi_{|mL|}$, and the generation of 1-jets is equivalent to the fact that $\Phi_{|mL|}$ is an immersion. □

(6.17) Lemma. *Let X be a compact complex manifold X equipped with a Kähler metric $\omega = i\sum_{1\le j,k\le n}\omega_{jk}(z)dz_j\wedge d\overline{z}_k$ and let $Y\subset X$ be an analytic subset of X. Then there exist globally defined quasi-psh potentials ψ and $(\psi_\varepsilon)_{\varepsilon\in]0,1]}$ on X, satisfying the following properties.*

(i) *The function ψ is smooth on $X\smallsetminus Y$, satisfies $i\partial\overline{\partial}\psi \ge -A\omega$ for some $A>0$, and ψ has logarithmic poles along Y, i.e., locally near Y*

$$\psi(z) \sim \log\sum_k |g_k(z)| + O(1)$$

where (g_k) is a local system of generators of the ideal sheaf $\mathcal{I}_Y$ of Y in X.

(ii) *We have $\psi = \lim_{\varepsilon\to 0}\downarrow\psi_\varepsilon$ where the ψ_ε are C^∞ and possess a uniform Hessian estimate*

$$i\partial\overline{\partial}\psi_\varepsilon \ge -A\omega \qquad \text{on } X.$$

Proof. By compactness of X, there is a covering of X by open coordinate balls B_j, $1\le j\le N$, such that $\mathcal{I}_Y$ is generated by finitely many holomorphic functions $(g_{j,k})_{1\le k\le m_j}$ on a neighborhood of $\overline{B}_j$. We take a partition of unity (θ_j) subordinate to (B_j) such that $\sum\theta_j^2 = 1$ on X, and define

$$\psi(z) = \frac{1}{2}\log\sum_j\theta_j(z)^2\sum_k|g_{j,k}(z)|^2,$$

$$\psi_\varepsilon(z) = \frac{1}{2}\log(e^{2\psi(z)}+\varepsilon^2) = \frac{1}{2}\log\Big(\sum_{j,k}\theta_j(z)^2|g_{j,k}(z)|^2+\varepsilon^2\Big).$$

Moreover, we consider the family of $(1,0)$-forms with support in B_j such that

$$\gamma_{j,k} = \theta_j \partial g_{j,k} + 2 g_{j,k} \partial\theta_j.$$

Straightforward calculations yield

$$\overline\partial\psi_\varepsilon = \frac{1}{2}\frac{\sum_{j,k}\theta_j g_{j,k}\overline{\gamma_{j,k}}}{e^{2\psi}+\varepsilon^2},$$

$$\text{(6.17 a)}\quad i\partial\overline\partial\psi_\varepsilon = \frac{i}{2}\left(\frac{\sum_{j,k}\gamma_{j,k}\wedge\overline{\gamma_{j,k}}}{e^{2\psi}+\varepsilon^2} - \frac{\sum_{j,k}\theta_j\overline{g_{j,k}}\gamma_{j,k}\wedge\sum_{j,k}\theta_j g_{j,k}\overline{\gamma_{j,k}}}{(e^{2\psi}+\varepsilon^2)^2}\right), + i\,\frac{\sum_{j,k}|g_{j,k}|^2(\theta_j\partial\overline\partial\theta_j - \partial\theta_j\wedge\overline\partial\theta_j)}{e^{2\psi}+\varepsilon^2}.$$

As $e^{2\psi} = \sum_{j,k}\theta_j^2|g_{j,k}|^2$, the first big sum in $i\partial\overline\partial\psi_\varepsilon$ is nonnegative by the Cauchy-Schwarz inequality; when viewed as a hermitian form, the value of this sum on a tangent vector $\xi\in T_X$ is simply
(6.17 b)

$$\frac{1}{2}\left(\frac{\sum_{j,k}|\gamma_{j,k}(\xi)|^2}{e^{2\psi}+\varepsilon^2} - \frac{\big|\sum_{j,k}\theta_j\overline{g_{j,k}}\gamma_{j,k}(\xi)\big|^2}{(e^{2\psi}+\varepsilon^2)^2}\right) \ge \frac{1}{2}\frac{\varepsilon^2}{(e^{2\psi}+\varepsilon^2)^2}\sum_{j,k}|\gamma_{j,k}(\xi)|^2.$$

Now, the second sum involving $\theta_j\partial\overline\partial\theta_j - \partial\theta_j\wedge\overline\partial\theta_j$ in (6.17 a) is uniformly bounded below by a fixed negative hermitian form $-A\omega$, $A\gg 0$, and therefore $i\partial\overline\partial\psi_\varepsilon \ge -A\omega$. Actually, for every pair of indices (j,j') we have a bound

$$C^{-1} \le \sum_k |g_{j,k}(z)|^2 / \sum_k |g_{j',k}(z)|^2 \le C \qquad \text{on } \overline B_j \cap \overline B_{j'},$$

since the generators $(g_{j,k})$ can be expressed as holomorphic linear combinations of the $(g_{j',k})$ by Cartan's theorem A (and vice versa). It follows easily that all terms $|g_{j,k}|^2$ are uniformly bounded by $e^{2\psi}+\varepsilon^2$. In particular, ψ and ψ_ε are quasi-psh, and we see that (i) and (ii) hold true. □

(6.18) Theorem (Chow [Chw49])**.** *Let $X\subset\mathbb{P}^N$ be a (closed) complex analytic subset of $\mathbb{P}^N$. Then X is algebraic and can be defined as the common zero set of a finite collection of homogeneous polynomials $P_j(z_0,z_1,\ldots,z_N)=0$, $1\le j\le k$.*

Proof. By Lemma 6.17 (i), there exists a quasi psh function ψ with logarithmic poles along X. Then $\mathcal{I}(2N\psi)\subset\mathcal{I}_X$, since $e^{-2N\psi}$ is certainly not integrable along Z. For $x\in\mathbb{P}^N\smallsetminus X$, we consider a quasi-psh weight

$$\varphi_x(z) = \psi(z) + \theta_x(z)(2N)\log|z-x|$$

with a suitable cut-off function with support on a neighborhood of x possessing holomorphic coordinates, in such a way that $\mathcal{I}(\varphi_x)_x = \mathfrak{m}_{\mathbb{P}^N,x}$. We can arrange that $\mathrm{i}\,\partial\overline\partial\varphi_x \ge -C\omega$ uniformly for all x. Then we have vanishing of $H^1(\mathbb{P}^N,\mathcal{O}(m)\otimes\mathcal{I}(\varphi_x))$ for $m\ge m_0$ and therefore we get a surjective map

$$H^0(\mathbb{P}^N,\mathcal{O}(m)) \to H^0(\mathbb{P}^N,\mathcal{O}(m)\otimes\mathcal{O}_{\mathbb{P}^N}/\mathcal{I}(\varphi_x)).$$

As a consequence, we can find a homogeneous polynomial of degree m which takes value 1 at x and vanishes on X (as prescribed by the ideal sheaf $\mathcal{I}(\psi)$). □

In the case of submanifolds, one can of course prove a slightly more precise result, by demanding that the polynomials P_j have non vanishing differentials along X.

(6.19) Theorem (Chow [Chw49]). *Let $X \subset \mathbb{P}^N$ be a complex analytic submanifold of dimension n. Then X is projective algebraic and can be defined by a collection of homogeneous polynomials $P_j(z_0, z_1, \ldots, z_N) = 0$, $1 \le j \le k$, such that the system of differentials (dP_j) has rank equal to* $\operatorname{codim} X$ *at every point of X.*

Proof. Put $r = \operatorname{codim} X = N - n$. There exists a quasi-psh function φ which has logarithmic poles along X. To see this, just take an open overing of $\mathbb{P}^N$ by open sets U_j where X is defined by $w_1^{(j)} = \ldots = w_r^{(j)} = 0$ and $(w_k^{(j)})_{1 \le k \le N}$ is a suitable coordinate system on U_j. As in the proof of Lemma 6.17, the function

$$\varphi(z) = \log\Big(\sum \theta_j^2(z)\big(|w_1^{(j)}|^2 + \ldots + |w_r^{(j)}|^2\big)\Big)$$

is quasi-psh if the functions θ_j are cut-off functions with support in U_j such that $\sum \theta_j^2 = 1$. An easy calculation also shows that $\mathcal{I}((r+1)\varphi) = \mathcal{I}_X^2$ where $\mathcal{I}_X$ is the reduced ideal sheaf of X in $\mathbb{P}^N$. Hence for $m \ge m_0$ large enough, we have

$$H^q(\mathbb{P}^N, \mathcal{O}(m) \otimes \mathcal{I}_X^2) = 0$$

for $q \ge 1$. The long-exact sequence associated with

$$0 \to \mathcal{I}_X^2 \to \mathcal{I}_X \to \mathcal{I}_X/\mathcal{I}_X^2 \to 0$$

twisted by $\mathcal{O}(m)$ implies the surjectivity of

$$H^0(\mathbb{P}^N, \mathcal{O}(m) \otimes \mathcal{I}_X) \to H^0(X, \mathcal{O}(m) \otimes \mathcal{I}_X/\mathcal{I}_X^2).$$

However, $\mathcal{I}_X/\mathcal{I}_X^2$ can be identified with the conormal bundle N_X^* (where $N_X = T_{\mathbb{P}^N|X}/T_X$), and we infer from this that $H^0(X, \mathcal{O}(m) \otimes \mathcal{I}_X/\mathcal{I}_X^2)$ is generated by its global sections for $m \ge m_1$ large enough by Theorem 6.14 c). This means that we can generate any 1-differential of N_X^* at any point $x \in X$ as the differential of a section

$$P \in H^0(\mathbb{P}^N, \mathcal{O}(m) \otimes \mathcal{I}_X) \subset H^0(\mathbb{P}^N, \mathcal{O}(m)),$$

i.e. a homogeneous polynomial of degree m vanishing on X, for $m \ge \max(m_0, m_1)$. □

(6.20) Corollary (Kodaira [Kod54]). *Let X be a compact complex manifold and* $\dim_{\mathbb{C}} X = n$. *The following conditions are equivalent.*

a) *X is projective algebraic, i.e. X can be embedded as an algebraic submanifold of the complex projective space $\mathbb{P}^N$ for N large.*

b) *X carries a positive line bundle L.*

c) *X carries a Hodge metric, i.e. a Kähler metric ω with rational cohomology class $\{\omega\} \in H^2(X, \mathbb{Q})$.*

Proof. a) ⇔ b). This follows from Theorems 6.16 and 6.19 combined.

b) ⇒ c). Take $\omega = \frac{\mathrm{i}}{2\pi}\Theta(L)$; then $\{\omega\}$ is the image of $c_1(L) \in H^2(X, \mathbb{Z})$.

c) $\Rightarrow$ b). We can multiply $\{\omega\}$ by a common denominator of its coefficients and suppose that $\{\omega\}$ is in the image of $H^2(X,\mathbb{Z})$. Then a classical result due to A. Weil shows that there exists a hermitian line bundle (L,h) such that $\frac{\mathrm{i}}{2\pi}\Theta_h(L)=\omega$. This bundle L is then positive. □

(6.21) Exercise (solution of the Levi problem)**.** Show that the following two properties are equivalent.

a) X is strongly pseudoconvex, i.e. X admits a strongly psh exhaustion function.

b) X is Stein, i.e. the global holomorphic functions $H^0(X,\mathcal{O}_X)$ separate points and yield local coordinates at any point, and X is holomorphically convex (this means that for any discrete sequence z_ν there is a function $f\in H^0(X,\mathcal{O}_X)$ such that $|f(z_\nu)|\to\infty$). □

(6.22) Remark. As long as forms of bidegree (n,q) are considered, the L^2 estimates can be extended to complex spaces with arbitrary singularities. In fact, if X is a complex space and φ is a psh weight function on X, we may still define a sheaf $K_X(\varphi)$ on X, such that the sections on an open set U are the holomorphic n-forms f on the regular part $U\cap X_{\mathrm{reg}}$, satisfying the integrability condition $\mathrm{i}^{n^2} f\wedge\overline{f}\,e^{-2\varphi}\in L^1_{\mathrm{loc}}(U)$. In this setting, the functoriality property 6.8 becomes

$$\mu_\star\big(K_{X'}(\varphi\circ\mu)\big)=K_X(\varphi)$$

for arbitrary complex spaces X, X' such that $\mu: X'\to X$ is a modification. If X is nonsingular we have $K_X(\varphi)=\mathcal{O}(K_X)\otimes\mathcal{I}(\varphi)$, however, if X is singular, the symbols K_X and $\mathcal{I}(\varphi)$ must not be dissociated. The statement of the Nadel vanishing theorem becomes $H^q(X,\mathcal{O}(F)\otimes K_X(\varphi))=0$ for $q\geq 1$, under the same assumptions (X Kähler and weakly pseudoconvex, curvature $\geq\varepsilon\omega$). The proof can be obtained by restricting everything to X_{reg}. Although in general X_{reg} is not weakly pseudoconvex (e.g. in case $\operatorname{codim} X_{\mathrm{sing}}\geq 2$), X_{reg} is always Kähler complete (the complement of a proper analytic subset in a Kähler weakly pseudoconvex space is complete Kähler, see e.g. [Dem82a]). As a consequence, Nadel's vanishing theorem is essentially insensitive to the presence of singularities. □

7. Numerically Effective and Pseudoeffective Line Bundles

Many problems of algebraic geometry (e.g. problems of classification of algebraic surfaces or higher dimensional varieties) lead in a natural way to the study of line bundles satisfying semipositivity conditions. It turns out that semipositivity in the sense of curvature (at least, as far as smooth metrics are considered) is not a very satisfactory notion. More flexible notions perfectly suitable for algebraic purposes are the notion of *numerical effectivity* and *pseudoeffectivity.* The goal of this section is to give the relevant algebraic definitions and to discuss their differential geometric counterparts. We first suppose that X is a projective algebraic manifold, $\dim X=n$.

(7.1) Definition. *A holomorphic line bundle L over a projective manifold X is said to be numerically effective, nef for short, if $L\cdot C=\int_C c_1(L)\geq 0$ for every curve $C\subset X$.*

If L is nef, it can be shown that $L^p \cdot Y = \int_Y c_1(L)^p \geq 0$ for any p-dimensional subvariety $Y \subset X$ (see e.g. [Har70]). In relation to this, let us recall the Nakai-Moishezon ampleness criterion: a line bundle L is ample if and only if $L^p \cdot Y > 0$ for every p-dimensional subvariety Y. From this, we easily infer

(7.2) Proposition. *Let L be a line bundle on a projective algebraic manifold X, on which an ample line bundle A and a hermitian metric ω are given. The following properties are equivalent:*

a) *L is nef*;

b) *for any integer $k \geq 1$, the line bundle $kL + A$ is ample*;

c) *for every $\varepsilon > 0$, there is a smooth metric h_ε on L such that* $\mathrm{i}\,\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$.

Proof. a) ⇒ b). If L is nef and A is ample then clearly $kL + A$ satisfies the Nakai-Moishezon criterion, hence $kL + A$ is ample.

b) ⇒ c). Condition c) is independent of the choice of the hermitian metric, so we may select a metric h_A on A with positive curvature and set $\omega = \mathrm{i}\,\Theta(A)$. If $kL + A$ is ample, this bundle has a metric h_{kL+A} of positive curvature. Then the metric $h_L = (h_{kL+A} \otimes h_A^{-1})^{1/k}$ has curvature

$$\mathrm{i}\,\Theta(L) = \frac{1}{k}\big(\mathrm{i}\,\Theta(kL + A) - \mathrm{i}\,\Theta(A)\big) \geq -\frac{1}{k}\mathrm{i}\,\Theta(A)\,;$$

in this way the negative part can be made smaller than $\varepsilon\,\omega$ by taking k large enough.

c) ⇒ a). Under hypothesis c), we get $L \cdot C = \int_C \frac{\mathrm{i}}{2\pi}\Theta_{h_\varepsilon}(L) \geq -\frac{\varepsilon}{2\pi}\int_C \omega$ for every curve C and every $\varepsilon > 0$, hence $L \cdot C \geq 0$ and L is nef. □

Let now X be an arbitrary compact complex manifold. Since there need not exist any curve in X, Property 7.2 c) is simply taken as a definition of nefness ([DPS94]):

(7.3) Definition. *A line bundle L on a compact complex manifold X is said to be nef if for every $\varepsilon > 0$, there is a smooth hermitian metric h_ε on L such that* $\mathrm{i}\,\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$.

In general, it is not possible to extract a smooth limit h_0 such that $\mathrm{i}\,\Theta_{h_0}(L) \geq 0$. The following simple example is given in [DPS94] (Example 1.7). Let E be a non trivial extension $0 \to \mathcal{O} \to E \to \mathcal{O} \to 0$ over an elliptic curve C and let $X = P(E)$ be the corresponding ruled surface over C. Then $L = \mathcal{O}_{P(E)}(1)$ is nef but does not admit any smooth metric of nonnegative curvature. This example answers negatively a question raised by [Fuj83].

Let us now introduce the important concept of *Kodaira-Iitaka dimension* of a line bundle.

(7.4) Definition. *If L is a line bundle, the Kodaira-Iitaka dimension $\kappa(L)$ is the supremum of the rank of the canonical maps*

$$\Phi_m : X \smallsetminus B_m \longrightarrow P(V_m^\star), \qquad x \longmapsto H_x = \{\sigma \in V_m\,;\,\sigma(x) = 0\}, \qquad m \geq 1$$

with $V_m = H^0(X, mL)$ *and* $B_m = \bigcap_{\sigma \in V_m} \sigma^{-1}(0) =$ *base locus of* V_m*. In case* $V_m = \{0\}$ *for all* $m \geq 1$*, we set* $\kappa(L) = -\infty$*.*
A line bundle is said to be big if $\kappa(L) = \dim X$*.*

The following lemma is well-known (the proof is a rather elementary consequence of the Schwarz lemma).

(7.5) Serre-Siegel lemma ([Ser54], [Sie55])**.** *Let* L *be any line bundle on a compact complex manifold. Then we have*

$$h^0(X, mL) \leq O(m^{\kappa(L)}) \qquad \textit{for } m \geq 1,$$

and $\kappa(L)$ *is the smallest constant for which this estimate holds.* □

We now discuss the various concepts of positive cones in the space of numerical classes of line bundles, and establish a simple dictionary relating these concepts to corresponding concepts in the context of differential geometry.

Let us recall that an integral cohomology class in $H^2(X, \mathbb{Z})$ is the first Chern class of a holomorphic (or algebraic) line bundle if and only if it lies in the *Neron-Severi* group

$$\mathrm{NS}(X) = \mathrm{Ker}\,\big(H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)\big)$$

(this fact is just an elementary consequence of the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\star \to 0$). If X is compact Kähler, as we will suppose from now on in this section, this is the same as saying that the class is of type $(1,1)$ with respect to Hodge decomposition.

Let $\mathrm{NS}_{\mathbb{R}}(X)$ be the real vector space $\mathrm{NS}(X) \otimes \mathbb{R} \subset H^2(X, \mathbb{R})$. We define four convex cones

$$\mathcal{K}_{\mathrm{NS}}^{\mathrm{amp}}(X) \subset \mathcal{E}_{\mathrm{NS}}^{\mathrm{eff}}(X) \subset \mathrm{NS}_{\mathbb{R}}(X),$$
$$\mathcal{K}_{\mathrm{NS}}^{\mathrm{nef}}(X) \subset \mathcal{E}_{\mathrm{NS}}^{\mathrm{psef}}(X) \subset \mathrm{NS}_{\mathbb{R}}(X)$$

which are, respectively, the *convex cones* generated by Chern classes $c_1(L)$ of ample and effective line bundles, resp. the *closure of the convex cones* generated by numerically effective and pseudo-effective line bundles; we say that L is effective if mL has a section for some $m > 0$, i.e. if $\mathcal{O}(mL) \simeq \mathcal{O}(D)$ for some effective divisor D; and we say that L pseudo-effective if $c_1(L)$ is the cohomology class of some closed positive current T, i.e. if L can be equipped with a singular hermitian metric h with $T = \frac{\mathrm{i}}{2\pi}\Theta_h(L) \geq 0$ as a current. For each of the ample, effective, nef and pseudo-effective cones, the first Chern class $c_1(L)$ of a line bundle L lies in the cone if and only if L has the corresponding property (for $\mathcal{E}_{\mathrm{NS}}^{\mathrm{psef}}$ use the fact that the space of positive currents of mass 1 is weakly compact; the case of all other cones is obvious).

(7.6) Proposition. *Let* (X, ω) *be a compact Kähler manifold. The numerical cones satisfy the following properties.*

a) $\mathcal{K}_{\mathrm{NS}}^{\mathrm{amp}} = (\mathcal{K}_{\mathrm{NS}}^{\mathrm{amp}})^\circ \subset (\mathcal{K}_{\mathrm{NS}}^{\mathrm{nef}})^\circ, \quad \mathcal{K}_{\mathrm{NS}}^{\mathrm{nef}} \subset \mathcal{E}_{\mathrm{NS}}^{\mathrm{psef}}$.

b) *If moreover* X *is projective algebraic, we have* $\mathcal{K}_{\mathrm{NS}}^{\mathrm{amp}} = (\mathcal{K}_{\mathrm{NS}}^{\mathrm{nef}})^\circ$ *(therefore* $\overline{\mathcal{K}_{\mathrm{NS}}^{\mathrm{amp}}} = \mathcal{K}_{\mathrm{NS}}^{\mathrm{nef}}$*), and* $\overline{\mathcal{E}_{\mathrm{NS}}^{\mathrm{eff}}} = \mathcal{E}_{\mathrm{NS}}^{\mathrm{psef}}$*.*

If L *is a line bundle on* X *and* h *denotes a hermitian metric on* L*, the following properties are equivalent:*

c) $c_1(L) \in \mathcal{K}_{\mathrm{NS}}^{\mathrm{amp}} \Leftrightarrow \exists\varepsilon > 0$, $\exists h$ *smooth such that* $\mathrm{i}\,\Theta_h(L) \ge \varepsilon\omega$.

d) $c_1(L) \in \mathcal{K}_{\mathrm{NS}}^{\mathrm{nef}} \Leftrightarrow \forall\varepsilon > 0$, $\exists h$ *smooth such that* $\mathrm{i}\,\Theta_h(L) \ge -\varepsilon\omega$.

e) $c_1(L) \in \mathcal{E}_{\mathrm{NS}}^{\mathrm{psef}} \Leftrightarrow \exists h$ *possibly singular such that* $\mathrm{i}\,\Theta_h(L) \ge 0$.

f) *If moreover X is projective algebraic, then*
$c_1(L) \in (\mathcal{E}_{\mathrm{NS}}^{\mathrm{eff}})^\circ \Leftrightarrow \kappa(L) = \dim X$
$\Leftrightarrow \exists\varepsilon > 0$, $\exists h$ *possibly singular such that* $\mathrm{i}\,\Theta_h(L) \ge \varepsilon\omega$.

Proof. c) and d) are already known and e) is a definition.

a) The ample cone $\mathcal{K}_{\mathrm{NS}}^{\mathrm{amp}}$ is always open by definition and contained in $\mathcal{K}_{\mathrm{NS}}^{\mathrm{nef}}$, so the first inclusion is obvious ($\mathcal{K}_{\mathrm{NS}}^{\mathrm{amp}}$ is of course empty if X is not projective algebraic). Let us now prove that $\mathcal{K}_{\mathrm{NS}}^{\mathrm{nef}} \subset \mathcal{E}_{\mathrm{NS}}^{\mathrm{psef}}$. Let L be a line bundle with $c_1(L) \in \mathcal{K}_{\mathrm{NS}}^{\mathrm{nef}}$. Then for every $\varepsilon > 0$, there is a current $T_\varepsilon = \frac{\mathrm{i}}{2\pi}\Theta_{h_\varepsilon}(L) \ge -\varepsilon\omega$. Then $T_\varepsilon + \varepsilon\omega$ is a closed positive current and the family is uniformly bounded in mass for $\varepsilon \in]0,1]$, since

$$\int_X (T_\varepsilon + \varepsilon\omega) \wedge \omega^{n-1} = \int_X c_1(L) \wedge \omega^{n-1} + \varepsilon \int_X \omega^n.$$

By weak compactness, some subsequence converges to a weak limit $T \ge 0$ and $T \in c_1(L)$ (the cohomology class $\{T\}$ of a current is easily shown to depend continuously on T with respect to the weak topology; use e.g. Poincaré duality to check this).

b) If X is projective, the equality $\mathcal{K}_{\mathrm{NS}}^{\mathrm{amp}} = (\mathcal{K}_{\mathrm{NS}}^{\mathrm{nef}})^\circ$ is a simple consequence of 7.2 b) and of the fact that ampleness (or positivity) is an open property. It remains to show that $\mathcal{E}_{\mathrm{NS}}^{\mathrm{psef}} \subset \overline{\mathcal{E}_{\mathrm{NS}}^{\mathrm{eff}}}$. Let L be a line bundle with $c_1(L) \in \mathcal{E}_{\mathrm{NS}}^{\mathrm{psef}}$ and let h_L be a singular hermitian on L such that $T = \frac{\mathrm{i}}{2\pi}\Theta(L) \ge 0$. Fix a point $x_0 \in X$ such that the Lelong number of T at x_0 is zero, and take a sufficiently positive line bundle A (replacing A by a multiple if necessary), such that $A - K_X$ has a singular metric h_{A-K_X} of curvature $\ge \varepsilon\omega$ and such that h_{A-K_X} is smooth on $X \smallsetminus \{x_0\}$ and has an isolated logarithmic pole of Lelong number $\ge n$ at x_0. Then apply Corollary 6.13 to $F = mL + A - K_X$ equipped with the metric $h_L^{\otimes m} \otimes h_{A-K_X}$. Since the weight φ of this metric has a Lelong number $\ge n$ at x_0 and a Lelong number equal to the Lelong number of $T = \frac{\mathrm{i}}{2\pi}\Theta(L)$ at nearby points, $\limsup_{x\to x_0} \nu(T,x) = \nu(T,x_0) = 0$, Corollary 6.13 implies that $H^0(X, K_X + F) = H^0(X, mL + A)$ has a section which does not vanish at x_0. Hence there is an effective divisor D_m such that $\mathcal{O}(mL+A) = \mathcal{O}(D_m)$ and $c_1(L) = \frac{1}{m}\{D_m\} - \frac{1}{m}c_1(A) = \lim \frac{1}{m}\{D_m\}$ is in $\overline{\mathcal{E}}_{\mathrm{NS}}^{\mathrm{eff}}$. □

f) Fix a nonsingular ample divisor A. If $c_1(L) \in (\mathcal{E}_{\mathrm{NS}}^{\mathrm{eff}})^\circ$, there is an integer $m > 0$ such that $c_1(L) - \frac{1}{m}c_1(A)$ is still effective, hence for m,p large we have $mpL - pA = D + F$ with an effective divisor D and a numerically trivial line bundle F. This implies $\mathcal{O}(kmpL) = \mathcal{O}(kpA + kD + kF) \supset \mathcal{O}(kpA + kF)$, hence $h^0(X, kmpL) \ge h^0(X, kpA+kF) \sim (kp)^n A^n/n!$ by the Riemann-Roch formula. Therefore $\kappa(L) = n$.

If $\kappa(L) = n$, then $h^0(X, kL) \ge ck^n$ for $k \ge k_0$ and $c > 0$. The exact cohomology sequence

$$0 \longrightarrow H^0(X, kL - A) \longrightarrow H^0(X, kL) \longrightarrow H^0(A, kL_{\restriction A})$$

where $h^0(A, kL_{\restriction A}) = O(k^{n-1})$ shows that $kL - A$ has non zero sections for k large. If D is the divisor of such a section, then $kL \simeq \mathcal{O}(A + D)$. Select a smooth metric on A such that $\frac{\mathrm{i}}{2\pi}\Theta(A) \ge \varepsilon_0\omega$ for some $\varepsilon_0 > 0$, and take the singular metric on $\mathcal{O}(D)$ with weight function $\varphi_D = \sum \alpha_j \log|g_j|$ described in Example 3.14. Then the metric with weight $\varphi_L = \frac{1}{k}(\varphi_A + \varphi_D)$ on L yields

$$\frac{\mathrm{i}}{2\pi}\Theta(L) = \frac{1}{k}\Big(\frac{\mathrm{i}}{2\pi}\Theta(A) + [D]\Big) \geq (\varepsilon_0/k)\,\omega,$$

as desired.

Finally, the curvature condition $\mathrm{i}\,\Theta_h(L) \geq \varepsilon\omega$ in the sense of currents yields by definition $c_1(L) \in (\mathcal{E}_{\mathrm{NS}}^{\mathrm{psef}})^\circ$. Moreover, b) implies $(\mathcal{E}_{\mathrm{NS}}^{\mathrm{psef}})^\circ = (\mathcal{E}_{\mathrm{NS}}^{\mathrm{eff}})^\circ$. □

Before going further, we need a lemma.

(7.7) Lemma. *Let X be a compact Kähler n-dimensional manifold, let L be a nef line bundle on X, and let E be an arbitrary holomorphic vector bundle. Then $h^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL)) = o(k^n)$ as $k \to +\infty$, for every $q \geq 1$. If X is projective algebraic, the following more precise bound holds:*

$$h^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL)) = O(k^{n-q}), \qquad \forall q \geq 0.$$

Proof. The Kähler case will be proved in Section 12, as a consequence of the holomorphic Morse inequalities. In the projective algebraic case, we proceed by induction on $n = \dim X$. If $n = 1$ the result is clear, as well as if $q = 0$. Now let A be a nonsingular ample divisor such that $E \otimes \mathcal{O}(A - K_X)$ is Nakano positive. Then the Nakano vanishing theorem applied to the vector bundle $F = E \otimes \mathcal{O}(kL + A - K_X)$ shows that $H^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL + A)) = 0$ for all $q \geq 1$. The exact sequence

$$0 \to \mathcal{O}(kL) \to \mathcal{O}(kL + A) \to \mathcal{O}(kL + A)_{\restriction A} \to 0$$

twisted by E implies

$$H^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL)) \simeq H^{q-1}(A, \mathcal{O}(E_{\restriction A} \otimes \mathcal{O}(kL + A)_{\restriction A}),$$

and we easily conclude by induction since $\dim A = n-1$. Observe that the argument does not work any more if X is not algebraic. It seems to be unknown whether the $O(k^{n-q})$ bound still holds in that case. □

(7.8) Corollary. *If L is nef, then L is big (i.e. $\kappa(L) = n$) if and only if $L^n > 0$. Moreover, if L is nef and big, then for every $\delta > 0$, L has a singular metric $h = e^{-2\varphi}$ such that $\max_{x\in X} \nu(\varphi, x) \leq \delta$ and $\mathrm{i}\,\Theta_h(L) \geq \varepsilon\,\omega$ for some $\varepsilon > 0$. The metric h can be chosen to be smooth on the complement of a fixed divisor D, with logarithmic poles along D.*

Proof. By Lemma 7.7 and the Riemann-Roch formula, we have $h^0(X, kL) = \chi(X, kL) + o(k^n) = k^n L^n/n! + o(k^n)$, whence the first statement. If L is big, the proof made in (7.6 f) shows that there is a singular metric h_1 on L such that

$$\frac{\mathrm{i}}{2\pi}\Theta_{h_1}(L) = \frac{1}{k}\Big(\frac{\mathrm{i}}{2\pi}\Theta(A) + [D]\Big)$$

with a positive line bundle A and an effective divisor D. Now, for every $\varepsilon > 0$, there is a smooth metric h_ε on L such that $\frac{\mathrm{i}}{2\pi}\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$, where $\omega = \frac{\mathrm{i}}{2\pi}\Theta(A)$. The convex combination of metrics $h'_\varepsilon = h_1^{k\varepsilon} h_\varepsilon^{1-k\varepsilon}$ is a singular metric with poles along D which satisfies

$$\frac{\mathrm{i}}{2\pi}\Theta_{h'_\varepsilon}(L) \geq \varepsilon(\omega + [D]) - (1 - k\varepsilon)\varepsilon\omega \geq k\varepsilon^2\omega.$$

Its Lelong numbers are $\varepsilon\nu(D,x)$ and they can be made smaller than δ by choosing $\varepsilon > 0$ small. □

We still need a few elementary facts about the numerical dimension of nef line bundles.

(7.9) Definition. *Let L be a nef line bundle on a compact Kähler manifold X. One defines the numerical dimension of L to be*

$$\nu(L) = \max\big\{k = 0, \ldots, n\,;\ c_1(L)^k \neq 0 \ \text{in}\ H^{2k}(X, \mathbb{R})\big\}.$$

By Corollary 7.8, we have $\kappa(L) = n$ if and only if $\nu(L) = n$. In general, we merely have an inequality.

(7.10) Proposition. *If L is a nef line bundle on a compact Kähler manifold, then $\kappa(L) \le \nu(L)$.*

Proof. By induction on $n = \dim X$. If $\nu(L) = n$ or $\kappa(L) = n$ the result is true, so we may assume $r := \kappa(L) \le n-1$ and $k := \nu(L) \le n-1$. Fix $m > 0$ so that $\Phi = \Phi_{|mL|}$ has generic rank r. Select a nonsingular ample divisor A in X such that the restriction of $\Phi_{|mL|}$ to A still has rank r (for this, just take A passing through a point $x \notin B_{|mL|}$ at which $\mathrm{rank}(d\Phi_x) = r < n$, in such a way that the tangent linear map $d\Phi_{x\restriction T_{A,x}}$ still has rank r). Then $\kappa(L_{\restriction A}) \ge r = \kappa(L)$ (we just have an equality because there might exist sections in $H^0(A, mL_{\restriction A})$ which do not extend to X). On the other hand, we claim that $\nu(L_{\restriction A}) = k = \nu(L)$. The inequality $\nu(L_{\restriction A}) \ge \nu(L)$ is clear. Conversely, if we set $\omega = \frac{\mathrm{i}}{2\pi}\Theta(A) > 0$, the cohomology class $c_1(L)^k$ can be represented by a closed positive current of bidegree (k,k)

$$T = \lim_{\varepsilon\to 0}\Big(\frac{\mathrm{i}}{2\pi}\Theta_{h_\varepsilon}(L) + \varepsilon\omega\Big)^k$$

after passing to some subsequence (there is a uniform bound for the mass thanks to the Kähler assumption, taking wedge products with ω^{n-k}). The current T must be non zero since $c_1(L)^k \neq 0$ by definition of $k = \nu(L)$. Then $\{[A]\} = \{\omega\}$ as cohomology classes, and

$$\int_A c_1(L_{\restriction A})^k \wedge \omega^{n-1-k} = \int_X c_1(L)^k \wedge [A] \wedge \omega^{n-1-k} = \int_X T \wedge \omega^{n-k} > 0.$$

This implies $\nu(L_{\restriction A}) \ge k$, as desired. The induction hypothesis with X replaced by A yields

$$\kappa(L) \le \kappa(L_{\restriction A}) \le \nu(L_{\restriction A}) \le \nu(L).$$ □

(7.11) Remark. It may happen that $\kappa(L) < \nu(L)$: take e.g.

$$L \to X = X_1 \times X_2$$

equal to the total tensor product of an ample line bundle L_1 on a projective manifold X_1 and of a unitary flat line bundle L_2 on an elliptic curve X_2 given by a representation $\pi_1(X_2) \to U(1)$ such that no multiple kL_2 with $k \neq 0$ is trivial. Then

$H^0(X, kL) = H^0(X_1, kL_1) \otimes H^0(X_2, kL_2) = 0$ for $k > 0$, and thus $\kappa(L) = -\infty$. However $c_1(L) = \mathrm{pr}_1^\star c_1(L_1)$ has numerical dimension equal to $\dim X_1$. The same example shows that the Kodaira dimension may increase by restriction to a subvariety (if $Y = X_1 \times \{\text{point}\}$, then $\kappa(L_{\restriction Y}) = \dim Y$). □

We now derive an algebraic version of the Nadel vanishing theorem in the context of nef line bundles. This algebraic vanishing theorem has been obtained independently by Kawamata [Kaw82] and Viehweg [Vie82], who both reduced it to the Kodaira-Nakano vanishing theorem by cyclic covering constructions. Since then, a number of other proofs have been given, one based on connections with logarithmic singularities [EV86], another on Hodge theory for twisted coefficient systems [Kol85], a third one on the Bochner technique [Dem89] (see also [EV92] for a general survey, and [Eno93] for an extension to the compact Kähler case). Since the result is best expressed in terms of multiplier ideal sheaves (avoiding then any unnecessary desingularization in the statement), we feel that the direct approach via Nadel's vanishing theorem is probably the most natural one.

If $D = \sum \alpha_j D_j \geq 0$ is an effective $\mathbb{Q}$-divisor, we define the *multiplier ideal sheaf* $\mathcal{I}(D)$ to be equal to $\mathcal{I}(\varphi)$ where $\varphi = \sum \alpha_j |g_j|$ is the corresponding psh function defined by generators g_j of $\mathcal{O}(-D_j)$; as we saw in Remark 6.9, the computation of $\mathcal{I}(D)$ can be made algebraically by using desingularizations $\mu : \widetilde{X} \to X$ such that $\mu^\star D$ becomes a divisor with normal crossings on $\widetilde{X}$.

(7.12) Kawamata-Viehweg vanishing theorem. *Let X be a projective algebraic manifold and let F be a line bundle over X such that some positive multiple mF can be written $mF = L + D$ where L is a nef line bundle and D an effective divisor. Then*

$$H^q\big(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(m^{-1}D)\big) = 0 \quad \textit{for} \quad q > n - \nu(L).$$

(7.13) Special case. *If F is a nef line bundle, then*

$$H^q\big(X, \mathcal{O}(K_X + F)\big) = 0 \quad \textit{for} \quad q > n - \nu(F).$$

Proof of Theorem 7.12. First suppose that $\nu(L) = n$, i.e. that L is big. By the proof of 7.5 f), there is a singular hermitian metric on L such that the corresponding weight $\varphi_{L,0}$ has algebraic singularities and

$$\mathrm{i}\,\Theta_0(L) = 2\mathrm{i}\,d'd''\varphi_L \geq \varepsilon_0\omega$$

for some $\varepsilon_0 > 0$. On the other hand, since L is nef, there are metrics given by weights $\varphi_{L,\varepsilon}$ such that $\frac{\mathrm{i}}{2\pi}\Theta_\varepsilon(L) \geq \varepsilon\omega$ for every $\varepsilon > 0$, ω being a Kähler metric. Let $\varphi_D = \sum \alpha_j \log|g_j|$ be the weight of the singular metric on $\mathcal{O}(D)$ described in Example 3.14. We define a singular metric on F by

$$\varphi_F = \frac{1}{m}\big((1-\delta)\varphi_{L,\varepsilon} + \delta\varphi_{L,0} + \varphi_D\big)$$

with $\varepsilon \ll \delta \ll 1$, δ rational. Then φ_F has algebraic singularities, and by taking δ small enough we find $\mathcal{I}(\varphi_F) = \mathcal{I}(\frac{1}{m}\varphi_D) = \mathcal{I}(\frac{1}{m}D)$. In fact, $\mathcal{I}(\varphi_F)$ can be computed by taking integer parts of $\mathbb{Q}$-divisors (as explained in Remark 6.9), and adding $\delta\varphi_{L,0}$

does not change the integer part of the rational numbers involved when δ is small. Now

$$\begin{aligned} dd^c\varphi_F &= \frac{1}{m}\big((1-\delta)dd^c\varphi_{L,\varepsilon} + \delta dd^c\varphi_{L,0} + dd^c\varphi_D\big) \\ &\geq \frac{1}{m}\big(-(1-\delta)\varepsilon\omega + \delta\varepsilon_0\omega + [D] \geq \frac{\delta\varepsilon}{m}\omega, \end{aligned}$$

if we choose $\varepsilon \leq \delta\varepsilon_0$. Nadel's theorem thus implies the desired vanishing result for all $q \geq 1$.

Now, if $\nu(L) < n$, we use hyperplane sections and argue by induction on $n = \dim X$. Since the sheaf $\mathcal{O}(K_X) \otimes \mathcal{I}(m^{-1}D)$ behaves functorially with respect to modifications (and since the L^2 cohomology complex is "the same" upstairs and downstairs), we may assume after blowing-up that D is a divisor with normal crossings. By Remark 6.9, the multiplier ideal sheaf $\mathcal{I}(m^{-1}D) = \mathcal{O}(-\lfloor m^{-1}D\rfloor)$ is locally free. By Serre duality, the expected vanishing is equivalent to

$$H^q(X, \mathcal{O}(-F) \otimes \mathcal{O}(\lfloor m^{-1}D\rfloor)) = 0 \qquad \text{for } q < \nu(L).$$

Then select a nonsingular ample divisor A such that A meets all components D_j transversally. Select A positive enough so that $\mathcal{O}(A + F - \lfloor m^{-1}D\rfloor)$ is ample. Then $H^q(X, \mathcal{O}(-A-F)\otimes\mathcal{O}(\lfloor m^{-1}D\rfloor)) = 0$ for $q < n$ by Kodaira vanishing, and the exact sequence $0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to (i_A)_\star\mathcal{O}_A \to 0$ twisted by $\mathcal{O}(-F) \otimes \mathcal{O}(\lfloor m^{-1}D\rfloor)$ yields an isomorphism

$$H^q(X, \mathcal{O}(-F) \otimes \mathcal{O}(\lfloor m^{-1}D\rfloor)) \simeq H^q(A, \mathcal{O}(-F_{\restriction A}) \otimes \mathcal{O}(\lfloor m^{-1}D_{\restriction A}\rfloor).$$

The proof of 6.8 showed that $\nu(L_{\restriction A}) = \nu(L)$, hence the induction hypothesis implies that the cohomology group on A on the right hand side is zero for $q < \nu(L)$. □

8. A Simple Algebraic Approach to Fujita's Conjecture

This section is devoted to a proof of various results related to the Fujita conjecture. The main ideas occuring here are inspired by a recent work of Y.T. Siu [Siu96]. His method, which is algebraic in nature and quite elementary, consists in a combination of the Riemann-Roch formula together with Nadel's vanishing theorem (in fact, only the algebraic case is needed, thus the original Kawamata-Viehweg vanishing theorem would be sufficient). Slightly later, Angehrn and Siu [AS95], [Siu95] introduced other closely related methods, producing better bounds for the global generation question; since their method is rather delicate, we can only refer the reader to the above references. In the sequel, X denotes a projective algebraic n-dimensional manifold. The first observation is the following well-known consequence of the Riemann-Roch formula.

(8.1) Special case of Riemann-Roch. *Let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent ideal sheaf on X such that the subscheme $Y = V(\mathcal{J})$ has dimension d (with possibly some lower dimensional components). Let $[Y] = \sum \lambda_j [Y_j]$ be the effective algebraic cycle of dimension d associated to the d dimensional components of Y (taking into account multiplicities λ_j given by the ideal $\mathcal{J}$). Then for any line bundle F, the Euler characteristic*

$$\chi(Y, \mathcal{O}(F + mL)_{\restriction Y}) = \chi(X, \mathcal{O}(F + mL) \otimes \mathcal{O}_X/\mathcal{J})$$

is a polynomial $P(m)$ of degree d and leading coefficient $L^d \cdot [Y]/d!$

The second fact is an elementary lemma about numerical polynomials (polynomials with rational coefficients, mapping $\mathbb{Z}$ into $\mathbb{Z}$).

(8.2) Lemma. *Let $P(m)$ be a numerical polynomial of degree $d > 0$ and leading coefficient $a_d/d!$, $a_d \in \mathbb{Z}$, $a_d > 0$. Suppose that $P(m) \geq 0$ for $m \geq m_0$. Then*

a) *For every integer $N \geq 0$, there exists $m \in [m_0, m_0 + Nd]$ such that $P(m) \geq N$.*

b) *For every $k \in \mathbb{N}$, there exists $m \in [m_0, m_0 + kd]$ such that $P(m) \geq a_d k^d/2^{d-1}$.*

c) *For every integer $N \geq 2d^2$, there exists $m \in [m_0, m_0 + N]$ such that $P(m) \geq N$.*

Proof. a) Each of the N equations $P(m) = 0$, $P(m) = 1$, ..., $P(m) = N - 1$ has at most d roots, so there must be an integer $m \in [m_0, m_0 + dN]$ which is not a root of these.

b) By Newton's formula for iterated differences $\Delta P(m) = P(m+1) - P(m)$, we get

$$\Delta^d P(m) = \sum_{1 \leq j \leq d} (-1)^j \binom{d}{j} P(m + d - j) = a_d, \qquad \forall m \in \mathbb{Z}.$$

Hence if $j \in \{0, 2, 4, \ldots, 2\lfloor d/2 \rfloor\} \subset [0, d]$ is the even integer achieving the maximum of $P(m_0 + d - j)$ over this finite set, we find

$$2^{d-1} P(m_0 + d - j) = \left(\binom{d}{0} + \binom{d}{2} + \ldots \right) P(m_0 + d - j) \geq a_d,$$

whence the existence of an integer $m \in [m_0, m_0 + d]$ with $P(m) \geq a_d/2^{d-1}$. The case $k = 1$ is thus proved. In general, we apply the above case to the polynomial $Q(m) = P(km - (k-1)m_0)$, which has leading coefficient $a_d k^d/d!$

c) If $d = 1$, part a) already yields the result. If $d = 2$, a look at the parabola shows that

$$\max_{m \in [m_0, m_0 + N]} P(m) \geq \begin{cases} a_2 N^2/8 & \text{if } N \text{ is even,} \\ a_2(N^2 - 1)/8 & \text{if } N \text{ is odd;} \end{cases}$$

thus $\max_{m \in [m_0, m_0+N]} P(m) \geq N$ whenever $N \geq 8$. If $d \geq 3$, we apply b) with k equal to the smallest integer such that $k^d/2^{d-1} \geq N$, i.e. $k = \lceil 2(N/2)^{1/d} \rceil$, where $\lceil x \rceil \in \mathbb{Z}$ denotes the round-up of $x \in \mathbb{R}$. Then $kd \leq (2(N/2)^{1/d} + 1)d \leq N$ whenever $N \geq 2d^2$, as a short computation shows. □

We now apply Nadel's vanishing theorem pretty much in the same way as Siu [Siu96], but with substantial simplifications in the technique and improvements in the bounds. Our method yields simultaneously a simple proof of the following basic result.

(8.3) Theorem. *If L is an ample line bundle over a projective n-fold X, then the adjoint line bundle $K_X + (n+1)L$ is nef.*

By using Mori theory and the base point free theorem ([Mor82], [Kaw84]), one can even show that $K_X + (n+1)L$ is semiample, i.e., there exists a positive integer

m such that $m(K_X + (n+1)L)$ is generated by sections (see [Kaw85] and [Fuj87]). The proof rests on the observation that $n+1$ is the maximal length of extremal rays of smooth projective n-folds. Our proof of (8.3) is different and will be given simultaneously with the proof of Th. (8.4) below.

(8.4) Theorem. *Let L be an ample line bundle and let G be a nef line bundle on a projective n-fold X. Then the following properties hold.*

a) *$2K_X + mL + G$ generates simultaneous jets of order $s_1, \ldots, s_p \in \mathbb{N}$ at arbitrary points $x_1, \ldots, x_p \in X$, i.e., there is a surjective map*

$$H^0(X, 2K_X + mL + G) \longrightarrow\!\!\!\!\!\rightarrow \bigoplus_{1\le j\le p} \mathcal{O}(2K_X + mL + G) \otimes \mathcal{O}_{X,x_j}/\mathfrak{m}_{X,x_j}^{s_j+1},$$

provided that $m \ge 2 + \displaystyle\sum_{1\le j\le p} \binom{3n + 2s_j - 1}{n}$.

In particular $2K_X + mL + G$ is very ample for $m \ge 2 + \binom{3n+1}{n}$.

b) *$2K_X + (n+1)L + G$ generates simultaneous jets of order $s_1, \ldots, s_p$ at arbitrary points $x_1, \ldots, x_p \in X$ provided that the intersection numbers $L^d \cdot Y$ of L over all d-dimensional algebraic subsets Y of X satisfy*

$$L^d \cdot Y > \frac{2^{d-1}}{\lfloor n/d \rfloor^d} \sum_{1\le j\le p} \binom{3n + 2s_j - 1}{n}.$$

Proof. The proofs of (8.3) and (8.4 a, b) go along the same lines, so we deal with them simultaneously (in the case of (8.3), we simply agree that $\{x_1, \ldots, x_p\} = \emptyset$). The idea is to find an integer (or rational number) m_0 and a singular hermitian metric h_0 on $K_X + m_0 L$ with strictly positive curvature current $\mathrm{i}\,\Theta_{h_0} \ge \varepsilon\omega$, such that $V(\mathcal{I}(h_0))$ is 0-dimensional and the weight φ_0 of h_0 satisfies $\nu(\varphi_0, x_j) \ge n + s_j$ for all j. As L and G are nef, $(m - m_0)L + G$ has for all $m \ge m_0$ a metric h' whose curvature $\mathrm{i}\,\Theta_{h'}$ has arbitrary small negative part (see [Dem90]), e.g., $\mathrm{i}\,\Theta_{h'} \ge -\frac{\varepsilon}{2}\omega$. Then $\mathrm{i}\,\Theta_{h_0} + \mathrm{i}\,\Theta_{h'} \ge \frac{\varepsilon}{2}\omega$ is again positive definite. An application of Corollary (6.12) to $F = K_X + mL + G = (K_X + m_0 L) + ((m - m_0)L + G)$ equipped with the metric $h_0 \otimes h'$ implies the existence of the desired sections in $K_X + F = 2K_X + mL + G$ for $m \ge m_0$.

Let us fix an embedding $\Phi_{|\mu L|} : X \to \mathbb{P}^N$, $\mu \gg 0$, given by sections $\lambda_0, \ldots, \lambda_N \in H^0(X, \mu L)$, and let h_L be the associated metric on L of positive definite curvature form $\omega = \frac{\mathrm{i}}{2\pi}\Theta(L)$. In order to obtain the desired metric h_0 on $K_X + m_0 L$, we fix $a \in \mathbb{N}^\star$ and use a double induction process to construct singular metrics $(h_{k,\nu})_{\nu\ge 1}$ on $aK_X + b_k L$ for a non increasing sequence of positive integers $b_1 \ge b_2 \ge \ldots \ge b_k \ge \ldots$. Such a sequence much be stationary and m_0 will just be the stationary limit $m_0 = \lim b_k/a$. The metrics $h_{k,\nu}$ are taken to satisfy the following properties:

α) $h_{k,\nu}$ is an algebraic metric of the form

$$\|\xi\|^2_{h_{k,\nu}} = \frac{|\tau_k(\xi)|^2}{\big(\sum_{1\le i\le\nu,\,0\le j\le N} \big|\tau_k^{(a+1)\mu}(\sigma_i^{a\mu} \cdot \lambda_j^{(a+1)b_k - am_i})\big|^2\big)^{1/(a+1)\mu}},$$

defined by sections $\sigma_i \in H^0(X, (a+1)K_X + m_iL)$, $m_i < \frac{a+1}{a}b_k$, $1 \le i \le \nu$, where $\xi \mapsto \tau_k(\xi)$ is an arbitrary local trivialization of $aK_X + b_kL$; note that $\sigma_i^{a\mu} \cdot \lambda_j^{(a+1)b_k - am_i}$ is a section of

$$a\mu((a+1)K_X + m_iL) + ((a+1)b_k - am_i)\mu L = (a+1)\mu(aK_X + b_kL).$$

β) $\mathrm{ord}_{x_j}(\sigma_i) \ge (a+1)(n+s_j)$ for all i, j;

γ) $\mathcal{I}(h_{k,\nu+1}) \supset \mathcal{I}(h_{k,\nu})$ and $\mathcal{I}(h_{k,\nu+1}) \ne \mathcal{I}(h_{k,\nu})$ whenever the zero variety $V(\mathcal{I}(h_{k,\nu}))$ has positive dimension.

The weight $\varphi_{k,\nu} = \frac{1}{2(a+1)\mu} \log \sum \big|\tau_k^{(a+1)\mu}(\sigma_i^{a\mu} \cdot \lambda_j^{(a+1)b_k - am_i})\big|^2$ of $h_{k,\nu}$ is plurisubharmonic and the condition $m_i < \frac{a+1}{a}b_k$ implies $(a+1)b_k - am_i \ge 1$, thus the difference $\varphi_{k,\nu} - \frac{1}{2(a+1)\mu}\log\sum|\tau(\lambda_j)|^2$ is also plurisubharmonic. Hence $\frac{\mathrm{i}}{2\pi}\Theta_{h_{k,\nu}}(aK_X + b_kL) = \frac{\mathrm{i}}{\pi}d'd''\varphi_{k,\nu} \ge \frac{1}{(a+1)}\omega$. Moreover, condition β) clearly implies $\nu(\varphi_{k,\nu}, x_j) \ge a(n+s_j)$. Finally, condition γ) combined with the strong Noetherian property of coherent sheaves ensures that the sequence $(h_{k,\nu})_{\nu\ge 1}$ will finally produce a zero dimensional subscheme $V(\mathcal{I}(h_{k,\nu}))$. We agree that the sequence $(h_{k,\nu})_{\nu\ge 1}$ stops at this point, and we denote by $h_k = h_{k,\nu}$ the final metric, such that $\dim V(\mathcal{I}(h_k)) = 0$.

For $k = 1$, it is clear that the desired metrics $(h_{1,\nu})_{\nu\ge 1}$ exist if b_1 is taken large enough (so large, say, that $(a+1)K_X + (b_1 - 1)L$ generates jets of order $(a+1)(n + \max s_j)$ at every point; then the sections $\sigma_1, \ldots, \sigma_\nu$ can be chosen with $m_1 = \ldots = m_\nu = b_1 - 1$). Suppose that the metrics $(h_{k,\nu})_{\nu\ge 1}$ and h_k have been constructed and let us proceed with the construction of $(h_{k+1,\nu})_{\nu\ge 1}$. We do this again by induction on ν, assuming that $h_{k+1,\nu}$ is already constructed and that $\dim V(\mathcal{I}(h_{k+1,\nu})) > 0$. We start in fact the induction with $\nu = 0$, and agree in this case that $\mathcal{I}(h_{k+1,0}) = 0$ (this would correspond to an infinite metric of weight identically equal to $-\infty$). By Nadel's vanishing theorem applied to

$$F_m = aK_X + mL = (aK_X + b_kL) + (m - b_k)L$$

with the metric $h_k \otimes (h_L)^{\otimes m - b_k}$, we get

$$H^q(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{I}(h_k)) = 0 \qquad \text{for } q \ge 1,\ m \ge b_k.$$

As $V(\mathcal{I}(h_k))$ is 0-dimensional, the sheaf $\mathcal{O}_X/\mathcal{I}(h_k)$ is a skyscraper sheaf, and the exact sequence $0 \to \mathcal{I}(h_k) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}(h_k) \to 0$ twisted with the invertible sheaf $\mathcal{O}((a+1)K_X + mL)$ shows that

$$H^q(X, \mathcal{O}((a+1)K_X + mL)) = 0 \qquad \text{for } q \ge 1,\ m \ge b_k.$$

Similarly, we find

$$H^q(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{I}(h_{k+1,\nu})) = 0 \qquad \text{for } q \ge 1,\ m \ge b_{k+1}$$

(also true for $\nu = 0$, since $\mathcal{I}(h_{k+1,0}) = 0$), and when $m \ge \max(b_k, b_{k+1}) = b_k$, the exact sequence $0 \to \mathcal{I}(h_{k+1,\nu}) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}(h_{k+1}, \nu) \to 0$ implies

$$H^q(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{O}_X/\mathcal{I}(h_{k+1,\nu})) = 0 \qquad \text{for } q \ge 1,\ m \ge b_k.$$

In particular, since the H^1 group vanishes, every section u' of $(a+1)K_X + mL$ on the subscheme $V(\mathcal{I}(h_{k+1,\nu}))$ has an extension u to X. Fix a basis $u'_1, \ldots, u'_N$ of the sections on $V(\mathcal{I}(h_{k+1,\nu}))$ and take arbitrary extensions $u_1, \ldots, u_N$ to X. Look at the linear map assigning the collection of jets of order $(a+1)(n+s_j)-1$ at all points x_j

$$u = \sum_{1\le j\le N} a_j u_j \longmapsto \bigoplus J_{x_j}^{(a+1)(n+s_j)-1}(u).$$

Since the rank of the bundle of s-jets is $\binom{n+s}{n}$, the target space has dimension

$$\delta = \sum_{1\le j\le p} \binom{n+(a+1)(n+s_j)-1}{n}.$$

In order to get a section $\sigma_{\nu+1} = u$ satisfying condition β) with non trivial restriction $\sigma'_{\nu+1}$ to $V(\mathcal{I}(h_{k+1,\nu}))$, we need at least $N = \delta + 1$ independent sections $u'_1, \ldots, u'_N$. This condition is achieved by applying Lemma (8.2) to the numerical polynomial

$$\begin{aligned} P(m) &= \chi(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{O}_X/\mathcal{I}(h_{k+1,\nu})) \\ &= h^0(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{O}_X/\mathcal{I}(h_{k+1,\nu})) \ge 0, \qquad m \ge b_k. \end{aligned}$$

The polynomial P has degree $d = \dim V(\mathcal{I}(h_{k+1,\nu})) > 0$. We get the existence of an integer $m \in [b_k, b_k+\eta]$ such that $N = P(m) \ge \delta + 1$ with some explicit integer $\eta \in \mathbb{N}$ (for instance $\eta = n(\delta+1)$ always works by (8.2 a), but we will also use other possibilities to find an optimal choice in each case). Then we find a section $\sigma_{\nu+1} \in H^0(X, (a+1)K_X + mL)$ with non trivial restriction $\sigma'_{\nu+1}$ to $V(\mathcal{I}(h_{k+1,\nu}))$, vanishing at order $\ge (a+1)(n+s_j)$ at each point x_j. We just set $m_{\nu+1} = m$, and the condition $m_{\nu+1} < \frac{a+1}{a} b_{k+1}$ is satisfied if $b_k + \eta < \frac{a+1}{a} b_{k+1}$. This shows that we can take inductively

$$b_{k+1} = \left\lfloor \frac{a}{a+1}(b_k+\eta) \right\rfloor + 1.$$

By definition, $h_{k+1,\nu+1} \le h_{k+1,\nu}$, hence $\mathcal{I}(h_{k+1,\nu+1}) \supset \mathcal{I}(h_{k+1,\nu})$. We necessarily have $\mathcal{I}(h_{k+1,\nu+1}) \ne \mathcal{I}(h_{k+1,\nu})$, for $\mathcal{I}(h_{k+1,\nu+1})$ contains the ideal sheaf associated with the zero divisor of $\sigma_{\nu+1}$, whilst $\sigma_{\nu+1}$ does not vanish identically on $V(\mathcal{I}(h_{k+1,\nu}))$. Now, an easy computation shows that the iteration of $b_{k+1} = \lfloor \frac{a}{a+1}(b_k+\eta) \rfloor + 1$ stops at $b_k = a(\eta+1)+1$ for any large initial value b_1. In this way, we obtain a metric h_∞ of positive definite curvature on $aK_X + (a(\eta+1)+1)L$, with $\dim V(\mathcal{I}(h_\infty)) = 0$ and $\nu(\varphi_\infty, x_j) \ge a(n+s_j)$ at each point x_j.

Proof of (8.3). In this case, the set $\{x_j\}$ is taken to be empty, thus $\delta = 0$. By (8.2 a), the condition $P(m) \ge 1$ is achieved for some $m \in [b_k, b_k+n]$ and we can take $\eta = n$. As μL is very ample, there exists on μL a metric with an isolated logarithmic pole of Lelong number 1 at any given point x_0 (e.g., the algebraic metric defined with all sections of μL vanishing at x_0). Hence

$$F'_a = aK_X + (a(n+1)+1)L + n\mu L$$

has a metric h'_a such that $V(\mathcal{I}(h'_a))$ is zero dimensional and contains $\{x_0\}$. By Corollary (6.12), we conclude that

$$K_X + F'_a = (a+1)K_X + (a(n+1)+1+n\mu)L$$

is generated by sections, in particular $K_X + \frac{a(n+1)+1+n\mu}{a+1} L$ is nef. As a tends to $+\infty$, we infer that $K_X + (n+1)L$ is nef. □

Proof of (8.4 a). Here, the choice $a = 1$ is sufficient for our purposes. Then

$$\delta = \sum_{1 \le j \le p} \binom{3n + 2s_j - 1}{n}.$$

If $\{x_j\} \neq \emptyset$, we have $\delta + 1 \ge \binom{3n-1}{n} + 1 \ge 2n^2$ for $n \ge 2$. Lemma (8.2 c) shows that $P(m) \ge \delta + 1$ for some $m \in [b_k, b_k + \eta]$ with $\eta = \delta + 1$. We can start in fact the induction procedure $k \mapsto k + 1$ with $b_1 = \eta + 1 = \delta + 2$, because the only property needed for the induction step is the vanishing property

$$H^0(X, 2K_X + mL) = 0 \qquad \text{for } q \ge 1,\ m \ge b_1,$$

which is true by the Kodaira vanishing theorem and the ampleness of $K_X + b_1 L$ (here we use Fujita's result (8.3), observing that $b_1 > n + 1$). Then the recursion formula $b_{k+1} = \lfloor \frac{1}{2}(b_k + \eta) \rfloor + 1$ yields $b_k = \eta + 1 = \delta + 2$ for all k, and (8.4 a) follows. □

Proof of (8.4 b). Quite similar to (8.4 a), except that we take $\eta = n$, $a = 1$ and $b_k = n+1$ for all k. By Lemma (8.2 b), we have $P(m) \ge a_d k^d / 2^{d-1}$ for some integer $m \in [m_0, m_0 + kd]$, where $a_d > 0$ is the coefficient of highest degree in P. By Lemma (8.2) we have $a_d \ge \inf_{\dim Y = d} L^d \cdot Y$. We take $k = \lfloor n/d \rfloor$. The condition $P(m) \ge \delta + 1$ can thus be realized for some $m \in [m_0, m_0 + kd] \subset [m_0, m_0 + n]$ as soon as

$$\inf_{\dim Y = d} L^d \cdot Y \ \lfloor n/d \rfloor^d / 2^{d-1} > \delta,$$

which is equivalent to the condition given in (8.4 b). □

(8.5) Corollary. *Let X be a smooth projective n-fold, let L be an ample line bundle and G a nef line bundle over X. Then $m(K_X + (n+2)L) + G$ is very ample for $m \ge \binom{3n+1}{n} - 2n$.*

Proof. Apply Theorem (8.4 a) with $G' = a(K_X + (n+1)L) + G$, so that

$$2K_X + mL + G' = (a+2)(K_X + (n+2)L) + (m - 2n - 4 - a)L + G,$$

and take $m = a + 2n + 4 \ge 2 + \binom{3n+1}{n}$. □

The main drawback of the above technique is that multiples of L at least equal to $(n+1)L$ are required to avoid zeroes of the Hilbert polynomial. In particular, it is not possible to obtain directly a very ampleness criterion for $2K_X + L$ in the statement of (8.4 b). Nevertheless, using different ideas from Angehrn-Siu [AS95], [Siu96] has obtained such a criterion. We derive here a slightly weaker version, thanks to the following elementary Lemma.

(8.6) Lemma. *Assume that for some integer $\mu \in \mathbb{N}^\star$ the line bundle μF generates simultaneously all jets of order $\mu(n + s_j) + 1$ at any point x_j in a subset $\{x_1, \ldots, x_p\}$ of X. Then $K_X + F$ generates simultaneously all jets of order s_j at x_j.*

Proof. Take the algebraic metric on F defined by a basis of sections $\sigma_1, \ldots, \sigma_N$ of μF which vanish at order $\mu(n + s_j) + 1$ at all points x_j. Since we are still free to choose the homogeneous term of degree $\mu(n + s_j) + 1$ in the Taylor expansion at x_j, we find that $x_1, \ldots, x_p$ are isolated zeroes of $\bigcap \sigma_j^{-1}(0)$. If φ is the weight of the metric of F near x_j, we thus have $\varphi(z) \sim (n + s_j + \frac{1}{\mu}) \log |z - x_j|$ in suitable coordinates. We replace φ in a neighborhood of x_j by

$$\varphi'(z) = \max\big(\varphi(z)\,,\, |z|^2 - C + (n + s_j) \log |z - x_j|\big)$$

and leave φ elsewhere unchanged (this is possible by taking $C > 0$ very large). Then $\varphi'(z) = |z|^2 - C + (n + s_j) \log |z - x_j|$ near x_j, in particular φ' is strictly plurisubharmonic near x_j. In this way, we get a metric h' on F with semipositive curvature everywhere on X, and with positive definite curvature on a neighborhood of $\{x_1, \ldots, x_p\}$. The conclusion then follows directly from Hörmander's L^2 estimates (6.1) and (6.2). □

(8.7) Theorem. *Let X be a smooth projective n-fold, and let L be an ample line bundle over X. Then $2K_X + L$ generates simultaneous jets of order $s_1, \ldots, s_p$ at arbitrary points $x_1, \ldots, x_p \in X$ provided that the intersection numbers $L^d \cdot Y$ of L over all d-dimensional algebraic subsets Y of X satisfy*

$$L^d \cdot Y > \frac{2^{d-1}}{\lfloor n/d \rfloor^d} \sum_{1 \le j \le p} \binom{(n+1)(4n + 2s_j + 1) - 2}{n}, \qquad 1 \le d \le n.$$

Proof. By Lemma (8.6) applied with $F = K_X + L$ and $\mu = n+1$, the desired jet generation of $2K_X + L$ occurs if $(n+1)(K_X + L)$ generates jets of order $(n+1)(n + s_j) + 1$ at x_j. By Lemma (8.6) again with $F = aK_X + (n+1)L$ and $\mu = 1$, we see by backward induction on a that we need the simultaneous generation of jets of order $(n+1)(n + s_j) + 1 + (n + 1 - a)(n + 1)$ at x_j. In particular, for $2K_X + (n+1)L$ we need the generation of jets of order $(n+1)(2n + s_j - 1) + 1$. Theorem (8.4 b) yields the desired condition. □

We now list a few immediate consequences of Theorem 8.4, in connection with some classical questions of algebraic geometry.

(8.8) Corollary. *Let X be a projective n-fold of general type with K_X ample. Then mK_X is very ample for $m \ge m_0 = \binom{3n+1}{n} + 4$.*

(8.9) Corollary. *Let X be a Fano n-fold, that is, a n-fold such that $-K_X$ is ample. Then $-mK_X$ is very ample for $m \ge m_0 = \binom{3n+1}{n}$.*

Proof. Corollaries 8.8, 8.9 follow easily from Theorem 8.4 a) applied to $L = \pm K_X$. Hence we get pluricanonical embeddings $\Phi : X \to \mathbb{P}^N$ such that $\Phi^\star \mathcal{O}(1) = \pm m_0 K_X$. The image $Y = \Phi(X)$ has degree

$$\deg(Y) = \int_Y c_1\big(\mathcal{O}(1)\big)^n = \int_X c_1\big(\pm m_0 K_X\big)^n = m_0^n |K_X^n|.$$

It can be easily reproved from this that there are only finitely many deformation types of Fano n-folds, as well as of n-folds of general type with K_X ample, corresponding to a given discriminant $|K_X^n|$ (from a theoretical viewpoint, this result is a consequence of Matsusaka's big theorem [Mat72] and [KoM83], but the bounds which can be obtained from it are probably extremely huge). In the Fano case, a fundamental result obtained indepedently by Kollár-Miyaoka-Mori [KoMM92] and Campana [Cam92] shows that the discriminant K_X^n is in fact bounded by a constant C_n depending only on n. Therefore, one can find an explicit bound C'_n for the degree of the embedding Φ, and it follows that there are only finitely many families of Fano manifolds in each dimension. □

In the case of surfaces, much more is known. We will content ourselves with a brief account of recent results. If X is a surface, the failure of an adjoint bundle $K_X + L$ to be globally generated or very ample is described in a very precise way by the following result of I. Reider [Rei88].

(8.10) Reider's Theorem. *Let X be a smooth projective surface and let L be a nef line bundle on X.*

a) *Assume that $L^2 \geq 5$ and let $x \in X$ be a given point. Then $K_X + L$ has a section which does not vanish at x, unless there is an effective divisor $D \subset X$ passing through x such that either*

$$\begin{aligned} L \cdot D = 0 \quad &\textit{and} \quad D^2 = -1\,; \qquad \textit{or} \\ L \cdot D = 1 \quad &\textit{and} \quad D^2 = 0. \end{aligned}$$

b) *Assume that $L^2 \geq 10$. Then any two points $x, y \in X$ (possibly infinitely near) are separated by sections of $K_X + L$, unless there is an effective divisor $D \subset X$ passing through x and y such that either*

$$\begin{aligned} L \cdot D = 0 \quad &\textit{and} \quad D^2 = -1 \textit{ or } -2\,; \qquad \textit{or} \\ L \cdot D = 1 \quad &\textit{and} \quad D^2 = 0 \quad \textit{or } -1\,; \qquad \textit{or} \\ L \cdot D = 2 \quad &\textit{and} \quad D^2 = 0. \end{aligned}$$ □

(8.11) Corollary. *Let L be an ample line bundle on a smooth projective surface X. Then $K_X + 3L$ is globally generated and $K_X + 4L$ is very ample. If $L^2 \geq 2$ then $K_X + 2L$ is globally generated and $K_X + 3L$ is very ample.* □

The case of higher order jets can be treated similarly. The most general result in this direction has been obtained by Beltrametti and Sommese [BeS93].

(8.12) Theorem ([BeS93]). *Let X be a smooth projective surface and let L be a nef line bundle on X. Let p be a positive integer such that $L^2 > 4p$. Then for every 0-dimensional subscheme $Z \subset X$ of length $h^0(Z, \mathcal{O}_Z) \leq p$ the restriction*

$$\rho_Z : H^0(X, \mathcal{O}_X(K_X + L)) \longrightarrow H^0(Z, \mathcal{O}_Z(K_X + L))$$

is surjective, unless there is an effective divisor $D \subset X$ intersecting the support $|Z|$ such that

$$L \cdot D - p \leq D^2 < \frac{1}{2} L \cdot D.$$ □

Proof (Sketch). The proof the above theorems rests in an essential way on the construction of rank 2 vector bundles sitting in an exact sequence

$$0 \to \mathcal{O}_X \to E \to L \otimes \mathcal{I}_Z \to 0.$$

Arguing by induction on the length of Z, we may assume that Z is a 0-dimensional subscheme such that ρ_Z is not surjective, but such that $\rho_{Z'}$ is surjective for every proper subscheme $Z' \subset Z$. The existence of E is obtained by a classical construction of Serre (unfortunately, this construction only works in dimension 2). The numerical condition on L^2 in the hypotheses ensures that $c_1(E)^2 - 4\,c_2(E) > 0$, hence E is unstable in the sense of Bogomolov. The existence of the effective divisor D asserted in 8.10 or 8.12 follows. We refer to [Rei88], [BeS93] and [Laz97] for details. The reader will find in [FdB95] a proof of the Bogomolov inequality depending only on the Kawamata-Viehweg vanishing theorem. □

(8.13) Exercise. Prove the Fujita conjecture in the case of dimension 1, according to the following steps.

a) By using Hodge theory, show that for every smooth function f on a compact Kähler manifold (X, ω), the equation $\Delta u = f$ has a solution if and only if $\int_X f\, dV_\omega = 0$.

b) Derive from (a), by using the local solvability of elliptic operators, that one has a similar result when f is a distribution.

c) If $X = C$ is a compact complex curve and L a positive line bundle, for every positive measure μ on X such that $\int_C \mu = \deg(L) = \int_C c_1(L)$, there exists a singular hermitian metric h on L such that $\frac{\mathrm{i}}{2\pi}\Theta_h(L) = \mu$ (with the obvious identification of measures and currents of bidegree $(1,1)$).

d) Given a finite collection of points $x_j \in C$ and integers $s_j > 0$, then $K_C + L$ generates jets of order s_j at all points x_j as soon as $\deg(L) > \sum_j (s_j + 1)$.

e) If L is positive on C, then $K_C + 2L$ is globally generated and $K_C + 3L$ is very ample.

(8.14) Exercise. The goal of the exercise is to prove the following weaker form of Theorems 8.10 and 8.12, by a simple direct method based on Nadel's vanishing theorem:

Let L be a nef line bundle on a smooth projective surface X. Fix points $x_1, \ldots, x_N$ and corresponding multiplicities $s_1, \ldots, s_N$, and set $p = \sum (2 + s_j)^2$. Then $H^0(X, K_X + L)$ generates simultaneously jets of order s_j at all points x_j provided that $L^2 > p$ and $L \cdot C > p$ for all curves C passing through one of the points x_j.

a) Using the Riemann-Roch formula, show that the condition $L^2 > p$ implies the existence of a section of a large multiple mL vanishing at order $> m(2 + s_j)$ at each of the points.

b) Construct a sequence of singular hermitian metrics on L with positive definite curvature, such that the weights φ_ν have algebraic singularities, $\nu(\varphi_\nu, x_j) \geq 2 + s_j$ at each point, and such that for some integer $m_1 > 0$ the multiplier ideal sheaves satisfy $\mathcal{I}(m_1 \varphi_{\nu+1}) \supsetneq \mathcal{I}(m_1 \varphi_\nu)$ if $V(\mathcal{I}(\varphi_\nu))$ is not 0-dimensional near some x_j.

Hint: a) starts the procedure. Fix $m_0 > 0$ such that $m_0 L - K_X$ is ample. Use Nadel's vanishing theorem to show that

$$H^q(X, \mathcal{O}((m + m_0)L) \otimes \mathcal{I}(\lambda m \varphi_\nu)) = 0 \qquad \text{for all } q \geq 1,\ m \geq 0,\ \lambda \in [0,1].$$

Let D_ν be the effective $\mathbb{Q}$-divisor describing the 1-dimensional singularities of φ_ν. Then $\mathcal{I}(\lambda m\varphi_\nu) \subset \mathcal{O}(-\lfloor \lambda m D_\nu \rfloor)$ and the quotient has 0-dimensional support, hence

$$H^q(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{O}(-\lfloor \lambda m D_\nu \rfloor)) = 0 \qquad \text{for all } q \geq 1,\ m \geq 0,\ \lambda \in [0,1].$$

By the Riemann-Roch formule again prove that

$$(\star)\quad h^0(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{O}/\mathcal{O}(-\lfloor \lambda m D_\nu \rfloor)) = \frac{m^2}{2}(2\lambda\, L\cdot D_\nu - \lambda^2 D_\nu^2) + O(m).$$

As the left hand side of $(\star)$ is increasing with λ, one must have $D_\nu^2 \leq L \cdot D_\nu$. If $V(\mathcal{I}(\varphi_\nu))$ is not 0-dimensional at x_j, then the coefficient of some component of D_ν passing through x_j is at least 1, hence

$$2\,L\cdot D_\nu - D_\nu^2 \geq L\cdot D_\nu \geq p+1.$$

Show the existence of an integer $m_1 > 0$ independent of ν such that

$$h^0(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{O}/\mathcal{O}(-\lfloor m D_\nu \rfloor)) > \sum_{1\leq j\leq N} \binom{(m+m_0)(2+s_j)+2}{2}$$

for $m \geq m_1$, and infer the existence of a suitable section of $(m_1+m_0)L$ which is not in $H^0(X, \mathcal{O}((m_1+m_0)L - \lfloor m_1 D_\nu \rfloor))$. Use this section to construct $\varphi_{\nu+1}$ such that $\mathcal{I}(m_1\varphi_{\nu+1}) \supsetneq \mathcal{I}(m_1\varphi_\nu)$.

9. Holomorphic Morse Inequalities

Let X be a compact Kähler manifold, E a holomorphic vector bundle of rank r and L a line bundle over X. If L is equipped with a smooth metric of curvature form $\Theta(L)$, we define the q-index set of L to be the open subset

$$(9.1)\qquad X(q,L) = \left\{ x \in X\ ;\ \mathrm{i}\,\Theta(L)_x \text{ has } \begin{matrix} q & \text{negative eigenvalues} \\ n-q & \text{positive eigenvalues} \end{matrix} \right\}$$

for $0 \leq q \leq n$. Hence X admits a partition $X = \Delta \cup \bigcup_q X(q,L)$ where $\Delta = \{x \in X\,;\, \det(\Theta(L)_x) = 0\}$ is the degeneracy set. We also introduce

$$(9.1')\qquad X(\leq q, L) = \bigcup_{0\leq j\leq q} X(j,L).$$

It is shown in [Dem85b] that the cohomology groups $H^q\big(X, E\otimes\mathcal{O}(kL)\big)$ satisfy the following asymptotic *weak Morse inequalities* as $k \to +\infty$

$$(9.2)\qquad h^q\big(X, E\otimes\mathcal{O}(kL)\big) \leq r\frac{k^n}{n!}\int_{X(q,L)} (-1)^q \Big(\frac{\mathrm{i}}{2\pi}\Theta(L)\Big)^n + o(k^n).$$

A sharper form is given by the *strong Morse inequalities*

$$(9.2')\qquad \sum_{0\leq j\leq q} (-1)^{q-j} h^j\big(X, E\otimes\mathcal{O}(kL)\big) \leq r\frac{k^n}{n!}\int_{X(\leq q,L)} (-1)^q \Big(\frac{\mathrm{i}}{2\pi}\Theta(L)\Big)^n + o(k^n).$$

These inequalities are a useful complement to the Riemann-Roch formula when information is needed about individual cohomology groups, and not just about the Euler-Poincaré characteristic.

One difficulty in the application of these inequalities is that the curvature integral is in general quite uneasy to compute, since it is neither a topological nor an algebraic invariant. However, the Morse inequalities can be reformulated in a more algebraic setting in which only algebraic invariants are involved. We give here two such reformulations.

(9.3) Theorem. *Let $L = F - G$ be a holomorphic line bundle over a compact Kähler manifold X, where F and G are numerically effective line bundles. Then for every $q = 0, 1, \ldots, n = \dim X$, there is an asymptotic strong Morse inequality*

$$\sum_{0\le j\le q} (-1)^{q-j} h^j(X, kL) \le \frac{k^n}{n!} \sum_{0\le j\le q} (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j + o(k^n).$$

Proof. By adding ε times a Kähler metric ω to the curvature forms of F and G, $\varepsilon > 0$ one can write $\frac{\mathrm{i}}{2\pi}\Theta(L) = \theta_\varepsilon(F) - \theta_\varepsilon(G)$ where $\theta_\varepsilon(F) = \frac{\mathrm{i}}{2\pi}\Theta(F) + \varepsilon\omega$ and $\theta_\varepsilon(G) = \frac{\mathrm{i}}{2\pi}\Theta(G) + \varepsilon\omega$ are positive definite. Let $\lambda_1 \ge \ldots \ge \lambda_n > 0$ be the eigenvalues of $\theta_\varepsilon(G)$ with respect to $\theta_\varepsilon(F)$. Then the eigenvalues of $\frac{\mathrm{i}}{2\pi}\Theta(L)$ with respect to $\theta_\varepsilon(F)$ are the real numbers $1 - \lambda_j$ and the set $X(\le q, L)$ is the set $\{\lambda_{q+1} < 1\}$ of points $x \in X$ such that $\lambda_{q+1}(x) < 1$. The strong Morse inequalities yield

$$\sum_{0\le j\le q} (-1)^{q-j} h^j(X, kL) \le \frac{k^n}{n!} \int_{\{\lambda_{q+1}<1\}} (-1)^q \prod_{1\le j\le n} (1 - \lambda_j)\theta_\varepsilon(F)^n + o(k^n).$$

On the other hand we have

$$\binom{n}{j} \theta_\varepsilon(F)^{n-j} \wedge \theta_\varepsilon(G)^j = \sigma_n^j(\lambda)\, \theta_\varepsilon(F)^n,$$

where $\sigma_n^j(\lambda)$ is the j-th elementary symmetric function in $\lambda_1, \ldots, \lambda_n$, hence

$$\sum_{0\le j\le q} (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j = \lim_{\varepsilon\to 0} \int_X \sum_{0\le j\le q} (-1)^{q-j} \sigma_n^j(\lambda)\, \theta_\varepsilon(F)^n.$$

Thus, to prove the Lemma, we only have to check that

$$\sum_{0\le j\le n} (-1)^{q-j} \sigma_n^j(\lambda) - \mathbf{1}_{\{\lambda_{q+1}<1\}} (-1)^q \prod_{1\le j\le n} (1 - \lambda_j) \ge 0$$

for all $\lambda_1 \ge \ldots \ge \lambda_n \ge 0$, where $\mathbf{1}_{\{\ldots\}}$ denotes the characteristic function of a set. This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda) = \sigma_{n-1}^j(\lambda) + \sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). □

In the case $q = 1$, we get an especially interesting lower bound (this bound has been observed and used by S. Trapani [Tra95] in a similar context).

(9.4) Consequence. $h^0(X, kL) - h^1(X, kL) \ge \frac{k^n}{n!}(F^n - nF^{n-1} \cdot G) - o(k^n)$. *Therefore some multiple kL has a section as soon as $F^n - nF^{n-1} \cdot G > 0$.*

(9.5) Remark. The weaker inequality

$$h^0(X, kL) \geq \frac{k^n}{n!}(F^n - nF^{n-1} \cdot G) - o(k^n)$$

is easy to prove if X is projective algebraic. Indeed, by adding a small ample $\mathbb{Q}$-divisor to F and G, we may assume that F, G are ample. Let $m_0 G$ be very ample and let k' be the smallest integer $\geq k/m_0$. Then $h^0(X, kL) \geq h^0(X, kF - k'm_0 G)$. We select k' smooth members G_j, $1 \leq j \leq k'$ in the linear system $|m_0 G|$ and use the exact sequence

$$0 \to H^0(X, kF - \sum G_j) \to H^0(X, kF) \to \bigoplus H^0(G_j, kF_{|G_j}).$$

Kodaira's vanishing theorem yields $H^q(X, kF) = 0$ and $H^q(G_j, kF_{|G_j}) = 0$ for $q \geq 1$ and $k \geq k_0$. By the exact sequence combined with Riemann-Roch, we get

$$\begin{aligned}
h^0(X, kL) &\geq h^0(X, kF - \sum G_j) \\
&\geq \frac{k^n}{n!}F^n - O(k^{n-1}) - \sum \Big(\frac{k^{n-1}}{(n-1)!}F^{n-1} \cdot G_j - O(k^{n-2})\Big) \\
&\geq \frac{k^n}{n!}\Big(F^n - n\frac{k'm_0}{k}F^{n-1} \cdot G\Big) - O(k^{n-1}) \\
&\geq \frac{k^n}{n!}\Big(F^n - n\,F^{n-1} \cdot G\Big) - O(k^{n-1}).
\end{aligned}$$

(This simple proof is due to F. Catanese.) □

(9.6) Corollary. *Suppose that F and G are nef and that F is big. Some multiple of $mF - G$ has a section as soon as*

$$m > n\,\frac{F^{n-1} \cdot G}{F^n}.$$

In the last condition, the factor n is sharp: this is easily seen by taking $X = \mathbb{P}_1^n$ and $F = \mathcal{O}(a, \ldots, a)$ and $G = \mathcal{O}(b_1, \ldots, b_n)$ over $\mathbb{P}_1^n$; the condition of the Corollary is then $m > \sum b_j/a$, whereas $k(mF - G)$ has a section if and only if $m \geq \sup b_j/a$; this shows that we cannot replace n by $n(1 - \varepsilon)$. □

10. Effective Version of Matsusaka's Big Theorem

An important problem of algebraic geometry is to find effective bounds m_0 such that multiples mL of an ample line bundle become very ample for $m \geq m_0$. From a theoretical point of view, this problem has been solved by Matsusaka [Mat72] and Kollár-Matsusaka [KoM83]. Their result is that there is a bound $m_0 = m_0(n, L^n, L^{n-1} \cdot K_X)$ depending only on the dimension and on the first two coefficients L^n and $L^{n-1} \cdot K_X$ in the Hilbert polynomial of L. Unfortunately, the original proof does not tell much on the actual dependence of m_0 in terms of these coefficients. The goal of this section is to find effective bounds for such an integer m_0, along the lines of [Siu93]. However, one of the technical lemmas used in [Siu93] to deal with dualizing sheaves can be sharpened. Using this sharpening of the lemma, Siu's bound will be here substantially improved. We first start with the simpler problem of obtaining merely a nontrivial section in mL. The idea, more generally, is to obtain a criterion for the ampleness of $mL-B$ when B is nef. In this way, one is able to subtract from mL any multiple of K_X which happens to get added by the application of Nadel's vanishing theorem (for this, replace B by B plus a multiple of $K_X+(n+1)L$).

(10.1) Proposition. *Let L be an ample line bundle over a projective n-fold X and let B be a nef line bundle over X. Then K_X+mL-B has a nonzero section for some integer m such that*

$$m \le n\,\frac{L^{n-1}\cdot B}{L^n}+n+1.$$

Proof. Let m_0 be the smallest integer $> n\,\frac{L^{n-1}\cdot B}{L^n}$. Then m_0L-B can be equipped with a singular hermitian metric of positive definite curvature. Let φ be the weight of this metric. By Nadel's vanishing theorem, we have

$$H^q(X,\mathcal{O}(K_X+mL-B)\otimes\mathcal{I}(\varphi))=0 \qquad \text{for } q\ge 1,$$

thus $P(m)=h^0(X,\mathcal{O}(K_X+mL-B)\otimes\mathcal{I}(\varphi))$ is a polynomial for $m\ge m_0$. Since P is a polynomial of degree n and is not identically zero, there must be an integer $m\in[m_0,m_0+n]$ which is not a root. Hence there is a nontrivial section in

$$H^0(X,\mathcal{O}(K_X+mL-B))\supset H^0(X,\mathcal{O}(K_X+mL-B)\otimes\mathcal{I}(\varphi))$$

for some $m\in[m_0,m_0+n]$, as desired. □

(10.2) Corollary. *If L is ample and B is nef, then $mL-B$ has a nonzero section for some integer*

$$m\le n\Big(\frac{L^{n-1}\cdot B+L^{n-1}\cdot K_X}{L^n}+n+1\Big).$$

Proof. By Fujita's result 9.3 a), $K_X+(n+1)L$ is nef. We can thus replace B by $B+K_X+(n+1)L$ in the result of Proposition 10.1. Corollary 10.2 follows. □

(10.3) Remark. We do not know if the above Corollary is sharp, but it is certainly not far from being so. Indeed, for $B=0$, the initial constant n cannot be replaced by anything smaller than $n/2$: take X to be a product of curves C_j of large genus g_j and $B=0$; our bound for $L=\mathcal{O}(a_1[p_1])\otimes\ldots\otimes\mathcal{O}(a_n[p_n])$ to have $|mL|\ne\emptyset$ becomes $m\le\sum(2g_j-2)/a_j+n(n+1)$, which fails to be sharp only by a factor 2 when $a_1=\ldots=a_n=1$ and $g_1\gg g_2\gg\ldots\gg g_n\to+\infty$. On the other hand, the additive constant $n+1$ is already best possible when $B=0$ and $X=\mathbb{P}^n$. □

So far, the method is not really sensitive to singularities (the Morse inequalities are indeed still true in the singular case as is easily seen by using desingularizations of the ambient variety). The same is true with Nadel's vanishing theorem, provided that K_X is replaced by the L^2 dualizing sheaf ω_X (according to the notation introduced in Remark 6.22, $\omega_X=K_X(0)$ is the sheaf of holomorphic n-forms u on $X_{\rm reg}$ such that $\mathrm{i}^{n^2}u\wedge\overline{u}$ is integrable in a neighborhood of the singular set). Then Proposition 10.1 can be generalized as

(10.4) Proposition. *Let L be an ample line bundle over a projective n-fold X and let B be a nef line bundle over X. For every p-dimensional (reduced) algebraic subvariety Y of X, there is an integer*

$$m \le p\frac{L^{p-1}\cdot B\cdot Y}{L^p\cdot Y}+p+1$$

such that the sheaf $\omega_Y\otimes\mathcal{O}_Y(mL-B)$ has a nonzero section. □

To proceed further, we need the following useful "upper estimate" about L^2 dualizing sheaves (this is one of the crucial steps in Siu's approach; unfortunately, it has the effect of producing rather large final bounds when the dimension increases).

(10.5) Proposition. *Let H be a very ample line bundle on a projective algebraic manifold X, and let $Y\subset X$ be a p-dimensional irreducible algebraic subvariety. If $\delta=H^p\cdot Y$ is the degree of Y with respect to H, the sheaf*

$$\mathcal{H}om\big(\omega_Y,\mathcal{O}_Y((\delta-p-2)H)\big)$$

has a nontrivial section.

Observe that if Y is a smooth hypersurface of degree δ in $(X,H)=(\mathbb{P}^{p+1},\mathcal{O}(1))$, then $\omega_Y=\mathcal{O}_Y(\delta-p-2)$ and the estimate is optimal. On the other hand, if Y is a smooth complete intersection of multidegree $(\delta_1,\ldots,\delta_r)$ in $\mathbb{P}^{p+r}$, then $\delta=\delta_1\ldots\delta_r$ whilst $\omega_Y=\mathcal{O}_Y(\delta_1+\ldots+\delta_r-p-r-1)$; in this case, Proposition (10.5) is thus very far from being sharp.

Proof. Let $X\subset\mathbb{P}^N$ be the embedding given by H, so that $H=\mathcal{O}_X(1)$. There is a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{p+1}$ whose restriction $\pi: Y\to\mathbb{P}^{p+1}$ to Y is a finite and regular birational map of Y onto an algebraic hypersurface Y' of degree δ in $\mathbb{P}^{p+1}$. Let $s\in H^0(\mathbb{P}^{p+1},\mathcal{O}(\delta))$ be the polynomial of degree δ defining Y'. We claim that for any small Stein open set $W\subset\mathbb{P}^{p+1}$ and any L^2 holomorphic p-form u on $Y'\cap W$, there is a L^2 holomorphic $(p+1)$-form $\widetilde{u}$ on W with values in $\mathcal{O}(\delta)$ such that $\widetilde{u}_{\restriction Y'\cap W}=u\wedge ds$. In fact, this is precisely the conclusion of the Ohsawa-Takegoshi extension theorem [OT87], [Ohs88] (see also [Man93] for a more general version); one can also invoke more standard local algebra arguments (see Hartshorne [Har77], Th. III-7.11). As $K_{\mathbb{P}^{p+1}}=\mathcal{O}(-p-2)$, the form $\widetilde{u}$ can be seen as a section of $\mathcal{O}(\delta-p-2)$ on W, thus the sheaf morphism $u\mapsto u\wedge ds$ extends into a global section of $\mathcal{H}om\big(\omega_{Y'},\mathcal{O}_{Y'}(\delta-p-2)\big)$. The pull-back by $\pi^\star$ yields a section of $\mathcal{H}om\big(\pi^\star\omega_{Y'},\mathcal{O}_Y((\delta-p-2)H)\big)$. Since π is finite and generically $1:1$, it is easy to see that $\pi^\star\omega_{Y'}=\omega_Y$. The Proposition follows. □

By an appropriate induction process based on the above results, we can now improve Siu's effective version of the Big Matsusaka Theorem [Siu93]. Our version depends on a constant λ_n such that $m(K_X+(n+2)L)+G$ is very ample for $m\ge\lambda_n$ and every nef line bundle G. Corollary (8.5) shows that $\lambda_n\le\binom{3n+1}{n}-2n$, and a similar argument involving the recent results of Angehrn-Siu [AS95] implies $\lambda_n\le n^3-n^2-n-1$ for $n\ge 2$. Of course, it is expected that $\lambda_n=1$ in view of the Fujita conjecture.

(10.6) Effective version of the Big Matsusaka Theorem. *Let L and B be nef line bundles on a projective n-fold X. Assume that L is ample and let H be the very ample line bundle $H = \lambda_n(K_X + (n+2)L)$. Then $mL - B$ is very ample for*

$$m \geq (2n)^{(3^{n-1}-1)/2} \frac{(L^{n-1}\cdot(B+H))^{(3^{n-1}+1)/2}(L^{n-1}\cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(L^n)^{3^{n-2}(n/2-1/4)+1/4}}.$$

In particular mL is very ample for

$$m \geq C_n\ (L^n)^{3^{n-2}} \left(n+2+\frac{L^{n-1}\cdot K_X}{L^n}\right)^{3^{n-2}(n/2+3/4)+1/4}$$

with $C_n = (2n)^{(3^{n-1}-1)/2}(\lambda_n)^{3^{n-2}(n/2+3/4)+1/4}$.

Proof. We use Proposition (10.4) and Proposition (10.5) to construct inductively a sequence of (non necessarily irreducible) algebraic subvarieties $X = Y_n \supset Y_{n-1} \supset \ldots \supset Y_2 \supset Y_1$ such that $Y_p = \bigcup_j Y_{p,j}$ is p-dimensional, and Y_{p-1} is obtained for each $p \geq 2$ as the union of zero sets of sections

$$\sigma_{p,j} \in H^0(Y_{p,j}, \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B))$$

with suitable integers $m_{p,j} \geq 1$. We proceed by induction on decreasing values of the dimension p, and find inductively upper bounds m_p for the integers $m_{p,j}$.

By Corollary (10.2), an integer m_n for $m_nL - B$ to have a section σ_n can be found with

$$m_n \leq n\,\frac{L^{n-1}\cdot(B+K_X+(n+1)L)}{L^n} \leq n\,\frac{L^{n-1}\cdot(B+H)}{L^n}.$$

Now suppose that the sections σ_n, ..., $\sigma_{p+1,j}$ have been constructed. Then we get inductively a p-cycle $\widetilde{Y}_p = \sum \mu_{p,j}Y_{p,j}$ defined by $\widetilde{Y}_p =$ sum of zero divisors of sections $\sigma_{p+1,j}$ in $\widetilde{Y}_{p+1,j}$, where the mutiplicity $\mu_{p,j}$ on $Y_{p,j} \subset Y_{p+1,k}$ is obtained by multiplying the corresponding multiplicity $\mu_{p+1,k}$ with the vanishing order of $\sigma_{p+1,k}$ along $Y_{p,j}$. As cohomology classes, we find

$$\widetilde{Y}_p \equiv \sum (m_{p+1,k}L - B)\cdot(\mu_{p+1,k}Y_{p+1,k}) \leq m_{p+1}L\cdot\widetilde{Y}_{p+1}.$$

Inductively, we thus have the numerical inequality

$$\widetilde{Y}_p \leq m_{p+1}\ldots m_nL^{n-p}.$$

Now, for each component $Y_{p,j}$, Proposition (10.4) shows that there exists a section of $\omega_{Y_{p,j}} \otimes \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ for some integer

$$m_{p,j} \leq p\frac{L^{p-1}\cdot B\cdot Y_{p,j}}{L^p\cdot Y_{p,j}} + p + 1 \leq pm_{p+1}\ldots m_n\,L^{n-1}\cdot B + p + 1.$$

Here, we have used the obvious lower bound $L^{p-1}\cdot Y_{p,j} \geq 1$ (this is of course a rather weak point in the argument). The degree of $Y_{p,j}$ with respect to H admits the upper bound

$$\delta_{p,j} := H^p\cdot Y_{p,j} \leq m_{p+1}\ldots m_nH^p\cdot L^{n-p}.$$

We use the Hovanski-Teissier concavity inequality ([Hov79], [Tei79, Tei82])

$$(L^{n-p} \cdot H^p)^{\frac{1}{p}} (L^n)^{1-\frac{1}{p}} \leq L^{n-1} \cdot H$$

to express our bounds in terms of the intersection numbers L^n and $L^{n-1} \cdot H$ only. We then get

$$\delta_{p,j} \leq m_{p+1} \ldots m_n \frac{(L^{n-1} \cdot H)^p}{(L^n)^{p-1}}.$$

By Proposition (10.5), there is a nontrivial section in

$$\mathcal{H}om\big(\omega_{Y_{p,j}}, \mathcal{O}_{Y_{p,j}}((\delta_{p,j} - p - 2)H)\big).$$

Combining this section with the section in $\omega_{Y_{p,j}} \otimes \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ already constructed, we get a section of $\mathcal{O}_{Y_{p,j}}(m_{p,j}L - B + (\delta_{p,j} - p - 2)H)$ on $Y_{p,j}$. Since we do not want H to appear at this point, we replace B with $B + (\delta_{p,j} - p - 2)H$ and thus get a section $\sigma_{p,j}$ of $\mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ with some integer $m_{p,j}$ such that

$$\begin{aligned} m_{p,j} &\leq p m_{p+1} \ldots m_n \, L^{n-1} \cdot (B + (\delta_{p,j} - p - 2)H) + p + 1 \\ &\leq p \, m_{p+1} \ldots m_n \, \delta_{p,j} \, L^{n-1} \cdot (B + H) \\ &\leq p \, (m_{p+1} \ldots m_n)^2 \frac{(L^{n-1} \cdot H)^p}{(L^n)^{p-1}} \, L^{n-1} \cdot (B + H). \end{aligned}$$

Therefore, by putting $M = n \, L^{n-1} \cdot (B + H)$, we get the recursion relation

$$m_p \leq M \, \frac{(L^{n-1} \cdot H)^p}{(L^n)^{p-1}} \, (m_{p+1} \ldots m_n)^2 \qquad \text{for } 2 \leq p \leq n-1,$$

with initial value $m_n \leq M/L^n$. If we let $(\overline{m}_p)$ be the sequence obtained by the same recursion formula with equalities instead of inequalities, we get $m_p \leq \overline{m}_p$ with $\overline{m}_{n-1} = M^3 (L^{n-1} \cdot H)^{n-1}/(L^n)^n$ and

$$\overline{m}_p = \frac{L^n}{L^{n-1} \cdot H} \, \overline{m}_{p+1}^2 \overline{m}_{p+1}$$

for $2 \leq p \leq n-2$. We then find inductively

$$m_p \leq \overline{m}_p = M^{3^{n-p}} \frac{(L^{n-1} \cdot H)^{3^{n-p-1}(n-3/2)+1/2}}{(L^n)^{3^{n-p-1}(n-1/2)+1/2}}.$$

We next show that $m_0 L - B$ is nef for

$$m_0 = \max\big(m_2 \,, \, m_3, \ldots, m_n \,, \, m_2 \ldots m_n \, L^{n-1} \cdot B\big).$$

In fact, let $C \subset X$ be an arbitrary irreducible curve. Either $C = Y_{1,j}$ for some j or there exists an integer $p = 2, \ldots, n$ such that C is contained in Y_p but not in Y_{p-1}. If $C \subset Y_{p,j} \smallsetminus Y_{p-1}$, then $\sigma_{p,j}$ does not vanish identically on C. Hence $(m_{p,j}L - B)_{\restriction C}$ has nonnegative degree and

$$(m_0 L - B) \cdot C \geq (m_{p,j} L - B) \cdot C \geq 0.$$

On the other hand, if $C = Y_{1,j}$, then

$$(m_0 L - B) \cdot C \geq m_0 - B \cdot \widetilde{Y}_1 \geq m_0 - m_2 \ldots m_n \, L^{n-1} \cdot B \geq 0.$$

By the definition of λ_n (and by Corollary (8.5) showing that such a constant exists), $H + G$ is very ample for every nef line bundle G, in particular $H + m_0 L - B$ is very ample. We thus replace again B with $B+H$. This has the effect of replacing M with $M = n\big(L^{n-1}\cdot(B+2H)\big)$ and m_0 with

$$m_0 = \max\big(m_n\,,\,m_{n-1},\ldots,m_2\,,\,m_2\ldots m_n\,L^{n-1}\cdot(B+H)\big).$$

The last term is the largest one, and from the estimate on $\overline{m}_p$, we get

$$\begin{aligned} m_0 &\le M^{(3^{n-1}-1)/2}\,\frac{(L^{n-1}\cdot H)^{(3^{n-2}-1)(n-3/2)/2+(n-2)/2}(L^{n-1}\cdot(B+H))}{(L^n)^{(3^{n-2}-1)(n-1/2)/2+(n-2)/2+1}} \\ &\le (2n)^{(3^{n-1}-1)/2}\,\frac{(L^{n-1}\cdot(B+H))^{(3^{n-1}+1)/2}(L^{n-1}\cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(L^n)^{3^{n-2}(n/2-1/4)+1/4}} \end{aligned}$$

□

(10.7) Remark. In the surface case $n=2$, one can take $\lambda_n=1$ and our bound yields mL very ample for

$$m \ge 4\,\frac{(L\cdot(K_X+4L))^2}{L^2}.$$

If one looks more carefully at the proof, the initial constant 4 can be replaced by 2. In fact, it has been shown recently by Fernández del Busto that mL is very ample for

$$m > \frac{1}{2}\left[\frac{(L\cdot(K_X+4L)+1)^2}{L^2}+3\right],$$

and an example of G. Xiao shows that this bound is essentially optimal (see [FdB96]).

11. Positivity concepts for vector bundles

In the course of the proof of Skoda's L^2 estimates, we will have to deal with dual bundles and exact sequences of hermitian vector bundles. The following fundamental differential geometric lemma will be needed.

(11.1) Lemma. *Let E be a hermitian holomorphic vector bundle of rank r on a complex n-dimensional manifold X. Then the Chern connections of E and $E^\star$ are related by $\Theta(E^\star) = -{}^t\Theta(E)$ where ^{t} denotes transposition. In other words, the associated hermitian forms $\widetilde\Theta(E)$ and $\widetilde\Theta(E^\star)$ are related by*

$$\widetilde\Theta(E)(\tau,\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu}, \qquad \tau=\sum_{j,\lambda}\tau_{j,\lambda}\frac{\partial}{\partial z_j}\otimes e_\lambda,$$

$$\widetilde\Theta(E^\star)(\tau,\tau) = -\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda}\tau^\star_{j\lambda}\overline{\tau}^\star_{k\mu}, \qquad \tau^\star=\sum_{j,\lambda}\tau^\star_{j,\lambda}\frac{\partial}{\partial z_j}\otimes e^\star_\lambda.$$

In particular $E >_{\rm Grif} 0$ if and only if $E^\star <_{\rm Grif} 0$.

Notice that the corresponding duality statement for Nakano positivity is wrong (because of the twist of indices, which is fortunately irrelevant in the case of decomposable tensors).

Proof. The Chern connections of E and $E^\star$ are related by the Leibnitz rule

$$d(\sigma\wedge s) = (D_{E^\star}\sigma)\wedge s + (-1)^{\deg\sigma}\sigma\wedge D_E s$$

whenever s, σ are forms with values in E, $E^\star$ respectively, and $\sigma\wedge s$ is computed using the pairing $E^\star\otimes E\to\mathbb{C}$. If we differentiate a second time, this yields the identity

$$0 = (D^2_{E^\star}\sigma)\wedge s + \sigma\wedge D^2_E s,$$

which is equivalent to the formula $\Theta(E^\star) = -{}^t\Theta(E)$. All other assertions follow. □

(11.2) Lemma. *Let*

$$0\longrightarrow S\xrightarrow{\;j\;}E\xrightarrow{\;g\;}Q\longrightarrow 0$$

be an exact sequence of holomorphic vector bundles. Assume that E is equipped with a smooth hermitian metric, and that S and Q are endowed with the metrics (restriction-metric and quotient-metric) induced by that of E. Then

$$j^\star\oplus g : E\to S\oplus Q,\qquad j\oplus g^\star : S\oplus Q\to E \tag{11.3}$$

are C^∞ isomorphisms of bundles, which are inverse of each other. In the C^∞-splitting $E\simeq S\oplus Q$, the Chern connection of E admits a matrix decomposition

$$D_E = \begin{pmatrix} D_S & -\beta^\star \\ \beta & D_Q \end{pmatrix} \tag{11.4}$$

in terms of the Chern connections of S and Q, where

$$\beta\in C^\infty\big(X,\Lambda^{1,0}T^\star_X\otimes\mathrm{Hom}(S,Q)\big),\qquad \beta^\star\in C^\infty\big(X,\Lambda^{0,1}T^\star_X\otimes\mathrm{Hom}(Q,S)\big).$$

The form β is called the second fundamental form associated with the exact sequence. It is uniquely defined by each of the two formulas

$$D'_{\mathrm{Hom}(S,E)}j = g^\star\circ\beta,\qquad j\circ\beta^\star = -D''_{\mathrm{Hom}(Q,E)}g^\star. \tag{11.5}$$

We have $D'_{\mathrm{Hom}(S,Q)}\beta = 0$, $D''_{\mathrm{Hom}(Q,S)}\beta^\star = 0$, and the curvature form of E splits as

$$\Theta(E) = \begin{pmatrix} \Theta(S)-\beta^\star\wedge\beta & -D'_{\mathrm{Hom}(Q,S)}\beta^\star \\ D''_{\mathrm{Hom}(S,Q)}\beta & \Theta(Q)-\beta\wedge\beta^\star \end{pmatrix}, \tag{11.6}$$

and the curvature forms of S and Q can be expressed as

$$\Theta(S) = \Theta(E)_{\restriction S} + \beta^\star\wedge\beta,\qquad \Theta(Q) = \Theta(E)_{\restriction Q} + \beta\wedge\beta^\star, \tag{11.7}$$

where $\Theta(E)_{\restriction S}$, $\Theta(E)_{\restriction Q}$ stand for $j^\star\circ\Theta(E)\circ j$ and $g\circ\Theta(E)\circ g^\star$.

Proof. Because of the uniqueness property of Chern connections, it is easy to see that we have a Leibnitz formula

$$D_F(f\wedge u) = (D_{\mathrm{Hom}(E,F)}f)\wedge u + (-1)^{\deg f} f\wedge D_E u$$

whenever u, f are forms with values in hermitian vector bundles E and $\mathrm{Hom}(E,F)$ (where $\mathrm{Hom}(E,F) = E^\star\otimes F$ is equipped with the tensor product metric and $f\wedge u$ incorporates the evaluation mapping $\mathrm{Hom}(E,F)\otimes E\to F$). In our case, given a form u with values in E, we write $u = ju_S + g^\star u_Q$ where $u_S = j^\star u$ and $u_Q = gu$ are the projections of u on S and Q. We then get

$$\begin{aligned}D_E u &= D_E(ju_S + g^\star u_Q)\\ &= (D_{\mathrm{Hom}(S,E)}j)\wedge u_S + j\cdot D_S u_S + (D_{\mathrm{Hom}(Q,E)}g^\star)\wedge u_Q + g^\star\cdot D_Q u_Q.\end{aligned}$$

Since j is holomorphic as well as $j^\star\circ j = \mathrm{Id}_S$, we find $D''_{Hom(S,E)}j = 0$ and

$$D''_{Hom(S,S)}\,\mathrm{Id}_S = 0 = D''_{\mathrm{Hom}(E,S)}j^\star\circ j.$$

By taking the adjoint, we see that $j^\star\circ D'_{\mathrm{Hom}(S,E)}j = 0$, hence $D'_{\mathrm{Hom}(S,E)}j$ takes values in $g^\star Q$ and we thus have a unique form β as in the Lemma such that $D'_{\mathrm{Hom}(S,E)}j = g^\star\circ\beta$. Similarly, g and $g\circ g^\star = \mathrm{Id}_Q$ are holomorphic, thus

$$D''_{Hom(Q,Q)}\,\mathrm{Id}_Q = 0 = g\circ D''_{\mathrm{Hom}(Q,E)}g^\star$$

and there is a form $\gamma\in C^\infty\big(X,\Lambda^{0,1}T^\star_X\otimes\mathrm{Hom}(Q,S)\big)$ such that $D''_{\mathrm{Hom}(Q,E)}g^\star = j\circ\gamma$. By adjunction, we get $D'_{\mathrm{Hom}(E,Q)}g = \gamma^\star\circ j^\star$ and $D''_{Hom(E,Q)}g = 0$ implies $D'_{\mathrm{Hom}(Q,E)}g^\star = 0$. If we differentiate $g\circ j = 0$ we then get

$$0 = D'_{\mathrm{Hom}(E,Q)}g\circ j + g\circ D'_{\mathrm{Hom}(S,E)}j = \gamma^\star\circ j^\star\circ j + g\circ g^\star\circ\beta = \gamma^\star + \beta,$$

thus $\gamma = -\beta^\star$ and $D''_{\mathrm{Hom}(Q,E)}g^\star = -j\circ\beta^\star$. Combining all this, we get

$$\begin{aligned}D_E u &= g^\star\beta\wedge u_S + j\cdot D_S u_S - j\beta^\star\wedge u_Q + g^\star\cdot D_Q u_Q\\ &= j\big(D_S u_S - \beta^\star\wedge u_Q\big) + g^\star\big(\beta\wedge u_S + D_Q u_Q\big),\end{aligned}$$

and the asserted matrix decomposition formula follows. By squaring the matrix, we get

$$D_E^2 = \begin{pmatrix} D_S^2 - \beta^\star\wedge\beta & -D_S\circ\beta^\star - \beta^\star\circ D_Q\\ D_Q\circ\beta + \beta\circ D_S & D_Q^2 - \beta\wedge\beta^\star\end{pmatrix}.$$

As $D_Q\circ\beta + \beta\circ D_S = D_{\mathrm{Hom}(S,Q)}\beta$ and $D_S\circ\beta^\star + \beta^\star\circ D_Q = D_{\mathrm{Hom}(Q,S)}\beta^\star$ by the Leibnitz rule, the curvature formulas follow (observe, since the Chern curvature form is of type $(1,1)$, that we must have $D'_{\mathrm{Hom}(S,Q)}\beta = 0$, $D''_{\mathrm{Hom}(Q,S)}\beta^\star = 0$). □

(11.8) Corollary. *Let $0\to S\to E\to Q\to 0$ be an exact sequence of hermitian vector bundles. Then*

a) $E\geq_{\mathrm{Grif}} 0 \implies Q\geq_{\mathrm{Grif}} 0$,

b) $E\leq_{\mathrm{Grif}} 0 \implies S\leq_{\mathrm{Grif}} 0$,

c) $E\leq_{\mathrm{Nak}} 0 \implies S\leq_{\mathrm{Nak}} 0$,

and analogous implications hold true for strict positivity.

Proof. If β is written $\sum dz_j\otimes\beta_j$, $\beta_j\in\mathrm{Hom}(S,Q)$, then formulas (11.7) yield

$$\begin{aligned}\mathrm{i}\,\Theta(S) &= \mathrm{i}\,\Theta(E)_{\restriction S} - \sum dz_j\wedge d\overline{z}_k\otimes\beta_k^\star\beta_j,\\ \mathrm{i}\,\Theta(Q) &= \mathrm{i}\,\Theta(E)_{\restriction Q} + \sum dz_j\wedge d\overline{z}_k\otimes\beta_j\beta_k^\star.\end{aligned}$$

Since $\beta\cdot(\xi\otimes s) = \sum\xi_j\beta_j\cdot s$ and $\beta^\star\cdot(\xi\otimes s) = \sum\overline{\xi}_k\beta_k^\star\cdot s$ we get

$$\widetilde{\Theta}(S)(\xi\otimes s,\xi'\otimes s') = \widetilde{\Theta}(E)(\xi\otimes s,\xi'\otimes s') - \sum_{j,k}\xi_j\overline{\xi}'_k\langle\beta_j\cdot s,\beta_k\cdot s'\rangle,$$

$$\widetilde{\Theta}(S)(u,u) = \widetilde{\Theta}(E)(u,u) - |\beta\cdot u|^2,$$
$$\widetilde{\Theta}(Q)(\xi\otimes s,\xi'\otimes s') = \widetilde{\Theta}(E)(\xi\otimes s,\xi'\otimes s') + \sum_{j,k}\xi_j\overline{\xi}'_k\langle\beta_k^\star\cdot s,\beta_j^\star\cdot s'\rangle,$$
$$\widetilde{\Theta}(Q)(\xi\otimes s,\xi\otimes s) = \widetilde{\Theta}(E)(\xi\otimes s,\xi\otimes s) = |\beta^\star\cdot(\xi\otimes s)|^2. \qquad \square$$

Next, we need positivity properties which somehow interpolate between Griffiths and Nakano positivity. This leads to the concept of m-tensor positivity.

(11.9) Definition. *Let T and E be complex vector spaces of dimensions n, r respectively, and let Θ be a hermitian form on $T\otimes E$.*

a) *A tensor $u\in T\otimes E$ is said to be of rank m if m is the smallest ≥ 0 integer such that u can be written*
$$u = \sum_{j=1}^{m}\xi_j\otimes s_j,\qquad \xi_j\in T,\ s_j\in E.$$

b) *Θ is said to be m-tensor positive (resp. m-tensor semi-positive) if $\Theta(u,u)>0$ (resp. $\Theta(u,u)\ge 0$) for every tensor $u\in T\otimes E$ of rank $\le m$, $u\ne 0$. In this case, we write*
$$\Theta >_m 0 \qquad (\textit{resp.}\ \ \Theta\ge_m 0).$$

We say that a hermitian vector bundle E is m-tensor positive if $\widetilde{\Theta}(E) >_m 0$. Griffiths positivity corresponds to $m=1$ and Nakano positivity to $m\ge\min(n,r)$. Recall from (5.2) that we have
$$\langle[\mathrm{i}\,\Theta(E),\Lambda]u,u\rangle = \sum_{|S|=q-1}\ \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\,u_{jS,\lambda}\overline{u}_{kS,\mu}$$
for every (n,q)-form $u=\sum u_{K,\lambda}\,dz_1\wedge\ldots\wedge dz_n\wedge d\overline{z}_K\otimes e_\lambda$ with values in E. Since $u_{jS,\lambda}=0$ for $j\in S$, the rank of the tensor $(u_{jS,\lambda})_{j,\lambda}\in\mathbb{C}^n\otimes\mathbb{C}^r$ is in fact $\le\min\{n-q+1,r\}$. We obtain therefore:

(11.10) Lemma. *Assume that $E\ge_m 0$ (resp. $E>_m 0$). Then the hermitian operator $[\mathrm{i}\,\Theta(E),\Lambda]$ is semipositive (resp. positive definite) on $\Lambda^{n,q}T^\star X\otimes E$ for $q\ge 1$ and $m\ge\min\{n-q+1,r\}$.*

The Nakano vanishing theorem can then be improved as follows.

(11.11) Theorem. *Let X be a weakly pseudoconvex Kähler manifold of dimension n and let E a hermitian vector bundle of rank r such that $\widetilde{\Theta}(E)>_m 0$ over X. Then*
$$H^{n,q}(X,E)=0 \quad \text{for}\ \ q\ge 1\ \ \text{and}\ \ m\ge\min\{n-q+1,r\}.$$

We next study some important relations which exist between the various positivity concepts. Our starting point is the following result of [DSk79].

(11.12) Theorem. *For any hermitian vector bundle E,*

$$E >_{\rm Grif} 0 \quad\Longrightarrow\quad E\otimes\det E >_{\rm Nak} 0.$$

To prove this result, we use the fact that

$$\Theta(\det E) = {\rm Tr}_E\,\Theta(E) \tag{11.13}$$

where ${\rm Tr}_E : {\rm Hom}(E,E)\to\mathbb{C}$ is the trace map, together with the identity

$$\Theta(E\otimes\det E) = \Theta(E) + {\rm Tr}_E(\Theta(E))\otimes{\rm Id}_E,$$

which is itself a consequence of (11.13) and of the standard formula

$$\Theta(E\otimes F) = \Theta(E)\otimes{\rm Id}_F + {\rm Id}_E\otimes\Theta(F).$$

In order to prove (11.13), for instance, we differentiate twice a wedge product, according to the formula

$$D_{\Lambda^p E}(s_1\wedge\cdots\wedge s_p) = \sum_{j=1}^p (-1)^{\deg s_1+\cdots+\deg s_{j-1}} s_1\wedge\cdots\wedge s_{j-1}\wedge D_E s_j\wedge\cdots\wedge s_p.$$

The corresponding hermitian forms on $T_X\otimes E$ are thus related by

$$\widetilde\Theta(E\otimes\det E) = \widetilde\Theta(E) + {\rm Tr}_E\,\widetilde\Theta(E)\otimes h,$$

where h denotes the hermitian metric on E and ${\rm Tr}_E\,\widetilde\Theta(E)$ is the hermitian form on T_X defined by

$${\rm Tr}_E\,\widetilde\Theta(E)(\xi,\xi) = \sum_{1\le\lambda\le r}\widetilde\Theta(E)(\xi\otimes e_\lambda,\xi\otimes e_\lambda),\qquad \xi\in T_X,$$

for any orthonormal frame $(e_1,\ldots,e_r)$ of E. Theorem 11.12 is now a consequence of the following simple property of hermitian forms on a tensor product of complex vector spaces.

(11.14) Proposition. *Let T, E be complex vector spaces of respective dimensions n, r, and h a hermitian metric on E. Then for every hermitian form Θ on $T\otimes E$*

$$\Theta >_{\rm Grif} 0 \quad\Longrightarrow\quad \Theta + {\rm Tr}_E\,\Theta\otimes h >_{\rm Nak} 0.$$

We first need a lemma analogous to Fourier inversion formula for discrete Fourier transforms.

(11.15) Lemma. *Let q be an integer ≥ 3, and $x_\lambda,\ y_\mu,\ 1\le\lambda,\mu\le r$, be complex numbers. Let σ describe the set U_q^r of r-tuples of q-th roots of unity and put*

$$x'_\sigma = \sum_{1\le\lambda\le r} x_\lambda\overline\sigma_\lambda,\qquad y'_\sigma = \sum_{1\le\mu\le r} y_\mu\overline\sigma_\mu,\qquad \sigma\in U_q^r.$$

Then for every pair (α,β), $1\le\alpha,\beta\le r$, the following identity holds:

$$q^{-r}\sum_{\sigma\in U_q^r} x'_\sigma \overline{y}'_\sigma \sigma_\alpha \overline{\sigma}_\beta = \begin{cases} x_\alpha \overline{y}_\beta & \textit{if}\ \ \alpha \neq \beta, \\ \displaystyle\sum_{1\le\mu\le r} x_\mu \overline{y}_\mu & \textit{if}\ \ \alpha = \beta. \end{cases}$$

Proof. The coefficient of $x_\lambda \overline{y}_\mu$ in the summation $q^{-r}\sum_{\sigma\in U_q^r} x'_\sigma \overline{y}'_\sigma \sigma_\alpha \overline{\sigma}_\beta$ is given by

$$q^{-r}\sum_{\sigma\in U_q^r} \sigma_\alpha \overline{\sigma}_\beta \overline{\sigma}_\lambda \sigma_\mu .$$

This coefficient equals 1 when the pairs $\{\alpha,\mu\}$ and $\{\beta,\lambda\}$ are equal (in which case $\sigma_\alpha \overline{\sigma}_\beta \overline{\sigma}_\lambda \sigma_\mu = 1$ for any one of the q^r elements of U_q^r). Hence, it is sufficient to prove that

$$\sum_{\sigma\in U_q^r} \sigma_\alpha \overline{\sigma}_\beta \overline{\sigma}_\lambda \sigma_\mu = 0$$

when the pairs $\{\alpha,\mu\}$ and $\{\beta,\lambda\}$ are distinct.

If $\{\alpha,\mu\} \neq \{\beta,\lambda\}$, then one of the elements of one of the pairs does not belong to the other pair. As the four indices α,β,λ,μ play the same role, we may suppose for example that $\alpha \notin \{\beta,\lambda\}$. Let us apply to σ the substitution $\sigma\mapsto\tau$, where τ is defined by

$$\tau_\alpha = e^{2\pi\mathrm{i}\,/q}\sigma_\alpha,\ \ \tau_\nu = \sigma_\nu \quad \text{for} \quad \nu\neq\alpha.$$

We get

$$\sum_\sigma \sigma_\alpha \overline{\sigma}_\beta \overline{\sigma}_\lambda \sigma_\mu = \sum_\tau = \begin{cases} e^{2\pi\mathrm{i}\,/q}\displaystyle\sum_\sigma & \text{if}\ \ \alpha\neq\mu, \\ e^{4\pi\mathrm{i}\,/q}\displaystyle\sum_\sigma & \text{if}\ \ \alpha=\mu, \end{cases}$$

Since $q\ge 3$ by hypothesis, it follows that

$$\sum_\sigma \sigma_\alpha \overline{\sigma}_\beta \overline{\sigma}_\lambda \sigma_\mu = 0.$$

Proof of Proposition 11.14. Let $(t_j)_{1\le j\le n}$ be a basis of T, $(e_\lambda)_{1\le\lambda\le r}$ an orthonormal basis of E and $\xi = \sum_j \xi_j t_j \in T$, $u=\sum_{j,\lambda} u_{j\lambda}\, t_j\otimes e_\lambda \in T\otimes E$. The coefficients $c_{jk\lambda\mu}$ of Θ with respect to the basis $t_j\otimes e_\lambda$ satisfy the symmetry relation $\overline{c}_{jk\lambda\mu} = c_{kj\mu\lambda}$, and we have the formulas

$$\Theta(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda}\overline{u}_{k\mu},$$

$$\mathrm{Tr}_E\,\Theta(\xi,\xi) = \sum_{j,k,\lambda} c_{jk\lambda\lambda}\xi_j\overline{\xi}_k,$$

$$(\Theta + \mathrm{Tr}_E\,\Theta\otimes h)(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda}\overline{u}_{k\mu} + c_{jk\lambda\lambda} u_{j\mu}\overline{u}_{k\mu}.$$

For every $\sigma\in U_q^r$ (cf. Lemma 11.15), put

$$u'_{j\sigma} = \sum_{1\le\lambda\le r} u_{j\lambda}\overline{\sigma}_\lambda \in \mathbb{C},$$

$$\widehat{u}_\sigma = \sum_j u'_{j\sigma} t_j \in T \quad , \quad \widehat{e}_\sigma = \sum_\lambda \sigma_\lambda e_\lambda \in E.$$

Lemma 11.15 implies

$$\begin{aligned} q^{-r}\sum_{\sigma\in U_q^r}\Theta(\widehat{u}_\sigma\otimes\widehat{e}_\sigma,\widehat{u}_\sigma\otimes\widehat{e}_\sigma) &= q^{-r}\sum_{\sigma\in U_q^r} c_{jk\lambda\mu}u'_{j\sigma}\overline{u}'_{k\sigma}\sigma_\lambda\overline{\sigma}_\mu \\ &= \sum_{j,k,\lambda\neq\mu} c_{jk\lambda\mu}u_{j\lambda}\overline{u}_{k\mu}+\sum_{j,k,\lambda,\mu} c_{jk\lambda\lambda}u_{j\mu}\overline{u}_{k\mu}. \end{aligned}$$

The Griffiths positivity assumption shows that the left hand side is ≥ 0, hence

$$(\Theta+\mathrm{Tr}_E\,\Theta\otimes h)(u,u)\geq\sum_{j,k,\lambda} c_{jk\lambda\lambda}u_{j\lambda}\overline{u}_{k\lambda}\geq 0$$

with strict positivity if $\Theta>_{\mathrm{Grif}}0$ and $u\neq 0$. □

We now relate Griffiths positivity to m-tensor positivity. The most useful result is the following

(11.16) Proposition. *Let T be a complex vector space and (E,h) a hermitian vector space of respective dimensions n,r with $r\geq 2$. Then for any hermitian form Θ on $T\otimes E$ and any integer $m\geq 1$*

$$\Theta>_{\mathrm{Grif}}0 \quad\Longrightarrow\quad m\,\mathrm{Tr}_E\,\Theta\otimes h-\Theta>_m 0.$$

Proof. Let us distinguish two cases.

a) $m=1$. Let $u\in T\otimes E$ be a tensor of rank 1. Then u can be written $u=\xi_1\otimes e_1$ with $\xi_1\in T$, $\xi_1\neq 0$, and $e_1\in E$, $|e_1|=1$. Complete e_1 into an orthonormal basis $(e_1,\ldots,e_r)$ of E. One gets immediately

$$\begin{aligned} (\mathrm{Tr}_E\,\Theta\otimes h)(u,u)=\mathrm{Tr}_E\,\Theta(\xi_1,\xi_1) &= \sum_{1\leq\lambda\leq r}\Theta(\xi_1\otimes e_\lambda,\xi_1\otimes e_\lambda) \\ &> \Theta(\xi_1\otimes e_1,\xi_1\otimes e_1)=\Theta(u,u). \end{aligned}$$

b) $m\geq 2$. Every tensor $u\in T\otimes E$ of rank $\leq m$ can be written

$$u=\sum_{1\leq\lambda\leq q}\xi_\lambda\otimes e_\lambda\quad,\quad \xi_\lambda\in T,$$

with $q=\min(m,r)$ and $(e_\lambda)_{1\leq\lambda\leq r}$ an orthonormal basis of E. Let F be the vector subspace of E generated by $(e_1,\ldots,e_q)$ and Θ_F the restriction of Θ to $T\otimes F$. The first part shows that

$$\Theta':=\mathrm{Tr}_F\,\Theta_F\otimes h-\Theta_F>_{\mathrm{Grif}}0.$$

Proposition 11.14 applied to Θ' on $T\otimes F$ yields

$$\Theta'+\mathrm{Tr}_F\,\Theta'\otimes h=q\,\mathrm{Tr}_F\,\Theta_F\otimes h-\Theta_F>_q 0.$$

Since $u\in T\otimes F$ is of rank $\leq q\leq m$, we get (for $u\neq 0$)

$$\begin{aligned} \Theta(u,u)=\Theta_F(u,u) &< q(\mathrm{Tr}_F\,\Theta_F\otimes h)(u,u) \\ &= q\sum_{1\leq j,\lambda\leq q}\Theta(\xi_j\otimes e_\lambda,\xi_j\otimes e_\lambda)\leq m\,\mathrm{Tr}_E\,\Theta\otimes h(u,u). \end{aligned}$$ □

Proposition 11.16 is of course also true in the semi-positive case. From these facts, we deduce

(11.17) Theorem. *Let E be a Griffiths (semi-)positive bundle of rank $r \geq 2$. Then for any integer $m \geq 1$*

$$E^\star \otimes (\det E)^m >_m 0 \qquad (\textit{resp.} \quad \geq_m 0).$$

Proof. We apply Proposition 11.16 to $\Theta = -\Theta(E^\star) = {}^t\Theta(E) \geq_{\rm Grif} 0$ on $T_X \otimes E^\star$ and observe that

$$\Theta(\det E) = \mathrm{Tr}_E\,\Theta(E) = \mathrm{Tr}_{E^\star}\,\Theta.$$

(11.18) Theorem. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of hermitian vector bundles. Then for any $m \geq 1$*

$$E >_m 0 \quad \Longrightarrow \quad S \otimes (\det Q)^m >_m 0.$$

Proof. Formulas (11.7) imply

$$\mathrm{i}\,\Theta(S) >_m \mathrm{i}\,\beta^\star \wedge \beta \quad , \quad \mathrm{i}\,\Theta(Q) >_m \mathrm{i}\,\beta \wedge \beta^\star,$$

$$\mathrm{i}\,\Theta(\det Q) = \mathrm{Tr}_Q(\mathrm{i}\,\Theta(Q)) > \mathrm{Tr}_Q(\mathrm{i}\,\beta \wedge \beta^\star).$$

If we write $\beta = \sum dz_j \otimes \beta_j$ as in the proof of Corollary 11.8, then

$$\begin{aligned}\mathrm{Tr}_Q(\mathrm{i}\,\beta \wedge \beta^\star) &= \sum idz_j \wedge d\overline{z}_k\,\mathrm{Tr}_Q(\beta_j\beta_k^\star)\\ &= \sum idz_j \wedge d\overline{z}_k\,\mathrm{Tr}_S(\beta_k^\star\beta_j) = \mathrm{Tr}_S(-\mathrm{i}\,\beta^\star \wedge \beta).\end{aligned}$$

Furthermore, it has been already proved that $-\mathrm{i}\,\beta^\star \wedge \beta \geq_{\rm Nak} 0$. By Proposition 11.16 applied to the corresponding hermitian form Θ on $T_X \otimes S$, we get

$$m\,\mathrm{Tr}_S(-\mathrm{i}\,\beta^\star \wedge \beta) \otimes \mathrm{Id}_S + \mathrm{i}\,\beta^\star \wedge \beta \geq_m 0,$$

and Theorem 11.18 follows. □

(11.19) Corollary. *Let X be a weakly pseudoconvex Kähler n-dimensional manifold, E a holomorphic vector bundle of rank $r \geq 2$ and $m \geq 1$ an integer. Then*

a) $E >_{\rm Grif} 0 \Rightarrow H^{n,q}(X, E \otimes \det\, E) = 0$ *for* $q \geq 1$;

b) $E >_{\rm Grif} 0 \Rightarrow H^{n,q}\big(X, E^\star \otimes (\det\, E)^m\big) = 0$ *for* $q \geq 1$ *and* $m \geq \min\{n - q + 1, r\}$;

c) *Let* $0 \to S \to E \to Q \to 0$ *be an exact sequence of vector bundles and* $m = \min\{n - q + 1, \mathrm{rk}\ S\}$, $q \geq 1$. *If* $E >_m 0$ *and if* L *is a line bundle such that* $L \otimes (\det Q)^{-m} \geq 0$, *then*

$$H^{n,q}(X, S \otimes L) = 0.$$

Proof. Immediate consequence of Theorem 11.11, in combination with 11.12 for a), 11.17 for b) and 11.18 for c). □

12. Skoda's L^2 Estimates for Surjective Bundle Morphisms

Let (X,ω) be a Kähler manifold, $\dim X = n$, and let $g : E \to Q$ a holomorphic morphism of hermitian vector bundles over X. Assume in the first instance that g is *surjective*. We are interested in conditions insuring that the induced morphisms $g : H^{n,k}(X,E) \longrightarrow H^{n,k}(X,Q)$ are also surjective (dealing with $(n,\bullet)$ bidegrees is always easier, since we have to understand positivity conditions for the curvature term). For that purpose, it is natural to consider the subbundle $S = \mathrm{Ker}\, g \subset E$ and the exact sequence

$$(12.1) \qquad 0 \longrightarrow S \xrightarrow{\ j\ } E \xrightarrow{\ g\ } Q \longrightarrow 0$$

where $j : S \to E$ is the inclusion. In fact, we need a little more flexibility to handle the curvature terms, so we take the tensor product of the exact sequence by a holomorphic line bundle L (whose properties will be specified later):

$$(12.2) \qquad 0 \longrightarrow S \otimes L \longrightarrow E \otimes L \xrightarrow{\ g\ } Q \otimes L \longrightarrow 0.$$

(12.3) Theorem. *Let k be an integer such that $0 \le k \le n$. Set $r = \mathrm{rk}\ E$, $q = \mathrm{rkQ}$, $s = \mathrm{rk}\ S = r - q$ and*

$$m = \min\{n-k, s\} = \min\{n-k, r-q\}.$$

Assume that (X,ω) possesses also a complete Kähler metric $\widehat{\omega}$, that $E \ge_m 0$, and that $L \longrightarrow X$ is a hermitian holomorphic line bundle such that

$$\mathrm{i}\,\Theta(L) - (m+\varepsilon)\mathrm{i}\,\Theta(\det Q) \ge 0$$

for some $\varepsilon > 0$. Then for every D''-closed form f of type (n,k) with values in $Q \otimes L$ such that $\|f\| < +\infty$, there exists a D''-closed form h of type (n,k) with values in $E \otimes L$ such that $f = g \cdot h$ and

$$\|h\|^2 \le (1 + m/\varepsilon)\,\|f\|^2.$$

The idea of the proof is essentially due to [Sko78], who actually proved the special case $k = 0$. The general case appeared in [Dem82b].

Proof. Let $j : S \to E$ be the inclusion morphism, $g^\star : Q \to E$ and $j^\star : E \to S$ the adjoints of g, j, and the matrix of D_E with respect to the orthogonal splitting $E \simeq S \oplus Q$ (cf. Lemma 11.2). Then $g^\star f$ is a lifting of f in $E \otimes L$. We will try to find h under the form

$$h = g^\star f + ju, \qquad u \in L^2(X, \Lambda^{n,k}T^\star_X \otimes S \otimes L).$$

As the images of S and Q in E are orthogonal, we have $|h|^2 = |f|^2 + |u|^2$ at every point of X. On the other hand $D''_{Q\otimes L} f = 0$ by hypothesis and $D'' g^\star = -j \circ \beta^\star$ by (11.5), hence

$$D''_{E\otimes L} h = -j(\beta^\star \wedge f) + j\, D''_{S\otimes L} = j(D''_{S\otimes L} - \beta^\star \wedge f).$$

We are thus led to solve the equation

$$D''_{S\otimes L}u = \beta^\star \wedge f, \tag{12.4}$$

and for that, we apply Theorem 6.1 to the $(n, k+1)$-form $\beta^\star \wedge f$. One now observes that the curvature of $S \otimes L$ can be expressed in terms of β. This remark will be used to prove:

(12.5) Lemma. *Let $A_k = [\mathrm{i}\,\Theta(S \otimes L), \Lambda]$ be the curvature operator acting as an hermitian operator on the bundle of $(n, k+1)$-forms. Then*

$$\langle A_k^{-1}(\beta^\star \wedge f), (\beta^\star \wedge f)\rangle \le (m/\varepsilon)\,|f|^2.$$

If the Lemma is taken for granted, Theorem 9.4 yields a solution u of (12.4) in $L^2(X, \Lambda^{n,q}T^\star_X \otimes S \otimes L)$ such that $\|u\|^2 \le (m/\varepsilon)\,\|f\|^2$. As $\|h\|^2 = \|f\|^2 + \|u\|^2$, the proof of Theorem 12.3 is complete. □

Proof of Lemma 12.5. Exactly as in the proof of Theorem 11.18, the formulas (11.7) yield

$$\mathrm{i}\,\Theta(S) \ge_m \mathrm{i}\,\beta^\star \wedge \beta, \qquad \mathrm{i}\,\Theta(\det Q) \ge \mathrm{Tr}_Q(\mathrm{i}\,\beta \wedge \beta^\star) = \mathrm{Tr}_S(-\mathrm{i}\,\beta^\star \wedge \beta).$$

Since $C^\infty(X, \Lambda^{1,1}T^\star_X \otimes \mathrm{Herm}\ S) \ni \Theta := -\mathrm{i}\,\beta^\star \wedge \beta \ge_{\mathrm{Grif}} 0$, Proposition 11.16 implies

$$m\ \mathrm{Tr}_S(-\mathrm{i}\,\beta^\star \wedge \beta) \otimes \mathrm{Id}_S + \mathrm{i}\,\beta^\star \wedge \beta \ge_m 0.$$

From the hypothesis on the curvature of L we get

$$\begin{aligned}
\mathrm{i}\,\Theta(S \otimes L) &\ge_m \mathrm{i}\,\Theta(S) \otimes \mathrm{Id}_L + (m+\varepsilon)\,\mathrm{i}\,\Theta(\det Q) \otimes \mathrm{Id}_{S\otimes L}\\
&\ge_m \big(\mathrm{i}\,\beta^\star \wedge \beta + (m+\varepsilon)\ \mathrm{Tr}_S(-\mathrm{i}\,\beta^\star \wedge \beta) \otimes \mathrm{Id}_S\big) \otimes \mathrm{Id}_L\\
&\ge_m (\varepsilon/m)\,(-\mathrm{i}\,\beta^\star \wedge \beta) \otimes \mathrm{Id}_S \otimes \mathrm{Id}_L\,.
\end{aligned}$$

For any $v \in \Lambda^{n,k+1}T^\star_X \otimes S \otimes L$, Lemma 11.10 implies

$$\langle A_k v, v\rangle \ge (\varepsilon/m)\,\langle -\mathrm{i}\,\beta^\star \wedge \beta \wedge \Lambda v, v\rangle, \tag{12.6}$$

because $\mathrm{rk}(S \otimes L) = s$ and $m = \min\{n-k, s\}$. Let $(dz_1, \ldots, dz_n)$ be an orthonormal basis of $T^\star_X$ at a given point $x_0 \in X$ and set

$$\beta = \sum_{1\le j\le n} dz_j \otimes \beta_j, \qquad \beta_j \in \mathrm{Hom}(S, Q).$$

The adjoint of the operator $\beta^\star \wedge \bullet = \sum d\overline{z}_j \wedge \beta_j^\star \bullet$ is the contraction operator $\beta \,\lrcorner\, \bullet$ defined by

$$\beta \,\lrcorner\, v = \sum \frac{\partial}{\partial \overline{z}_j} \,\lrcorner\, (\beta_j v) = \sum -\mathrm{i}\,dz_j \wedge \Lambda(\beta_j v) = -\mathrm{i}\,\beta \wedge \Lambda v.$$

Consequently, we get $\langle -\mathrm{i}\,\beta^\star \wedge \beta \wedge \Lambda v, v\rangle = |\beta \,\lrcorner\, v|^2$ and (12.6) implies

$$|\langle \beta^\star \wedge f, v\rangle|^2 = |\langle f, \beta \,\lrcorner\, v\rangle|^2 \le |f|^2\,|\beta \,\lrcorner\, v|^2 \le (m/\varepsilon)\langle A_k v, v\rangle\,|f|^2.$$

This is equivalent to the estimate asserted in the lemma. □

If X has a plurisubharmonic exhaustion function ψ, we can select a convex increasing function $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ and multiply the metric of L by the weight $\exp(-\chi\circ\psi)$ in order to make the L^2 norm of f converge. Theorem 12.3 implies therefore:

(12.7) Corollary. *Let (X,ω) be a weakly pseudoconvex Kähler manifold, $g : E \to Q$ a surjective bundle morphism with $r = \mathrm{rk}\, E$, $q = \mathrm{rk}\, Q$, and $L \to X$ a hermitian holomorphic line bundle. We set $m = \min\{n-k, r-q\}$ and assume that $E \geq_m 0$ and*

$$\mathrm{i}\,\Theta(L) - (m+\varepsilon)\,\mathrm{i}\,\Theta(\det Q) \geq 0$$

for some $\varepsilon > 0$. Then g induces a surjective map

$$H^{n,k}(X, E\otimes L) \longrightarrow H^{n,k}(X, Q\otimes L).$$

The most remarkable feature of this result is that it does not require any strict positivity assumption on the curvature (for instance E can be a flat bundle). A careful examination of the proof shows that it amounts to verify that the image of the coboundary morphism

$$-\beta^\star \wedge \bullet \ :\ H^{n,k}(X, Q\otimes L) \longrightarrow H^{n,k+1}(X, S\otimes L)$$

vanishes; however the cohomology group $H^{n,k+1}(X, S\otimes L)$ itself does not necessarily vanish, as it would do under a strict positivity assumption.

We want now to get estimates also when Q is endowed with a metric given a priori, that can be distinct from the quotient metric of E by g. Then the map $g^\star(gg^\star)^{-1} \ :\ Q \longrightarrow E$ is the lifting of Q orthogonal to $S = \mathrm{Ker}\, g$. The quotient metric $|\bullet|'$ on Q is therefore defined in terms of the original metric $|\bullet|$ by

$$|v|'^2 = |g^\star(gg^\star)^{-1}v|^2 = \langle (gg^\star)^{-1}v, v\rangle = \det(gg^\star)^{-1}\,\langle \widetilde{gg^\star}v, v\rangle$$

where $\widetilde{gg^\star} \in \mathrm{End}(Q)$ denotes the endomorphism of Q whose matrix is the transposed comatrix of $gg^\star$. For every $w \in \det Q$, we find

$$|w|'^2 = \det(gg^\star)^{-1}\,|w|^2.$$

If Q' denotes the bundle Q with the quotient metric, we get

$$\mathrm{i}\,\Theta(\det Q') = \mathrm{i}\,\Theta(\det Q) + \mathrm{i}\,d'd''\log\det(gg^\star).$$

In order that the hypotheses of Theorem 12.3 be satisfied, we are led to define a new metric $|\bullet|'$ on L by $|u|'^2 = |u|^2\,\big(\det(gg^\star)\big)^{-m-\varepsilon}$. Then

$$\mathrm{i}\,\Theta(L') = \mathrm{i}\,\Theta(L) + (m+\varepsilon)\,\mathrm{i}\,d'd''\log\det(gg^\star) \geq (m+\varepsilon)\,\mathrm{i}\,\Theta(\det Q').$$

Theorem 12.3 applied to (E, Q', L') can now be reformulated:

(12.8) Theorem. *Let X be a complete Kähler manifold equipped with a Kähler metric ω on X, let $E \to Q$ be a surjective morphism of hermitian vector bundles and let $L \to X$ be a hermitian holomorphic line bundle. Set $r = \mathrm{rk}\, E$, $q = \mathrm{rk}\, Q$ and $m = \min\{n-k, r-q\}$, and assume that $E \geq_m 0$ and*

$$\mathrm{i}\,\Theta(L) - (m+\varepsilon)\mathrm{i}\,\Theta(\det Q) \geq 0$$

for some $\varepsilon > 0$. *Then for every* D''*-closed form* f *of type* (n,k) *with values in* $Q\otimes L$ *such that*

$$I = \int_X \langle \widetilde{gg^\star} f, f\rangle\,(\det gg^\star)^{-m-1-\varepsilon}\,dV < +\infty,$$

there exists a D''*-closed form* h *of type* (n,k) *with values in* $E\otimes L$ *such that* $f = g\cdot h$ *and*

$$\int_X |h|^2\,(\det gg^\star)^{-m-\varepsilon}\,dV \le (1+m/\varepsilon)\,I. \qquad \square$$

Our next goal is to extend Theorem 12.8 in the case when $g : E \longrightarrow Q$ is only *generically* surjective; this means that the analytic set

$$Y = \{x\in X\ ;\ g_x\ :\ E_x \longrightarrow Q_x \text{ is not surjective }\}$$

defined by the equation $\Lambda^q g = 0$ is nowhere dense in X. Here $\Lambda^q g$ is a section of the bundle $\mathrm{Hom}(\Lambda^q E, \det Q)$. The idea is to apply the above Theorem 12.8 to $X \smallsetminus Y$. For this, we have to know whether $X\smallsetminus Y$ has a complete Kähler metric.

(12.9) Lemma. *Let* (X,ω) *be a Kähler manifold, and* $Y = \sigma^{-1}(0)$ *an analytic subset defined by a section of a hermitian vector bundle* $E\to X$. *If* X *is weakly pseudoconvex and exhausted by* $X_c = \{x\in X\ ;\ \psi(x) < c\}$, *then* $X_c \smallsetminus Y$ *has a complete Kähler metric for all* $c\in\mathbb{R}$. *The same conclusion holds for* $X\smallsetminus Y$ *if* (X,ω) *is complete and if for some constant* $C\ge 0$ *we have* $\Theta_E \le_{\mathrm{Grif}} C\,\omega\otimes\langle\ ,\ \rangle_E$ *on* X.

Proof. Set $\tau = \log|\sigma|^2$. Then $d'\tau = \{D'\sigma,\sigma\}/|\sigma|^2$ and $D''D'\sigma = D^2\sigma = \Theta(E)\sigma$, thus

$$\mathrm{i}\,d'd''\tau = \mathrm{i}\,\frac{\{D'\sigma, D'\sigma\}}{|\sigma|^2} - \mathrm{i}\,\frac{\{D'\sigma,\sigma\}\wedge\{\sigma, D'\sigma\}}{|\sigma|^4} - \frac{\{\mathrm{i}\,\Theta(E)\sigma,\sigma\}}{|\sigma|^2}.$$

For every $\xi\in T_X$, we find therefore

$$\begin{aligned} H\tau(\xi) &= \frac{|\sigma|^2\,|D'\sigma\cdot\xi|^2 - |\langle D'\sigma\cdot\xi,\sigma\rangle|^2}{|\sigma|^4} - \frac{\widetilde\Theta(E)(\xi\otimes\sigma,\xi\otimes\sigma)}{|\sigma|^2}\\ &\ge -\frac{\widetilde\Theta(E)(\xi\otimes\sigma,\xi\otimes\sigma)}{|\sigma|^2}\end{aligned}$$

by the Cauchy-Schwarz inequality. If C is a bound for the coefficients of $\widetilde\Theta(E)$ on the compact subset $\overline{X}_c$, we get $\mathrm{i}\,d'd''\tau \ge -C\omega$ on X_c. Let $\chi\in C^\infty(\mathbb{R},\mathbb{R})$ be a convex increasing function. We set

$$\widehat\omega = \omega + \mathrm{i}\,d'd''(\chi\circ\tau) = \omega + \mathrm{i}\,\big(\chi'\circ\tau\ d'd''\tau + \chi''\circ\tau\ d'\tau\wedge d''\tau\big).$$

We thus see that $\widehat\omega$ is positive definite if $\chi'\le 1/2C$, and by a computation similar to the one preceding Theorem 6.2, we check that $\widehat\omega$ is complete near $Y = \tau^{-1}(-\infty)$ as soon as

$$\int_{-\infty}^0 \sqrt{\chi''(t)}\,dt = +\infty.$$

One can choose for example χ such that $\chi(t) = \frac{1}{5C}(t - \log|t|)$ for $t\le -1$. In order to obtain a complete Kähler metric on $X_c\smallsetminus Y$, we also need the metric to be complete near ∂X_c. If $\widehat\omega$ is not, such a metric can be defined by

$$\begin{aligned}\widetilde{\omega} = \widehat{\omega} + \mathrm{i}\, d'd'' \log(c-\psi)^{-1} &= \widehat{\omega} + \frac{\mathrm{i}\, d'd''\psi}{c-\psi} + \frac{\mathrm{i}\, d'\psi\wedge d''\psi}{(c-\psi)^2}\\ &\geq \mathrm{i}\, d'\log(c-\psi)^{-1}\wedge d''\log(c-\psi)^{-1}\ ;\end{aligned}$$

$\widetilde{\omega}$ is complete on $X_c \smallsetminus \Omega$ because $\log(c-\psi)^{-1}$ tends to $+\infty$ on ∂X_c. □

We also need another elementary lemma dealing with the extension of partial differential equalities across analytic sets.

(12.10) Lemma. *Let Ω be an open subset of $\mathbb{C}^n$ and Y an analytic subset of Ω. Assume that v is a $(p,q-1)$-form with L^2_{loc} coefficients and w a (p,q)-form with L^1_{loc} coefficients such that $d''v = w$ on $\Omega \smallsetminus Y$ (in the sense of distribution theory). Then $d''v = w$ on Ω.*

Proof. An induction on the dimension of Y shows that it is sufficient to prove the result in a neighborhood of a regular point $a \in Y$. By using a local analytic isomorphism, the proof is reduced to the case where Y is contained in the hyperplane $z_1 = 0$, with $a = 0$. Let $\lambda \in C^\infty(\mathbb{R},\mathbb{R})$ be a function such that $\lambda(t) = 0$ for $t \leq \frac{1}{2}$ and $\lambda(t) = 1$ for $t \geq 1$. We must show that

$$\int_\Omega w\wedge\alpha = (-1)^{p+q}\int_\Omega v\wedge d''\alpha \tag{12.11}$$

for all $\alpha \in \mathcal{D}(\Omega, \Lambda^{n-p,n-q}T^\star_\Omega)$. Set $\lambda_\varepsilon(z) = \lambda(|z_1|/\varepsilon)$ and replace α in the integral by $\lambda_\varepsilon\alpha$. Then $\lambda_\varepsilon\alpha \in \mathcal{D}(\Omega\smallsetminus Y, \Lambda^{n-p,n-q}T^\star_\Omega)$ and the hypotheses imply

$$\int_\Omega w\wedge\lambda_\varepsilon\alpha = (-1)^{p+q}\int_\Omega v\wedge d''(\lambda_\varepsilon\alpha) = (-1)^{p+q}\int_\Omega v\wedge(d''\lambda_\varepsilon\wedge\alpha + \lambda_\varepsilon d''\alpha).$$

As w and v have L^1_{loc} coefficients on Ω, the integrals of $w\wedge\lambda_\varepsilon\alpha$ and $v\wedge\lambda_\varepsilon d''\alpha$ converge respectively to the integrals of $w\wedge\alpha$ and $v\wedge d''\alpha$ as ε tends to 0. The remaining term can be estimated by means of the Cauchy-Schwarz inequality:

$$\Big|\int_\Omega v\wedge d''\lambda_\varepsilon\wedge\alpha\Big|^2 \leq \int_{|z_1|\leq\varepsilon}|v\wedge\alpha|^2\,dV.\ \int_{\mathrm{Supp}\,\alpha}|d''\lambda_\varepsilon|^2\,dV\ ;$$

as $v\in L^2_{\mathrm{loc}}(\Omega)$, the integral $\int_{|z_1|\leq\varepsilon}|v\wedge\alpha|^2\,dV$ converges to 0 with ε, whereas

$$\int_{\mathrm{Supp}\,\alpha}|d''\lambda_\varepsilon|^2\,dV \leq \frac{C}{\varepsilon^2}\,\mathrm{Vol}\big(\mathrm{Supp}\,\alpha\cap\{|z_1|\leq\varepsilon\}\big)\leq C'.$$

Equality (12.11) follows when ε tends to 0. □

(12.12) Theorem. *The existence statement and the estimates of Theorem* 12.8 *remain true for a generically surjective morphism $g : E \to Q$, provided that X is weakly pseudoconvex.*

Proof. Apply Theorem 12.8 to each relatively compact domain $X_c \smallsetminus Y$ (these domains are complete Kähler by Lemma 12.9). From a sequence of solutions on $X_c\smallsetminus Y$ we can extract a subsequence converging weakly on $X\smallsetminus Y$ as c tends to $+\infty$. One gets a form h satisfying the estimates, such that $D''h = 0$ on $X\smallsetminus Y$ and $f = g\cdot h$.

In order to see that $D''h = 0$ on X, it suffices to apply Lemma 12.10 and to observe that h has L^2_{loc} coefficients on X by our estimates. □

A very special but interesting case is obtained for the trivial bundles $E = \Omega\times\mathbb{C}^r$, $Q = \Omega\times\mathbb{C}$ over a pseudoconvex open set $\Omega\subset\mathbb{C}^n$. Then the morphism g is given by a r-tuple $(g_1,\ldots,g_r)$ of holomorphic functions on Ω. Let us take $k = 0$ and $L = \Omega\times\mathbb{C}$ with the metric given by a weight $e^{-\varphi}$. If we observe that $\widetilde{gg^\star} = \mathrm{Id}$ when $\mathrm{rk}\ Q = 1$, Theorem 12.8 applied on $X = \Omega\smallsetminus g^{-1}(0)$ and Lemmas 12.9, 12.10 give:

(12.13) Theorem (Skoda [Sko72b])**.** *Let Ω be a complete Kähler open subset of $\mathbb{C}^n$ and φ a plurisubharmonic function on Ω. Set $m = \min\{n, r-1\}$. Then for every holomorphic function f on Ω such that*

$$I = \int_{\Omega\smallsetminus Z} |f|^2\,|g|^{-2(m+1+\varepsilon)}e^{-\varphi}\,dV < +\infty,$$

where $Z = g^{-1}(0)$, there exist holomorphic functions $(h_1,\ldots,h_r)$ on Ω such that $f = \sum g_j h_j$ and

$$\int_{\Omega\smallsetminus Y} |h|^2\,|g|^{-2(m+\varepsilon)}e^{-\varphi}\,dV \le (1+m/\varepsilon)I. \qquad \square$$

We now show that Theorem 12.13 can be applied to get deep results concerning ideals of the local ring $\mathcal{O}_n = \mathbb{C}\{z_1,\ldots,z_n\}$ of germs of holomorphic functions on $(\mathbb{C}^n, 0)$. Let $\mathcal{I} = (g_1,\ldots,g_r) \neq (0)$ be an ideal of $\mathcal{O}_n$.

(12.14) Definition. *Let $k\in\mathbb{R}_+$. We associate to $\mathcal{I}$ the following ideals:*

a) *the ideal $\overline{\mathcal{I}}^{(k)}$ of germs $u\in\mathcal{O}_n$ such that $|u| \le C|g|^k$ for some constant $C\ge 0$, where $|g|^2 = |g_1|^2+\cdots+|g_r|^2$.*

b) *the ideal $\widehat{\mathcal{I}}^{(k)}$ of germs $u\in\mathcal{O}_n$ such that*

$$\int_\Omega |u|^2\,|g|^{-2(k+\varepsilon)}\,dV < +\infty$$

on a small ball Ω centered at 0, if $\varepsilon > 0$ is small enough.

(12.15) Proposition. *For all $k, l\in\mathbb{R}_+$ we have*

a) $\overline{\mathcal{I}}^{(k)} \subset \widehat{\mathcal{I}}^{(k)}$;

b) $\mathcal{I}^k \subset \overline{\mathcal{I}}^{(k)}$ *if* $k\in\mathbb{N}$;

c) $\overline{\mathcal{I}}^{(k)}.\overline{\mathcal{I}}^{(l)} \subset \overline{\mathcal{I}}^{(k+l)}$;

d) $\overline{\mathcal{I}}^{(k)}.\widehat{\mathcal{I}}^{(l)} \subset \widehat{\mathcal{I}}^{(k+l)}$.

All properties are immediate from the definitions except a) which is a consequence of the integrability of $|g|^{-\varepsilon}$ for $\varepsilon > 0$ small (exercise to the reader!). Before stating the main result, we need a simple lemma.

(12.16) Lemma. *If $\mathcal{I} = (g_1, \ldots, g_r)$ and $r > n$, we can find elements $\widetilde{g}_1, \ldots, \widetilde{g}_n \in \mathcal{I}$ such that $C^{-1}|g| \le |\widetilde{g}| \le C|g|$ on a neighborhood of 0. Each $\widetilde{g}_j$ can be taken to be a linear combination*

$$\widetilde{g}_j = a_j . g = \sum_{1 \le k \le r} a_{jk} g_k, \qquad a_j \in \mathbb{C}^r \smallsetminus \{0\}$$

where the coefficients $([a_1], \ldots, [a_n])$ are chosen in the complement of a proper analytic subset of $(\mathbb{P}^{r-1})^n$.

It follows from the Lemma that the ideal $\mathcal{J} = (\widetilde{g}_1, \ldots, \widetilde{g}_n) \subset \mathcal{I}$ satisfies $\overline{\mathcal{J}}^{(k)} = \overline{\mathcal{I}}^{(k)}$ and $\widehat{\mathcal{J}}^{(k)} = \widehat{\mathcal{I}}^{(k)}$ for all k.

Proof. Assume that $g \in \mathcal{O}(\Omega)^r$. Consider the analytic subsets in $\Omega \times (\mathbb{P}^{r-1})^n$ defined by

$$A = \big\{(z, [w_1], \ldots, [w_n])\,;\, w_j . g(z) = 0\big\},$$
$$A^\star = \bigcup \text{irreducible components of } A \text{ not contained in } g^{-1}(0) \times (\mathbb{P}^{r-1})^n.$$

For $z \notin g^{-1}(0)$ the fiber $A_z = \{([w_1], \ldots, [w_n])\,;\, w_j . g(z) = 0\} = A^\star_z$ is a product of n hyperplanes in $\mathbb{P}^{r-1}$, hence $A \cap (\Omega \smallsetminus g^{-1}(0)) \times (\mathbb{P}^{r-1})^n$ is a fiber bundle with base $\Omega \smallsetminus g^{-1}(0)$ and fiber $(\mathbb{P}^{r-2})^n$. As $A^\star$ is the closure of this set in $\Omega \times (\mathbb{P}^{r-1})^n$, we have

$$\dim A^\star = n + n(r-2) = n(r-1) = \dim(\mathbb{P}^{r-1})^n.$$

It follows that the zero fiber

$$A^\star_0 = A^\star \cap \big(\{0\} \times (\mathbb{P}^{r-1})^n\big)$$

is a proper subset of $\{0\} \times (\mathbb{P}^{r-1})^n$. Choose $(a_1, \ldots, a_n) \in (\mathbb{C}^r \smallsetminus \{0\})^n$ such that $(0, [a_1], \ldots, [a_n])$ is not in $A^\star_0$. By an easy compactness argument the set $A^\star \cap \big(\overline{B}(0,\varepsilon) \times (\mathbb{P}^{r-1})^n\big)$ is disjoint from the neighborhood $B(0,\varepsilon) \times \prod [B(a_j, \varepsilon)]$ of $(0, [a_1], \ldots, [a_n])$ for ε small enough. For $z \in B(0,\varepsilon)$ we have $|a_j . g(z)| \ge \varepsilon |g(z)|$ for some j, otherwise the inequality $|a_j . g(z)| < \varepsilon |g(z)|$ would imply the existence of $h_j \in \mathbb{C}^r$ with $|h_j| < \varepsilon$ and $a_j . g(z) = h_j . g(z)$. Since $g(z) \ne 0$, we would have

$$(z, [a_1 - h_1], \ldots, [a_n - h_n]) \in A^\star \cap \big(B(0,\varepsilon) \times (\mathbb{P}^{r-1})^n\big),$$

a contradiction. We obtain therefore

$$\varepsilon |g(z)| \le \max |a_j . g(z)| \le (\max |a_j|)\, |g(z)| \quad \text{on } B(0,\varepsilon). \qquad \square$$

(12.17) Theorem (Briançon-Skoda [BSk74]). *Set $p = \min\{n-1, r-1\}$. Then*

a) $\widehat{\mathcal{I}}^{(k+1)} = \mathcal{I}\widehat{\mathcal{I}}^{(k)} = \overline{\mathcal{I}}\widehat{\mathcal{I}}^{(k)}$ *for $k \ge p$.*

b) $\overline{\mathcal{I}}^{(k+p)} \subset \widehat{\mathcal{I}}^{(k+p)} \subset \mathcal{I}^k$ *for all $k \in \mathbb{N}$.*

Proof. a) The inclusions $\mathcal{I}\widehat{\mathcal{I}}^{(k)} \subset \overline{\mathcal{I}}\widehat{\mathcal{I}}^{(k)} \subset \widehat{\mathcal{I}}^{(k+1)}$ are obvious thanks to Proposition 12.15, so we only have to prove that $\widehat{\mathcal{I}}^{(k+1)} \subset \mathcal{I}\widehat{\mathcal{I}}^{(k)}$. Assume first that $r \le n$. Let $f \in \widehat{\mathcal{I}}^{(k+1)}$ be such that

$$\int_\Omega |f|^2\,|g|^{-2(k+1+\varepsilon)}\,dV < +\infty.$$

For $k \ge p-1$, we can apply Theorem 12.13 with $m = r-1$ and with the weight $\varphi = (k-m)\log|g|^2$. Hence f can be written $f = \sum g_j h_j$ with

$$\int_\Omega |h|^2\,|g|^{-2(k+\varepsilon)}\,dV < +\infty,$$

thus $h_j \in \widehat{\mathcal{I}}^{(k)}$ and $f \in \mathcal{I}\widehat{\mathcal{I}}^{(k)}$. When $r > n$, Lemma 12.16 shows that there is an ideal $\mathcal{J} \subset \mathcal{I}$ with n generators such that $\widehat{\mathcal{J}}^{(k)} = \widehat{\mathcal{I}}^{(k)}$. We find

$$\widehat{\mathcal{I}}^{(k+1)} = \widehat{\mathcal{J}}^{(k+1)} \subset \mathcal{J}\,\widehat{\mathcal{J}}^{(k)} \subset \mathcal{I}\widehat{\mathcal{I}}^{(k)} \qquad \text{for } \ k \ge n-1.$$

b) Property a) implies inductively $\widehat{\mathcal{I}}^{(k+p)} = \mathcal{I}^k\,\widehat{\mathcal{I}}^{(p)}$ for all $k \in \mathbb{N}$. This gives in particular $\widehat{\mathcal{I}}^{(k+p)} \subset \mathcal{I}^k$. □

(12.18) Corollary.

a) *The ideal $\overline{\mathcal{I}}$ is the integral closure of $\mathcal{I}$, i.e. by definition the set of germs $u \in \mathcal{O}_n$ which satisfy an equation*

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0, \qquad a_s \in \mathcal{I}^s, \qquad 1 \le s \le d.$$

b) *Similarly, $\overline{\mathcal{I}}^{(k)}$ is the set of germs $u \in \mathcal{O}_n$ which satisfy an equation*

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0, \qquad a_s \in \mathcal{I}^{\lceil ks\rceil}, \qquad 1 \le s \le d,$$

where $\lceil t\rceil$ denotes the smallest integer $\ge t$.

As the ideal $\overline{\mathcal{I}}^{(k)}$ is finitely generated, property b) shows that there always exists a rational number $l \ge k$ such that $\overline{\mathcal{I}}^{(l)} = \overline{\mathcal{I}}^{(k)}$.

Proof. a) If $u \in \mathcal{O}_n$ satisfies a polynomial equation with coefficients $a_s \in \mathcal{I}^s$, then clearly $|a_s| \le C_s\,|g|^s$ and the usual elementary bound

$$|\text{roots}| \le 2 \max_{1\le s\le d} |a_s|^{1/s}$$

for the roots of a monic polynomial implies $|u| \le C\,|g|$.

Conversely, assume that $u \in \overline{\mathcal{I}}$. The ring $\mathcal{O}_n$ is Noetherian, so the ideal $\widehat{\mathcal{I}}^{(p)}$ has a finite number of generators $v_1, \ldots, v_N$. For every j we have $uv_j \in \overline{\mathcal{I}}\widehat{\mathcal{I}}^{(p)} = \mathcal{I}\widehat{\mathcal{I}}^{(p)}$, hence there exist elements $b_{jk} \in \mathcal{I}$ such that

$$uv_j = \sum_{1\le k\le N} b_{jk} v_k.$$

The matrix $(u\delta_{jk} - b_{jk})$ has the non zero vector (v_j) in its kernel, thus u satisfies the equation $\det(u\delta_{jk} - b_{jk}) = 0$, which is of the required type.

b) Observe that $v_1, \ldots, v_N$ satisfy simultaneously some integrability condition $\int_\Omega |v_j|^{-2(p+\varepsilon)} < +\infty$, thus $\widehat{\mathcal{I}}^{(p)} = \widehat{\mathcal{I}}^{(p+\eta)}$ for $\eta \in [0, \varepsilon[$. Let $u \in \overline{\mathcal{I}}^{(k)}$. For every integer $m \in \mathbb{N}$ we have

$$u^m v_j \in \overline{\mathcal{I}}^{(km)}\,\widehat{\mathcal{I}}^{(p+\eta)} \subset \widehat{\mathcal{I}}^{(km+\eta+p)}.$$

If $k \notin \mathbb{Q}$, we can find m such that $d(km+\varepsilon/2,\mathbb{Z}) < \varepsilon/2$, thus $km+\eta \in \mathbb{N}$ for some $\eta \in \,]0,\varepsilon[$. If $k \in \mathbb{Q}$, we take m such that $km \in \mathbb{N}$ and $\eta = 0$. Then

$$u^m v_j \in \widehat{\mathcal{I}}^{(N+p)} = \mathcal{I}^N\,\widehat{\mathcal{I}}^{(p)} \quad \text{with} \quad N = km+\eta \in \mathbb{N},$$

and the reasoning made in a) gives $\det(u^m\delta_{jk} - b_{jk}) = 0$ for some $b_{jk} \in \mathcal{I}^N$. This is an equation of the type described in b), where the coefficients a_s vanish when s is not a multiple of m and $a_{ms} \in \mathcal{I}^{Ns} \subset \mathcal{I}^{\lceil kms\rceil}$. □

Let us mention that Briançon and Skoda's result 12.17 b) is optimal for $k = 1$. Take for example $\mathcal{I} = (g_1,\ldots,g_r)$ with $g_j(z) = z_j^r$, $1 \le j \le r$, and $f(z) = z_1\ldots z_r$. Then $|f| \le C|g|$ and 12.17 b) yields $f^r \in \mathcal{I}$; however, it is easy to verify that $f^{r-1} \notin \mathcal{I}$. The theorem also gives an answer to the following conjecture made by J. Mather.

(12.19) Corollary. *Let $f \in \mathcal{O}_n$ and $\mathcal{I}_f = (z_1\partial f/\partial z_1,\ldots,z_n\partial f/\partial z_n)$. Then $f \in \overline{\mathcal{I}}_f$, and for every integer $k \ge 0$, $f^{k+n-1} \in \mathcal{I}_f^k$.*

The Corollary is also optimal for $k = 1$: for example, one can verify that the function $f(z) = (z_1\ldots z_n)^3 + z_1^{3n-1} + \cdots + z_n^{3n-1}$ is such that $f^{n-1} \notin \mathcal{I}_f$.

Proof. Set $g_j(z) = z_j\,\partial f/\partial z_j$, $1 \le j \le n$. By 12.17 b), it suffices to show that $|f| \le C|g|$. For every germ of analytic curve $\mathbb{C} \ni t \longmapsto \gamma(t)$, $\gamma \not\equiv 0$, the vanishing order of $f\circ\gamma(t)$ at $t = 0$ is the same as that of

$$t\,\frac{d(f\circ\gamma)}{dt} = \sum_{1\le j\le n} t\,\gamma_j'(t)\,\frac{\partial f}{\partial z_j}\big(\gamma(t)\big).$$

We thus obtain

$$|f\circ\gamma(t)| \le C_1\,|t|\,\Big|\frac{d(f\circ\gamma)}{dt}\Big| \le C_2 \sum_{1\le j\le n} |t\,\gamma_j'(t)|\,\Big|\frac{\partial f}{\partial z_j}\big(\gamma(t)\big)\Big| \le C_3\,|g\circ\gamma(t)|$$

and conclude by the following elementary lemma. □

(12.20) Curve selection lemma. *Let $f, g_1,\ldots,g_r \in \mathcal{O}_n$ be germs of holomorphic functions vanishing at 0. Then we have $|f| \le C|g|$ for some constant C if and only if for every germ of analytic curve γ through 0 there exists a constant C_γ such that $|f\circ\gamma| \le C_\gamma|g\circ\gamma|$.*

Proof. If the inequality $|f| \le C|g|$ does not hold on any neighborhood of 0, the germ of analytic set $(A,0) \subset (\mathbb{C}^{n+r},0)$ defined by

$$g_j(z) = f(z)z_{n+j}, \qquad 1 \le j \le r,$$

contains a sequence of points $\big(z_\nu, g_j(z_\nu)/f(z_\nu)\big)$ converging to 0 as ν tends to $+\infty$, with $f(z_\nu) \ne 0$. Hence $(A,0)$ contains an irreducible component on which $f \not\equiv 0$ and there is a germ of curve $\widetilde{\gamma} = (\gamma,\gamma_{n+j}) : (\mathbb{C},0) \to (\mathbb{C}^{n+r},0)$ contained in $(A,0)$ such that $f \circ \gamma \not\equiv 0$. We get $g_j \circ \gamma = (f \circ \gamma)\gamma_{n+j}$, hence $|g \circ \gamma(t)| \le C|t|\,|f \circ \gamma(t)|$ and the inequality $|f \circ \gamma| \le C_\gamma |g \circ \gamma|$ does not hold. □

13. The Ohsawa-Takegoshi L^2 Extension Theorem

We address here the following extension problem: let Y be a complex analytic submanifold of a complex manifold X ; given a holomorphic function f on Y satisfying suitable L^2 conditions on Y, find a holomorphic extension F of f to X, together with a good L^2 estimate for F on X. The first satisfactory solution has been obtained only rather recently by Ohsawa-Takegoshi [OT87, Ohs88]. We follow here a more geometric approach due to Manivel [Man93], which provides a generalized extension theorem in the general framework of vector bundles. As in Ohsawa-Takegoshi's fundamental paper, the main idea is to use a modified Bochner-Kodaira-Nakano inequality. Such inequalities were originally introduced in the work of Donnelly-Fefferman [DF83] and Donnelly-Xavier [DX84]. The main a priori inequality we are going to use is a simplified (and slightly extended) version of the original Ohsawa-Takegoshi a priori inequality, as proposed recently by Ohsawa [Ohs95].

(13.1) Lemma (Ohsawa [Ohs95]). *Let E be a hermitian vector bundle on a complex manifold X equipped with a Kähler metric ω. Let $\eta,\ \lambda > 0$ be smooth functions on X. Then for every form $u \in \mathcal{D}(X, \Lambda^{p,q}T^\star_X \otimes E)$ with compact support we have*

$$\begin{aligned}\|(\eta^{\frac12} + \lambda^{\frac12})D''^\star u\|^2 + \|\eta^{\frac12} D'' u\|^2 + \|\lambda^{\frac12} D' u\|^2 + 2\|\lambda^{-\frac12} d'\eta \wedge u\|^2 \\ \ge \langle\!\langle [\eta\,\mathrm{i}\,\Theta(E) - \mathrm{i}\ d'd''\eta - \mathrm{i}\,\lambda^{-1} d'\eta \wedge d''\eta, \Lambda]u, u\rangle\!\rangle.\end{aligned}$$

Proof. Let us consider the "twisted" Laplace-Beltrami operators

$$\begin{aligned}D'\eta D'^\star + D'^\star\eta D' &= \eta[D', D'^\star] + [D', \eta]D'^\star + [D'^\star, \eta]D' \\ &= \eta\Delta' + (d'\eta)D'^\star - (d'\eta)^* D', \\ D''\eta D''^\star + D''^\star\eta D'' &= \eta[D'', D''^\star] + [D'', \eta]D''^\star + [D''^\star, \eta]D'' \\ &= \eta\Delta'' + (d''\eta)D''^\star - (d''\eta)^* D'',\end{aligned}$$

where η, $(d'\eta)$, $(d''\eta)$ are abbreviated notations for the multiplication operators $\eta\bullet$, $(d'\eta)\wedge\bullet$, $(d''\eta)\wedge\bullet$. By subtracting the above equalities and taking into account the Bochner-Kodaira-Nakano identity $\Delta'' - \Delta' = [\mathrm{i}\,\Theta(E), \Lambda]$, we get

$$\begin{aligned}&D''\eta D''^\star + D''^\star\eta D'' - D'\eta D'^\star - D'^\star\eta D' \\ &\quad = \eta[\mathrm{i}\,\Theta(E), \Lambda] + (d''\eta)D''^\star - (d''\eta)^\star D'' + (d'\eta)^\star D' - (d'\eta)D'^\star.\end{aligned}\tag{13.2}$$

Moreover, the Jacobi identity yields

$$[D'', [d'\eta, \Lambda]] - [d'\eta, [\Lambda, D'']] + [\Lambda, [D'', d'\eta]] = 0,$$

whilst $[\Lambda, D''] = -\mathrm{i}\,D'^\star$ by the basic commutation relations 4.22. A straightforward computation shows that $[D'', d'\eta] = -(d'd''\eta)$ and $[d'\eta, \Lambda] = \mathrm{i}\,(d''\eta)^\star$. Therefore we get

$$\mathrm{i}\,[D'', (d''\eta)^\star] + \mathrm{i}\,[d'\eta, D'^\star] - [\Lambda, (d'd''\eta)] = 0,$$

that is,

$$[\mathrm{i}\ d'd''\eta, \Lambda] = [D'', (d''\eta)^\star] + [D'^\star, d'\eta] = D''(d''\eta)^\star + (d''\eta)^\star D'' + D'^\star(d'\eta) + (d'\eta)D'^\star.$$

After adding this to (13.2), we find

$$\begin{aligned} &D''\eta D''^\star + D''^\star\eta D'' - D'\eta D'^\star - D'^\star\eta D' + [\mathrm{i}\ d'd''\eta, \Lambda] \\ &\qquad = \eta[\mathrm{i}\,\Theta(E), \Lambda] + (d''\eta)D''^\star + D''(d''\eta)^\star + (d'\eta)^\star D' + D'^\star(d'\eta). \end{aligned}$$

We apply this identity to a form $u \in \mathcal{D}(X, \Lambda^{p,q}T_X^\star \otimes E)$ and take the inner bracket with u. Then

$$\langle\!\langle (D''\eta D''^\star)u, u\rangle\!\rangle = \langle\!\langle \eta D''^\star u, D''^\star u\rangle\!\rangle = \|\eta^{\frac12} D''^\star u\|^2,$$

and likewise for the other similar terms. The above equalities imply

$$\begin{aligned} &\|\eta^{\frac12} D''^\star u\|^2 + \|\eta^{\frac12} D'' u\|^2 - \|\eta^{\frac12} D' u\|^2 - \|\eta^{\frac12} D'^\star u\|^2 = \\ &\qquad \langle\!\langle [\eta\,\mathrm{i}\,\Theta(E) - \mathrm{i}\ d'd''\eta, \Lambda]u, u\rangle\!\rangle + 2\,\mathrm{Re}\,\langle\!\langle D''^\star u, (d''\eta)^\star u\rangle\!\rangle + 2\,\mathrm{Re}\,\langle\!\langle D'u, d'\eta\wedge u\rangle\!\rangle. \end{aligned}$$

By neglecting the negative terms $-\|\eta^{\frac12} D'u\|^2 - \|\eta^{\frac12} D'^\star u\|^2$ and adding the squares

$$\begin{aligned} \|\lambda^{\frac12} D''^\star u\|^2 + 2\,\mathrm{Re}\,\langle\!\langle D''^\star u, (d''\eta)^\star u\rangle\!\rangle + \|\lambda^{-\frac12}(d''\eta)^\star u\|^2 \geq 0, \\ \|\lambda^{\frac12} D'u\|^2 + 2\,\mathrm{Re}\,\langle\!\langle D'u, d'\eta\wedge u\rangle\!\rangle + \|\lambda^{-\frac12} d'\eta\wedge u\|^2 \geq 0 \end{aligned}$$

we get

$$\begin{aligned} &\|(\eta^{\frac12} + \lambda^{\frac12})D''^\star u\|^2 + \|\eta^{\frac12} D''u\|^2 + \|\lambda^{\frac12} D'u\|^2 + \|\lambda^{-\frac12} d'\eta\wedge u\|^2 + \|\lambda^{-\frac12}(d''\eta)^\star u\|^2 \\ &\qquad \geq \langle\!\langle [\eta\,\mathrm{i}\,\Theta(E) - \mathrm{i}\ d'd''\eta, \Lambda]u, u\rangle\!\rangle. \end{aligned}$$

Finally, we use the identities

$$\begin{aligned} (d'\eta)^\star(d'\eta) - (d''\eta)(d''\eta)^\star &= \mathrm{i}\,[d''\eta, \Lambda](d'\eta) + \mathrm{i}\,(d''\eta)[d'\eta, \Lambda] = [\mathrm{i}\,d''\eta\wedge d'\eta, \Lambda], \\ \|\lambda^{-\frac12} d'\eta\wedge u\|^2 - \|\lambda^{-\frac12}(d''\eta)^\star u\|^2 &= -\langle\!\langle [\mathrm{i}\,\lambda^{-1} d'\eta\wedge d''\eta, \Lambda]u, u\rangle\!\rangle, \end{aligned}$$

The inequality asserted in Lemma 13.1 follows by adding the second identity to our last inequality. □

In the special case of (n, q)-forms, the forms $D'u$ and $d'\eta\wedge u$ are of bidegree $(n+1, q)$, hence the estimate takes the simpler form

(13.3)
$$\|(\eta^{\frac12} + \lambda^{\frac12})D''^\star u\|^2 + \|\eta^{\frac12} D''u\|^2 \geq \langle\!\langle [\eta\,\mathrm{i}\,\Theta(E) - \mathrm{i}\ d'd''\eta - \mathrm{i}\,\lambda^{-1}\, d'\eta\wedge d''\eta, \Lambda]u, u\rangle\!\rangle.$$

(13.4) Proposition. *Let X be a complete Kähler manifold equipped with a (non necessarily complete) Kähler metric ω, and let E be a hermitian vector bundle over X. Assume that there are smooth and bounded functions η, $\lambda > 0$ on X such that the (hermitian) curvature operator $B = B^{n,q}_{E,\omega,\eta} = [\eta\,\mathrm{i}\,\Theta(E) - \mathrm{i}\ d'd''\eta - \mathrm{i}\,\lambda^{-1} d'\eta\wedge d''\eta, \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{n,q}T_X^\star\otimes E$, for some $q \geq 1$. Then for every form $g \in L^2(X, \Lambda^{n,q}T_X^\star\otimes E)$ such that $D''g = 0$ and $\int_X \langle B^{-1}g, g\rangle\, dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{n,q-1}T_X^\star\otimes E)$ such that $D''f = g$ and*

$$\int_X (\eta+\lambda)^{-1}|f|^2\, dV_\omega \leq 2\int_X \langle B^{-1}g, g\rangle\, dV_\omega.$$

Proof. The proof is almost identical to the proof of Theorem 6.1, except that we use (13.3) instead of (5.1). Assume first that ω is complete. With the same notation as in 8.4, we get for every $v = v_1 + v_2 \in (\mathrm{Ker}\, D'') \oplus (\mathrm{Ker}\, D'')^\perp$ the inequalities

$$|\langle g, v\rangle|^2 = |\langle g, v_1\rangle|^2 \le \int_X \langle B^{-1}g, g\rangle\, dV_\omega \int_X \langle Bv_1, v_1\rangle\, dV_\omega,$$

and

$$\int_X \langle Bv_1, v_1\rangle\, dV_\omega \le \|(\eta^{\frac12} + \lambda^{\frac12})D''^\star v_1\|^2 + \|\eta^{\frac12} D'' v_1\|^2 = \|(\eta^{\frac12} + \lambda^{\frac12})D''^\star v\|^2$$

provided that $v \in \mathrm{Dom}\, D''^\star$. Combining both, we find

$$|\langle g, v\rangle|^2 \le \Big(\int_X \langle B^{-1}g, g\rangle\, dV_\omega\Big)\|(\eta^{\frac12} + \lambda^{\frac12})D''^\star v\|^2.$$

This shows the existence of an element $w \in L^2(X, \Lambda^{n,q}T^\star_X \otimes E)$ such that

$$\|w\|^2 \le \int_X \langle B^{-1}g, g\rangle\, dV_\omega \qquad \text{and}$$
$$\langle\!\langle v, g\rangle\!\rangle = \langle\!\langle (\eta^{\frac12} + \lambda^{\frac12})D''^\star v, w\rangle\!\rangle \qquad \forall g \in \mathrm{Dom}\, D'' \cap \mathrm{Dom}\, D''^\star.$$

As $(\eta^{1/2} + \lambda^{\frac12})^2 \le 2(\eta + \lambda)$, it follows that $f = (\eta^{1/2} + \lambda^{\frac12})w$ satisfies $D''f = g$ as well as the desired L^2 estimate. If ω is not complete, we set $\omega_\varepsilon = \omega + \varepsilon\widehat{\omega}$ with some complete Kähler metric $\widehat{\omega}$. The final conclusion is then obtained by passing to the limit and using a monotonicity argument (the integrals are monotonic with respect to ε). □

(13.5) Remark. We will also need a variant of the L^2-estimate, so as to obtain approximate solutions with weaker requirements on the data: given $\delta > 0$ and $g \in L^2(X, \Lambda^{n,q}T^\star_X \otimes E)$ such that $D''g = 0$ and $\int_X \langle (B + \delta I)^{-1}g, g\rangle\, dV_\omega < +\infty$, there exists an approximate solution $f \in L^2(X, \Lambda^{n,q-1}T^\star_X \otimes E)$ and a correcting term $h \in L^2(X, \Lambda^{n,q}T^\star_X \otimes E)$ such that $D''f + \delta^{1/2}h = g$ and

$$\int_X (\eta + \lambda)^{-1}|f|^2\, dV_\omega + \int_X |h|^2\, dV_\omega \le 2\int_X \langle (B + \delta I)^{-1}g, g\rangle\, dV_\omega.$$

The proof is almost unchanged, we rely instead on the estimates

$$|\langle g, v_1\rangle|^2 \le \int_X \langle (B + \delta I)^{-1}g, g\rangle\, dV_\omega \int_X \langle (B + \delta I)v_1, v_1\rangle\, dV_\omega,$$

and

$$\int_X \langle (B + \delta I)v_1, v_1\rangle\, dV_\omega \le \|(\eta^{\frac12} + \lambda^{\frac12})D''^\star v\|^2 + \delta\|v\|^2. \qquad □$$

(13.6) Theorem. *Let X be a weakly pseudoconvex n-dimensional complex manifold equipped with a Kähler metric ω, let L (resp. E) be a hermitian holomorphic line bundle (resp. a hermitian holomorphic vector bundle of rank r over X), and s a global holomorphic section of E. Assume that s is generically transverse to the zero section, and let*

$$Y = \big\{x \in X\,;\ s(x) = 0, \Lambda^r ds(x) \ne 0\big\}, \qquad p = \dim Y = n - r.$$

Moreover, assume that the $(1,1)$*-form* $\mathrm{i}\,\Theta(L) + r\,\mathrm{i}\,d'd''\log|s|^2$ *is semipositive and that there is a continuous function* $\alpha \ge 1$ *such that the following two inequalities hold everywhere on* X :

a) $\mathrm{i}\,\Theta(L) + r\,\mathrm{i}\,d'd''\log|s|^2 \ge \alpha^{-1}\dfrac{\{\mathrm{i}\,\Theta(E)s, s\}}{|s|^2}$,

b) $|s| \le e^{-\alpha}$.

Then for every smooth D''*-closed* $(0,q)$*-form* f *over* Y *with values in the line bundle* $\Lambda^n T_X^\star \otimes L$ *(restricted to* Y*), such that* $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} dV_\omega < +\infty$*, there exists a* D''*-closed* $(0,q)$*-form* F *over* X *with values in* $\Lambda^n T_X^\star \otimes L$*, such that* F *is smooth over* $X \smallsetminus \{s = \Lambda^r(ds) = 0\}$*, satisfies* $F_{\restriction Y} = f$ *and*

$$\int_X \frac{|F|^2}{|s|^{2r}(-\log|s|)^2}\, dV_{X,\omega} \le C_r \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega}\,,$$

where C_r *is a numerical constant depending only on* r*.*

Observe that the differential ds (which is intrinsically defined only at points where s vanishes) induces a vector bundle isomorphism $ds : T_X/T_Y \to E$ along Y, hence a non vanishing section $\Lambda^r(ds)$, taking values in

$$\Lambda^r(T_X/T_Y)^\star \otimes \det E \subset \Lambda^r T_X^\star \otimes \det E.$$

The norm $|\Lambda^r(ds)|$ is computed here with respect to the metrics on $\Lambda^r T_X^\star$ and $\det E$ induced by the Kähler metric ω and by the given metric on E. Also notice that if hypothesis a) is satisfied for some α, one can always achieve b) by multiplying the metric of E with a sufficiently small weight $e^{-\chi\circ\psi}$ (with ψ a psh exhaustion on X and χ a convex increasing function; property a) remains valid after we multiply the metric of L by $e^{-(r+\alpha_0^{-1})\chi\circ\psi}$, where $\alpha_0 = \inf_{x\in X}\alpha(x)$.

Proof. Let us first assume that the singularity set $\Sigma = \{s = 0\} \cap \{\Lambda^r(ds) = 0\}$ is empty, so that Y is closed and nonsingular. We claim that there exists a smooth section

$$F_\infty \in C^\infty(X, \Lambda^{n,q} T_X^\star \otimes L) = C^\infty(X, \Lambda^{0,q} T_X^\star \otimes \Lambda^n T_X^\star \otimes L)$$

such that

(a) F_∞ coincides with f in restriction to Y,

(b) $|F_\infty| = |f|$ at every point of Y,

(c) $D''F_\infty = 0$ at every point of Y.

For this, consider coordinates patches $U_j \subset X$ biholomorphic to polydiscs such that

$$U_j \cap Y = \{z \in U_j\,;\ z_1 = \ldots = z_r = 0\}$$

in the corresponding coordinates. We can find a section $\widetilde{f}$ in $C^\infty(X, \Lambda^{n,q} T_X^\star \otimes L)$ which achieves a) and b), since the restriction map $(\Lambda^{0,q} T_X^*)_{\restriction Y} \to \Lambda^{0,q} T_Y^*$ can be viewed as an orthogonal projection onto a C^∞-subbundle of $(\Lambda^{0,q} T_X^*)_{\restriction Y}$. It is enough to extend this subbundle from $U_j \cap Y$ to U_j (e.g. by extending each component of a frame), and then to extend f globally via local smooth extensions and a partition of unity. For any such extension $\widetilde{f}$ we have

$$(D''\widetilde{f})_{\restriction Y} = (D''\widetilde{f}_{\restriction Y}) = D''f = 0.$$

It follows that we can divide $D''\widetilde{f} = \sum_{1\le\lambda\le r} g_{j,\lambda}(z)\wedge d\overline{z}_\lambda$ on $U_j\cap Y$, with suitable smooth $(0,q)$-forms $g_{j,\lambda}$ which we also extend arbitrarily from $U_j\cap Y$ to U_j. Then

$$F_\infty := \widetilde{f} - \sum_j \theta_j(z) \sum_{1\le\lambda\le r} \overline{z}_\lambda g_{j,\lambda}(z)$$

coincides with $\widetilde{f}$ on Y and satisfies (c). Since we do not know about F_∞ except in an infinitesimal neighborhood of Y, we will consider a truncation F_ε of F_∞ with support in a small tubular neighborhood $|s|<\varepsilon$ of Y, and solve the equation $D''u_\varepsilon = D''F_\varepsilon$ with the constraint that u_ε should be 0 on Y. As $\operatorname{codim} Y = r$, this will be the case if we can guarantee that $|u_\varepsilon|^2|s|^{-2r}$ is locally integrable near Y. For this, we will apply Proposition 13.4 with a suitable choice of the functions η and λ, and an additional weight $|s|^{-2r}$ in the metric of L.

Let us consider the smooth strictly convex function $\chi_0 : \,]-\infty,0]\to\,]-\infty,0]$ defined by $\chi_0(t) = t - \log(1-t)$ for $t\le 0$, which is such that $\chi_0(t)\le t$, $1\le\chi_0'\le 2$ and $\chi_0''(t) = 1/(1-t)^2$. We set

$$\sigma_\varepsilon = \log(|s|^2+\varepsilon^2), \qquad \eta_\varepsilon = \varepsilon - \chi_0(\sigma_\varepsilon).$$

As $|s|\le e^{-\alpha}\le e^{-1}$, we have $\sigma_\varepsilon\le 0$ for ε small, and

$$\eta_\varepsilon \ge \varepsilon - \sigma_\varepsilon \ge \varepsilon - \log(e^{-2\alpha}+\varepsilon^2).$$

Given a relatively compact subset $X_c = \{\psi < c\}\subset\subset X$, we thus have $\eta_\varepsilon\ge 2\alpha$ for $\varepsilon<\varepsilon(c)$ small enough. Simple calculations yield

$$\begin{aligned}
\mathrm{i}\,d'\sigma_\varepsilon &= \frac{\mathrm{i}\,\{D's,s\}}{|s|^2+\varepsilon^2},\\
\mathrm{i}\,d'd''\sigma_\varepsilon &= \frac{\mathrm{i}\,\{D's,D's\}}{|s|^2+\varepsilon^2} - \frac{\mathrm{i}\,\{D's,s\}\wedge\{s,D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{\{\mathrm{i}\,\Theta(E)s,s\}}{|s|^2+\varepsilon^2}\\
&\ge \frac{\varepsilon^2}{|s|^2}\frac{\mathrm{i}\,\{D's,s\}\wedge\{s,D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{\{\mathrm{i}\,\Theta(E)s,s\}}{|s|^2+\varepsilon^2}\\
&\ge \frac{\varepsilon^2}{|s|^2}\mathrm{i}\,d'\sigma_\varepsilon\wedge d''\sigma_\varepsilon - \frac{\{\mathrm{i}\,\Theta(E)s,s\}}{|s|^2+\varepsilon^2},
\end{aligned}$$

thanks to Lagrange's inequality $\mathrm{i}\,\{D's,s\}\wedge\{s,D's\}\le|s|^2\mathrm{i}\,\{D's,D's\}$. On the other hand, we have $d'\eta_\varepsilon = -\chi_0'(\sigma_\varepsilon)d\sigma_\varepsilon$ with $1\le\chi_0'(\sigma_\varepsilon)\le 2$, hence

$$\begin{aligned}
-\mathrm{i}\,d'd''\eta_\varepsilon &= \chi_0'(\sigma_\varepsilon)\mathrm{i}\,d'd''\sigma_\varepsilon + \chi_0''(\sigma_\varepsilon)\mathrm{i}\,d'\sigma_\varepsilon\wedge d''\sigma_\varepsilon\\
&\ge \Big(\frac{1}{\chi_0'(\sigma_\varepsilon)}\frac{\varepsilon^2}{|s|^2} + \frac{\chi_0''(\sigma_\varepsilon)}{\chi_0'(\sigma_\varepsilon)^2}\Big)\mathrm{i}\,d'\eta_\varepsilon\wedge d''\eta_\varepsilon - \chi_0'(\sigma_\varepsilon)\frac{\{\mathrm{i}\,\Theta_E s,s\}}{|s|^2+\varepsilon^2}.
\end{aligned}$$

We consider the original metric of L multiplied by the weight $|s|^{-2r}$. In this way, we get a curvature form

$$\mathrm{i}\,\Theta_L + r\,\mathrm{i}\,d'd''\log|s|^2 \ge \frac{1}{2}\chi_0'(\sigma_\varepsilon)\alpha^{-1}\frac{\{\mathrm{i}\,\Theta_E s,s\}}{|s|^2+\varepsilon^2}$$

by hypothesis a), thanks to the semipositivity of the left hand side and the fact that $\frac{1}{2}\chi_0'(\sigma_\varepsilon)\frac{1}{|s|^2+\varepsilon^2}\le\frac{1}{|s|^2}$. As $\eta_\varepsilon\ge 2\alpha$ on X_c for ε small, we infer

$$\eta_\varepsilon(\mathrm{i}\,\Theta_L+\mathrm{i}\,d'd''\log|s|^2)-\mathrm{i}\,d'd''\eta_\varepsilon-\frac{\chi_0''(\sigma_\varepsilon)}{\chi_0'(\sigma_\varepsilon)^2}\mathrm{i}\,d'\eta_\varepsilon\wedge d''\eta_\varepsilon\geq\frac{\varepsilon^2}{\chi_0'(\sigma_\varepsilon)|s|^2}\mathrm{i}\,d'\eta_\varepsilon\wedge d''\eta_\varepsilon$$

on X_c. Hence, if $\lambda_\varepsilon=\chi_0'(\sigma_\varepsilon)^2/\chi_0''(\sigma_\varepsilon)$, we obtain

$$\begin{aligned}B_\varepsilon&:=\big[\eta_\varepsilon(\mathrm{i}\,\Theta_L+\mathrm{i}\,d'd''\log|s|^2)-\mathrm{i}\,d'd''\eta_\varepsilon-\lambda_\varepsilon^{-1}\mathrm{i}\,d'\eta_\varepsilon\wedge d''\eta_\varepsilon\ ,\ \Lambda\big]\\&\geq\Big[\frac{\varepsilon^2}{\chi_0'(\sigma_\varepsilon)|s|^2}\mathrm{i}\,d'\eta_\varepsilon\wedge d''\eta_\varepsilon\ ,\ \Lambda\Big]=\frac{\varepsilon^2}{\chi_0'(\sigma_\varepsilon)|s|^2}(d''\eta_\varepsilon)(d''\eta_\varepsilon)^\star\end{aligned}$$

as an operator on (n,q)-forms (see the proof of Lemma 13.1).

Let $\theta:\mathbb{R}\to[0,1]$ be a smooth cut-off function such that $\theta(t)=1$ on $]-\infty,1/2]$, $\operatorname{Supp}\theta\subset\ ]-\infty,1[$ and $|\theta'|\leq 3$. For $\varepsilon>0$ small, we consider the (n,q)-form $F_\varepsilon=\theta(\varepsilon^{-2}|s|^2)F_\infty$ and its D''-derivative

$$g_\varepsilon=D''F_\varepsilon=(1+\varepsilon^{-2}|s|^2)\theta'(\varepsilon^{-2}|s|^2)d''\sigma_\varepsilon\wedge F_\infty+\theta(\varepsilon^{-2}|s|^2)D''F_\infty$$

[as is easily seen from the equality $1+\varepsilon^{-2}|s|^2=\varepsilon^{-2}e^{\sigma_\varepsilon}$]. We observe that g_ε has its support contained in the tubular neighborhood $|s|<\varepsilon$; moreover, as $\varepsilon\to 0$, the second term in the right hand side converges uniformly to 0 on every compact set; it will therefore produce no contribution in the limit. On the other hand, the first term has the same order of magnitude as $d''\sigma_\varepsilon$ and $d''\eta_\varepsilon$, and can be controlled in terms of B_ε. In fact, for any (n,q)-form u and any $(n,q+1)$-form v we have

$$\begin{aligned}|\langle d''\eta_\varepsilon\wedge u,v\rangle|^2&=|\langle u,(d''\eta_\varepsilon)^\star v\rangle|^2\leq|u|^2|(d''\eta_\varepsilon)^\star v|^2=|u|^2\langle(d''\eta_\varepsilon)(d''\eta_\varepsilon)^\star v,v\rangle\\&\leq\frac{\chi_0'(\sigma_\varepsilon)|s|^2}{\varepsilon^2}|u|^2\langle B_\varepsilon v,v\rangle.\end{aligned}$$

This implies

$$\langle B_\varepsilon^{-1}(d''\eta_\varepsilon\wedge u),(d''\eta_\varepsilon\wedge u)\rangle\leq\frac{\chi_0'(\sigma_\varepsilon)|s|^2}{\varepsilon^2}|u|^2.$$

The main term in g_ε can be written

$$g_\varepsilon^{(1)}:=(1+\varepsilon^{-2}|s|^2)\theta'(\varepsilon^{-2}|s|^2)\chi_0'(\sigma_\varepsilon)^{-1}d''\eta_\varepsilon\wedge F_\infty.$$

On $\operatorname{Supp}g_\varepsilon^{(1)}\subset\{|s|<\varepsilon\}$, since $\chi_0'(\sigma_\varepsilon)\geq 1$, we thus find

$$\langle B_\varepsilon^{-1}g_\varepsilon^{(1)},g_\varepsilon^{(1)}\rangle\leq(1+\varepsilon^{-2}|s|^2)^2\,\theta'(\varepsilon^{-2}|s|^2)^2|F_\infty|^2.$$

Instead of working on X itself, we will work rather on the relatively compact subset $X_c\smallsetminus Y_c$, where $Y_c=Y\cap X_c=Y\cap\{\psi<c\}$. We know that $X_c\smallsetminus Y_c$ is again complete Kähler by Lemma 12.9. In this way, we avoid the singularity of the weight $|s|^{-2r}$ along Y. We find

$$\int_{X_c\smallsetminus Y_c}\langle B_\varepsilon^{-1}g_\varepsilon^{(1)},g_\varepsilon^{(1)}\rangle\,|s|^{-2r}dV_\omega\leq\int_{X_c\smallsetminus Y_c}|F_\infty|^2(1+\varepsilon^{-2}|s|^2)^2\theta'(\varepsilon^{-2}|s|^2)^2|s|^{-2r}dV_\omega.$$

Now, we let $\varepsilon\to 0$ and view s as "transverse local coordinates" around Y. As F_∞ coincides with f on Y, it is not hard to see that the right hand side converges to $c_r\int_{Y_c}|f|^2|\Lambda^r(ds)|^{-2}dV_{Y,\omega}$ where c_r is the "universal" constant

$$c_r=\int_{z\in\mathbb{C}^r,\ |z|\leq 1}(1+|z|^2)^2\theta'(|z|^2)^2\frac{\mathrm{i}^{r^2}\Lambda^r(dz)\wedge\Lambda^r(d\overline{z})}{|z|^{2r}}<+\infty$$

depending only on r. The second term

$$g_\varepsilon^{(2)} = \theta(\varepsilon^{-2}|s|^2)d''F_\infty$$

in g_ε satisfies $\mathrm{Supp}(g_\varepsilon^{(2)}) \subset \{|s| < \varepsilon\}$ and $|g_\varepsilon^{(2)}| = O(|s|)$ (just look at the Taylor expansion of $d''F_\infty$ near Y). From this we easily conclude that

$$\int_{X_c \smallsetminus Y_c} \langle B_\varepsilon^{-1} g_\varepsilon^{(2)}, g_\varepsilon^{(2)}\rangle\, |s|^{-2r} dV_{X,\omega} = O(\varepsilon^2),$$

provided that B_ε remains locally uniformly bounded below near Y (this is the case for instance if we have strict inequalities in the curvature assumption a)). If this holds true, we apply Proposition 8.4 on $X_c \smallsetminus Y_c$ with the additional weight factor $|s|^{-2r}$. Otherwise, we use the modified estimate stated in Remark 8.5 in order to solve the approximate equation $D''u + \delta^{1/2}h = g_\varepsilon$ with $\delta > 0$ small. This yields sections $u = u_{c,\varepsilon,\delta}$, $h = h_{c,\varepsilon,\delta}$ such that

$$\begin{aligned}\int_{X_c \smallsetminus Y_c} (\eta_\varepsilon + \lambda_\varepsilon)^{-1} |u_{c,\varepsilon,\delta}|^2 |s|^{-2r}\, dV_\omega + \int_{X_c \smallsetminus Y_c} |h_{c,\varepsilon,\delta}|^2 |s|^{-2r}\, dV_\omega \\ \le 2 \int_{X_c \smallsetminus Y_c} \langle (B_\varepsilon + \delta I)^{-1} g_\varepsilon, g_\varepsilon\rangle |s|^{-2r}\, dV_\omega,\end{aligned}$$

and the right hand side is under control in all cases. The extra error term $\delta^{1/2}h$ can be removed at the end by letting δ tend to 0. Since there is essentially no additional difficulty involved in this process, we will assume for simplicity of exposition that we do have the required lower bound for B_ε and the estimates of $g_\varepsilon^{(1)}$ and $g_\varepsilon^{(2)}$ as above. For $\delta = 0$, the above estimate provides a solution $u_{c,\varepsilon}$ of the equation $D''u_{c,\varepsilon} = g_\varepsilon = D''F_\varepsilon$ on $X_c \smallsetminus Y_c$, such that

$$\begin{aligned}\int_{X_c \smallsetminus Y_c} (\eta_\varepsilon + \lambda_\varepsilon)^{-1} |u_{c,\varepsilon}|^2 |s|^{-2r} dV_{X,\omega} &\le 2 \int_{X_c \smallsetminus Y_c} \langle B_\varepsilon^{-1} g_\varepsilon, g_\varepsilon\rangle\, |s|^{-2r} dV_{X,\omega} \\ &\le 2\, c_r \int_{Y_c} \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega} + O(\varepsilon).\end{aligned}$$

Here we have

$$\begin{aligned}\sigma_\varepsilon &= \log(|s|^2 + \varepsilon^2) \le \log(e^{-2\alpha} + \varepsilon^2) \le -2\alpha + O(\varepsilon^2) \le -2 + O(\varepsilon^2), \\ \eta_\varepsilon &= \varepsilon - \chi_0(\sigma_\varepsilon) \le (1 + O(\varepsilon))\sigma_\varepsilon^2, \\ \lambda_\varepsilon &= \frac{\chi_0'(\sigma_\varepsilon)^2}{\chi_0''(\sigma_\varepsilon)} = (1 - \sigma_\varepsilon)^2 + (1 - \sigma_\varepsilon) \le (3 + O(\varepsilon))\sigma_\varepsilon^2, \\ \eta_\varepsilon + \lambda_\varepsilon &\le (4 + O(\varepsilon))\sigma_\varepsilon^2 \le (4 + O(\varepsilon))\big(-\log(|s|^2 + \varepsilon^2)\big)^2.\end{aligned}$$

As F_ε is uniformly bounded with support in $\{|s| < \varepsilon\}$, we conclude from an obvious volume estimate that

$$\int_{X_c} \frac{|F_\varepsilon|^2}{(|s|^2 + \varepsilon^2)^r (-\log(|s|^2 + \varepsilon^2))^2} dV_{X,\omega} \le \frac{\mathrm{Const}}{(\log \varepsilon)^2}.$$

Therefore, thanks to the usual inequality $|t+u|^2 \le (1+k)|t|^2 + (1+k^{-1})|u|^2$ applied to the sum $F_{c,\varepsilon} = \widetilde{f}_\varepsilon - u_{c,\varepsilon}$ with $k = |\log \varepsilon|$, we obtain from our previous estimates

$$\int_{X_c\smallsetminus Y_c}\frac{|F_{c,\varepsilon}|^2}{(|s|^2+\varepsilon^2)^r(-\log(|s|^2+\varepsilon^2))^2}dV_{X,\omega}\le 8\,c_r\int_{Y_c}\frac{|f|^2}{|\Lambda^r(ds)|^2}dV_{Y,\omega}+O(|\log\varepsilon|^{-1}).$$

In addition to this, we have $d''F_{c,\varepsilon}=0$ by construction, and this equation extends from $X_c\smallsetminus Y_c$ to X_c by Lemma 12.10.

If $q=0$, then $u_{c,\varepsilon}$ must also be smooth, and the non integrability of the weight $|s|^{-2r}$ along Y shows that $u_{c,\varepsilon}$ vanishes on Y, therefore

$$F_{c,\varepsilon\restriction Y}=F_{\varepsilon\restriction Y}=F_{\infty\restriction Y}=f.$$

The theorem and its final estimate are thus obtained by extracting weak limits, first as $\varepsilon\to 0$, and then as $c\to+\infty$. The initial assumption that $\Sigma=\{s=\Lambda^r(ds)=0\}$ is empty can be easily removed in two steps: i) the result is true if X is Stein, since we can always find a complex hypersurface Z in X such that $\Sigma\subset\overline{Y}\cap Z\subsetneq\overline{Y}$, and then apply the extension theorem on the Stein manifold $X\smallsetminus Z$, in combination with Lemma 12.10 ; ii) the whole procedure still works when Σ is nowhere dense in $\overline{Y}$ (and possibly nonempty). Indeed local L^2 extensions $\widetilde{f}_j$ still exist by step i) applied on small coordinate balls U_j ; we then set $F_\infty=\sum\theta_j\widetilde{f}_j$ and observe that $|D''F_\infty|^2|s|^{-2r}$ is locally integrable, thanks to the estimate $\int_{U_j}|\widetilde{f}_j|^2|s|^{-2r}(\log|s|)^{-2}dV<+\infty$ and the fact that $|\sum d''\theta_j\wedge\widetilde{f}_j|=O(|s|^\delta)$ for suitable $\delta>0$ [as follows from Hilbert's Nullstensatz applied to $\widetilde{f}_j-\widetilde{f}_k$ at singular points of $\overline{Y}$].

When $q\ge 1$, the arguments needed to get a smooth solution involve more delicate considerations, and we will skip the details, which are extremely technical and not very enlightening.

(13.7) Remarks.

a) When $q=0$, the estimates provided by Theorem 13.6 are independent of the Kähler metric ω. In fact, if f and F are holomorphic sections of $\Lambda^nT^\star_X\otimes L$ over Y (resp. X), viewed as $(n,0)$-forms with values in L, we can "divide" f by $\Lambda^r(ds)\in\Lambda^r(TX/TY)^\star\otimes\det E$ to get a section $f/\Lambda^r(ds)$ of $\Lambda^pT^\star_Y\otimes L\otimes(\det E)^{-1}$ over Y. We then find

$$|F|^2dV_{X,\omega}=\mathrm{i}^{\,n^2}\{F,F\},$$
$$\frac{|f|^2}{|\Lambda^r(ds)|^2}dV_{Y,\omega}=\mathrm{i}^{\,p^2}\{f/\Lambda^r(ds),f/\Lambda^r(ds)\},$$

where $\{\bullet,\bullet\}$ is the canonical bilinear pairing described in (3.3).

b) The hermitian structure on E is not really used in depth. In fact, one only needs E to be equipped with a *Finsler metric*, that is, a smooth complex homogeneous function of degree 2 on E [or equivalently, a smooth hermitian metric on the tautological bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $P(E)$]. The section s of E induces a section $[s]$ of $P(E)$ over $X\smallsetminus s^{-1}(0)$ and a corresponding section $\widetilde{s}$ of the pull-back line bundle $[s]^\star\mathcal{O}_{P(E)}(-1)$. A trivial check shows that Theorem 13.6 as well as its proof extend to the case of a Finsler metric on E, if we replace everywhere $\{\mathrm{i}\,\Theta(E)s,s\}$ by $\{\mathrm{i}\,\Theta([s]^\star\mathcal{O}_{P(E)}(-1))\widetilde{s},\widetilde{s}\}$ (especially in hypothesis 13.6 b)). A minor issue is that $|\Lambda^r(ds)|$ is (a priori) no longer defined, since no obvious hermitian norm exists on $\det E$. A posteriori, we have the following ad hoc definition of a metric on $(\det E)^\star$ which makes the L^2 estimates work as before: for $x\in X$ and $\xi\in\Lambda^rE^\star_x$, we set

$$|\xi|_x^2 = \frac{1}{c_r}\int_{z\in E_x}(1+|z|^2)^2\theta'(|z|^2)^2\,\frac{\mathrm{i}^{r^2}\xi\wedge\overline{\xi}}{|z|^{2r}}$$

where $|z|$ is the Finsler norm on E_x [the constant c_r is there to make the result agree with the hermitian case; it is not hard to see that this metric does not depend on the choice of θ].

c) Even when $q=0$, the regularity of $u_{c,\varepsilon,\delta}$ requires some explanations, in case $\delta>0$. In fact, the equation

$$D''u_{c,\varepsilon,\delta}+\delta^{1/2}h_{c,\varepsilon,\delta}=g_\varepsilon=D''F_\varepsilon$$

does not immediately imply smoothness of $u_{c,\varepsilon,\delta}$ (since $h_{c,\varepsilon,\delta}$ need not be smooth in general). However, if we take the pair $(u_{c,\varepsilon,\delta},h_{c,\varepsilon,\delta})$ to be the minimal L^2 solution orthogonal to the kernel of $D''\oplus\delta^{1/2}\,\mathrm{Id}$, then it must be in the closure of the image of the adjoint operator $D''^*\oplus\delta^{1/2}\,\mathrm{Id}$, i.e. it must satisfy the additional condition $D''^*h_{c,\varepsilon,\delta}=\delta^{1/2}u_{c,\varepsilon,\delta}$, whence $(\Delta''+\delta\,\mathrm{Id})h_{c,\varepsilon,\delta}=(D''D''^*+\delta\,\mathrm{Id})h_{c,\varepsilon,\delta}=\delta^{1/2}D''F_\varepsilon$, and therefore $h_{c,\varepsilon,\delta}$ is smooth by the ellipticity of Δ''. □

We now present a few interesting corollaries. The first one is a surjectivity theorem for restriction morphisms in Dolbeault cohomology.

(13.8) Corollary. *Let X be a projective algebraic manifold and E a holomorphic vector bundle of rank r over X, s a holomorphic section of E which is everywhere transverse to the zero section, $Y=s^{-1}(0)$, and let L be a holomorphic line bundle such that $F=L^{1/r}\otimes E^\star$ is Griffiths positive (we just mean formally that $\frac{1}{r}\mathrm{i}\,\Theta(L)\otimes\mathrm{Id}_E-\mathrm{i}\,\Theta(E)>_{\rm Grif}0$). Then the restriction morphism*

$$H^{0,q}(X,\Lambda^nT_X^\star\otimes L)\to H^{0,q}(Y,\Lambda^nT_X^\star\otimes L)$$

is surjective for every $q\ge 0$.

Proof. A short computation gives

$$\begin{aligned}\mathrm{i}\,d'd''\log|s|^2&=\mathrm{i}\,d'\Big(\frac{\{s,D's\}}{|s|^2}\Big)\\&=\mathrm{i}\Big(\frac{\{D's,D's\}}{|s|^2}-\frac{\{D's,s\}\wedge\{s,D's\}}{|s|^4}+\frac{\{s,\Theta(E)s\}}{|s|^2}\Big)\ge-\frac{\{\mathrm{i}\,\Theta(E)s,s\}}{|s|^2}\end{aligned}$$

thanks to Lagrange's inequality and the fact that $\Theta(E)$ is antisymmetric. Hence, if δ is a small positive constant such that

$$-\mathrm{i}\,\Theta(E)+\frac{1}{r}\mathrm{i}\,\Theta(L)\otimes\mathrm{Id}_E\ge_{\rm Grif}\delta\,\omega\otimes\mathrm{Id}_E>0,$$

we find

$$\mathrm{i}\,\Theta(L)+r\,\mathrm{i}\,d'd''\log|s|^2\ge r\delta\,\omega.$$

The compactness of X implies $\mathrm{i}\,\Theta(E)\le C\omega\otimes\mathrm{Id}_E$ for some $C>0$. Theorem 13.6 can thus be applied with $\alpha=r\delta/C$ and Corollary 13.8 follows. By remark 13.7 b), the above surjectivity property even holds if $L^{1/r}\otimes E^\star$ is just assumed to be ample (in the sense that the associated line bundle $\pi^\star L^{1/r}\otimes\mathcal{O}_{P(E)}(1)$ is positive on the projectivized bundle $\pi:P(E)\to X$ of lines of E). □

Another interesting corollary is the following special case, dealing with bounded pseudoconvex domains $\Omega \subset\subset \mathbb{C}^n$. Even this simple version retains highly interesting information on the behavior of holomorphic and plurisubharmonic functions.

(13.9) Corollary. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and let $Y \subset X$ be a nonsingular complex submanifold defined by a section s of some hermitian vector bundle E with* bounded *curvature tensor on Ω. Assume that s is everywhere transverse to the zero section and that $|s| \le e^{-1}$ on Ω. Then there is a constant $C > 0$ (depending only on E), with the following property: for every psh function φ on Ω, every holomorphic function f on Y with $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} e^{-\varphi} dV_Y < +\infty$, there exists an extension F of f to Ω such that*

$$\int_\Omega \frac{|F|^2}{|s|^{2r}(-\log|s|)^2} e^{-\varphi} dV_\Omega \le C \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} e^{-\varphi} dV_Y.$$

Proof. We apply essentially the same idea as for the previous corollary, in the special case when $L = \Omega \times \mathbb{C}$ is the trivial bundle equipped with a weight function $e^{-\varphi - A|z|^2}$. The choice of a sufficiently large constant $A > 0$ guarantees that the curvature assumption 13.6 a) is satisfied (A just depends on the presupposed bound for the curvature tensor of E). □

(13.10) Remark. The special case when $Y = \{z_0\}$ is a point is especially interesting. In that case, we just take $s(z) = (e \operatorname{diam} \Omega)^{-1}(z - z_0)$, viewed as a section of the rank $r = n$ trivial vector bundle $\Omega \times \mathbb{C}^n$ with $|s| \le e^{-1}$. We take $\alpha = 1$ and replace $|s|^{2n}(-\log|s|)^2$ in the denominator by $|s|^{2(n-\varepsilon)}$, using the inequality

$$-\log|s| = \frac{1}{\varepsilon} \log|s|^{-\varepsilon} \le \frac{1}{\varepsilon}|s|^{-\varepsilon}, \qquad \forall \varepsilon > 0.$$

For any given value f_0, we then find a holomorphic function f such that $f(z_0) = f_0$ and

$$\int_\Omega \frac{|f(z)|^2}{|z - z_0|^{2(n-\varepsilon)}} e^{-\varphi(z)} dV_\Omega \le \frac{C_n}{\varepsilon^2 (\operatorname{diam} \Omega)^{2(n-\varepsilon)}} |f_0|^2 e^{-\varphi(z_0)}.$$

We prove here, as an application of the Ohsawa-Takegoshi extension theorem, that every psh function on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ can be approximated very accurately by functions of the form $c \log|f|$, where $c > 0$ and f is a holomorphic function. The main idea is taken from [Dem92]. For other applications to algebraic geometry, see [Dem93b] and Demailly-Kollár [DK99]. Recall that the Lelong number of a function $\varphi \in \mathrm{Psh}(\Omega)$ at a point x_0 is defined to be

$$\nu(\varphi, x_0) = \liminf_{z \to x_0} \frac{\log \varphi(z)}{\log|z - x_0|} = \lim_{r \to 0_+} \frac{\sup_{B(x_0,r)} \varphi}{\log r}.$$

In particular, if $\varphi = \log|f|$ with $f \in \mathcal{O}(\Omega)$, then $\nu(\varphi, x_0)$ is equal to the vanishing order $\mathrm{ord}_{x_0}(f) = \sup\{k \in \mathbb{N}\,;\, D^\alpha f(x_0) = 0,\ \forall |\alpha| < k\}$.

(13.11) Theorem. *Let φ be a plurisubharmonic function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi} d\lambda < +\infty$ and let $\varphi_m = \frac{1}{2m} \log \sum |\sigma_\ell|^2$*

where (σ_ℓ) *is an orthonormal basis of* $\mathcal{H}_\Omega(m\varphi)$. *Then there are constants* $C_1, C_2 > 0$ *independent of* m *such that*

a) $\varphi(z) - \dfrac{C_1}{m} \le \varphi_m(z) \le \sup\limits_{|\zeta - z| < r} \varphi(\zeta) + \dfrac{1}{m} \log \dfrac{C_2}{r^n}$

for every $z \in \Omega$ *and* $r < d(z, \partial\Omega)$. *In particular,* φ_m *converges to* φ *pointwise and in* L^1_{loc} *topology on* Ω *when* $m \to +\infty$ *and*

b) $\nu(\varphi, z) - \dfrac{n}{m} \le \nu(\varphi_m, z) \le \nu(\varphi, z)$ *for every* $z \in \Omega$.

Proof. Note that $\sum |\sigma_\ell(z)|^2$ is the square of the norm of the evaluation linear form $f \mapsto f(z)$ on $\mathcal{H}_\Omega(m\varphi)$. As φ is locally bounded above, the L^2 topology is actually stronger than the topology of uniform convergence on compact subsets of Ω. It follows that the series $\sum |\sigma_\ell|^2$ converges uniformly on Ω and that its sum is real analytic. Moreover we have

$$\varphi_m(z) = \sup_{f \in B(1)} \frac{1}{m} \log |f(z)|$$

where $B(1)$ is the unit ball of $\mathcal{H}_\Omega(m\varphi)$. For $r < d(z, \partial\Omega)$, the mean value inequality applied to the psh function $|f|^2$ implies

$$\begin{aligned} |f(z)|^2 &\le \frac{1}{\pi^n r^{2n}/n!} \int_{|\zeta - z| < r} |f(\zeta)|^2 d\lambda(\zeta) \\ &\le \frac{1}{\pi^n r^{2n}/n!} \exp\Big(2m \sup_{|\zeta - z| < r} \varphi(\zeta)\Big) \int_\Omega |f|^2 e^{-2m\varphi} d\lambda. \end{aligned}$$

If we take the supremum over all $f \in B(1)$ we get

$$\varphi_m(z) \le \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{2m} \log \frac{1}{\pi^n r^{2n}/n!}$$

and the second inequality in a) is proved. Conversely, the Ohsawa-Takegoshi extension theorem (Corollary 13.9) applied to the 0-dimensional subvariety $\{z\} \subset \Omega$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on Ω such that $f(z) = a$ and

$$\int_\Omega |f|^2 e^{-2m\varphi} d\lambda \le C_3 |a|^2 e^{-2m\varphi(z)},$$

where C_3 only depends on n and $\operatorname{diam} \Omega$. We fix a such that the right hand side is 1. This gives the other inequality

$$\varphi_m(z) \ge \frac{1}{m} \log |a| = \varphi(z) - \frac{\log C_3}{2m}.$$

The above inequality implies $\nu(\varphi_m, z) \le \nu(\varphi, z)$. In the opposite direction, we find

$$\sup_{|x - z| < r} \varphi_m(x) \le \sup_{|\zeta - z| < 2r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}.$$

Divide by $\log r$ and take the limit as r tends to 0. The quotient by $\log r$ of the supremum of a psh function over $B(x, r)$ tends to the Lelong number at x. Thus we obtain

$$\nu(\varphi_m, x) \ge \nu(\varphi, x) - \frac{n}{m}.$$

□

Theorem 13.11 implies in a straightforward manner the deep result of [Siu74] on the analyticity of the Lelong number sublevel sets.

(13.12) Corollary. *Let φ be a plurisubharmonic function on a complex manifold X. Then, for every $c > 0$, the Lelong number sublevel set*

$$E_c(\varphi) = \big\{z \in X\,;\ \nu(\varphi, z) \ge c\big\}$$

is an analytic subset of X.

Proof. Since analyticity is a local property, it is enough to consider the case of a psh function φ on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. The inequalities obtained in 13.11 b) imply that

$$E_c(\varphi) = \bigcap_{m \ge m_0} E_{c-n/m}(\varphi_m).$$

Now, it is clear that $E_c(\varphi_m)$ is the analytic set defined by the equations $\sigma_\ell^{(\alpha)}(z) = 0$ for all multi-indices α such that $|\alpha| < mc$. Thus $E_c(\varphi)$ is analytic as a (countable) intersection of analytic sets. □

We now translate Theorem 13.11 into a more geometric setting. Let X be a projective manifold and L a line bundle over X. A singular hermitian metric h on L is a metric such that the weight function φ of h is $L^1_{\rm loc}$ in any local trivialization (such that $L_{|U} \simeq U \times \mathbb{C}$ and $\|\xi\|_h = |\xi| e^{-\varphi(x)}$, $\xi \in L_x \simeq \mathbb{C}$). The curvature of L can then be computed in the sense of distributions

$$T = \frac{\mathrm{i}}{2\pi} \Theta_h(L) = \frac{\mathrm{i}}{\pi} \partial\overline{\partial}\varphi,$$

and L is said to be pseudoeffective if L admits a singular hermitian metric h such that the curvature current $T = \frac{\mathrm{i}}{2\pi}\Theta_h(L)$ is semipositive [The weight functions φ of L are thus plurisubharmonic]. Our goal is to approximate T in the weak topology by divisors which have roughly the same Lelong numbers as T. The existence of weak approximations by divisors has already been proved in [Lel72] for currents defined on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ with $H^2(\Omega, \mathbb{R}) = 0$, and in [Dem92, 93b] in the situation considered here (cf. also [Dem82b], although the argument given there is somewhat incorrect). We take the opportunity to present here a slightly simpler derivation. In what follows, we use an additive notation for $\mathrm{Pic}(X)$, i.e. kL is meant for the line bundle $L^{\otimes k}$.

(13.13) Proposition. *For any $T = \frac{\mathrm{i}}{2\pi}\Theta_h(L) \ge 0$ and any ample line bundle F, there is a sequence of non zero sections $h_s \in H^0(X, p_sF + q_sL)$ with $p_s, q_s > 0$, $\lim q_s = +\infty$ and $\lim p_s/q_s = 0$, such that the divisors $D_s = \frac{1}{q_s}\,\mathrm{div}(h_s)$ satisfy $T = \lim D_s$ in the weak topology and $\sup_{x\in X} |\nu(D_s, x) - \nu(T, x)| \to 0$ as $s \to +\infty$.*

(13.14) Remark. The proof will actually show, with very slight modifications, that Proposition 13.13 also holds when X is a Stein manifold and L is an arbitrary holomorphic line bundle.

Proof. We first use Hörmander's L^2 estimates to construct a suitable family of holomorphic sections and combine this with some ideas of [Lel72] in a second step. Select a smooth metric with positive curvature on F, choose $\omega = \frac{\mathrm{i}}{2\pi}\Theta(F) > 0$ as a Kähler metric on X and fix some large integer k (how large k must be will be specified later). For all $m \geq 1$ we define

$$w_m(z) = \sup_{1\leq j\leq N} \frac{1}{m} \log \|f_j(z)\|,$$

where $(f_1, \ldots, f_N)$ is an orthonormal basis of the space of sections of $\mathcal{O}(kF + mL)$ with finite global L^2 norm $\int_X \|f\|^2 dV_\omega$. Let e_F and e_L be non vanishing holomorphic sections of F and L on a trivializing open set Ω, and let $e^{-\psi} = \|e_F\|$, $e^{-\varphi} = \|e_L\|$ be the corresponding weights. If f is a section of $\mathcal{O}(kF + mL)$ and if we still denote by f the associated complex valued function on Ω with respect to the holomorphic frame $e_F^k \otimes e_L^m$, we have $\|f(z)\| = |f(z)|e^{-k\psi(z)-m\varphi(z)}$; here φ is plurisubharmonic, ψ is smooth and strictly plurisubharmonic, and $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$, $\omega = \frac{i}{\pi}\partial\overline{\partial}\psi$. In Ω, we can write

$$w_m(z) = \sup_{1\leq j\leq N} \frac{1}{m} \log |f_j(z)| - \varphi(z) - \frac{k}{m}\psi(z).$$

In particular $T_m := \frac{i}{\pi}\partial\overline{\partial}w_m + T + \frac{k}{m}\omega$ is a closed positive current belonging to the cohomology class $c_1(L) + \frac{k}{m}c_1(F)$.

Step 1. We claim that T_m converges to T as m tends to $+\infty$ and that T_m satisfies the inequalities

$$\nu(T, x) - \frac{n}{m} \leq \nu(T_m, x) \leq \nu(T, x) \tag{13.15}$$

at every point $x \in X$. Note that T_m is defined on Ω by $T_m = \frac{i}{\pi}\partial\overline{\partial}v_{m,\Omega}$ with

$$v_{m,\Omega}(z) = \sup_{1\leq j\leq N} \frac{1}{m} \log |f_j(z)|, \qquad \int_\Omega |f_j|^2 e^{-2k\psi-2m\varphi} dV_\omega \leq 1.$$

We proceed in the same way as for the proof of Theorem 13.11. We suppose here that Ω is a coordinate open set with analytic coordinates $(z_1, \ldots, z_n)$. Take $z \in \Omega' \subset\subset \Omega$ and $r \leq r_0 = \frac{1}{2}d(\Omega', \partial\Omega)$. By the L^2 estimate and the mean value inequality for subharmonic functions, we obtain

$$|f_j(z)|^2 \leq \frac{C_1}{r^{2n}} \int_{|\zeta-z|<r} |f_j(\zeta)|^2 d\lambda(\zeta) \leq \frac{C_2}{r^{2n}} \sup_{|\zeta-z|<r} e^{2m\varphi(\zeta)}$$

with constants C_1, C_2 independent of m and r (the smooth function ψ is bounded on any compact subset of Ω). Hence we infer

$$v_{m,\Omega}(z) \leq \sup_{|\zeta-z|<r} \varphi(\zeta) + \frac{1}{2s} \log \frac{C_2}{r^{2n}}. \tag{13.16}$$

If we choose for example $r = 1/m$ and use the upper semi-continuity of φ, we infer $\limsup_{s\to+\infty} v_{m,\Omega} \leq \varphi$. Moreover, if $\gamma = \nu(\varphi, x) = \nu(T, x)$, then $\varphi(\zeta) \leq \gamma \log |\zeta - x| + O(1)$ near x. By taking $r = |z - x|$ in (13.16), we find

$$v_{m,\Omega}(z) \leq \sup_{|\zeta-x|<2r} \varphi(\zeta) - \frac{n}{m} \log r + O(1) \leq \Big(\gamma - \frac{n}{m}\Big) \log |z - x| + O(1),$$

$$\nu(T_m, x) = \nu(v_{m,\Omega}, x) \geq \Big(\gamma - \frac{n}{m}\Big)_+ \geq \nu(T, x) - \frac{n}{m}.$$

In the opposite direction, the inequalities require deeper arguments since we actually have to construct sections in $H^0(X, kF+mL)$. Assume that Ω is chosen isomorphic to a bounded pseudoconvex open set in $\mathbb{C}^n$. By the Ohsawa-Takegoshi L^2 extension theorem (Corollary 13.9), for every point $x \in \Omega$, there is a holomorphic function g on Ω such that $g(x) = e^{m\varphi(x)}$ and

$$\int_\Omega |g(z)|^2 e^{-2m\varphi(z)} d\lambda(z) \le C_3,$$

where C_3 depends only on n and $\mathrm{diam}(\Omega)$. For $x \in \Omega'$, set

$$\sigma(z) = \theta\big(|z-x|/r\big)\, g(z)\, e_F(z)^k \otimes e_L(z)^m, \qquad r = \min\big(1, 2^{-1}d(\Omega', \partial\Omega)\big),$$

where $\theta : \mathbb{R} \to [0,1]$ is a cut-off function such that $\theta(t) = 1$ for $t < 1/2$ and $\theta(t) = 0$ for $t \ge 1$. We solve the global equation $\overline{\partial} u = v$ on X with $v = \overline{\partial}\sigma$, after multiplication of the metric of $kF+mL$ with the weight

$$e^{-2n\rho_x(z)}, \qquad \rho_x(z) = \theta\big(|z-x|/r\big) \log|z-x| \le 0.$$

The $(0,1)$-form v can be considered as a $(n,1)$-form with values in the line bundle $\mathcal{O}(-K_X + kF + mL)$ and the resulting curvature form of this bundle is

$$\mathrm{Ricci}(\omega) + k\omega + mT + n\frac{i}{\pi}\partial\overline{\partial}\rho_x.$$

Here the first two summands are smooth, $i\partial\overline{\partial}\rho_x$ is smooth on $X \smallsetminus \{x\}$ and ≥ 0 on $B(x, r/2)$, and T is a positive current. Hence by choosing k large enough, we can suppose that this curvature form is $\ge \omega$, uniformly for $x \in \Omega'$. By Hörmander's standard L^2 estimates [AV65, Hör65, 66], we get a solution u on X such that

$$\int_X \|u\|^2 e^{-2n\rho_x} dV_\omega \le C_4 \int_{r/2<|z-x|<r} |g|^2 e^{-2k\psi - 2m\varphi - 2n\rho_x} dV_\omega \le C_5\,;$$

to get the estimate, we observe that v has support in the corona $r/2 < |z-x| < r$ and that ρ_x is bounded there. Thanks to the logarithmic pole of ρ_x, we infer that $u(x) = 0$. Moreover

$$\int_\Omega \|\sigma\|^2 dV_\omega \le \int_{\Omega' + B(0,r/2)} |g|^2 e^{-2k\psi - 2m\varphi} dV_\omega \le C_6,$$

hence $f = \sigma - u \in H^0(X, kF+mL)$ satisfies $\int_X \|f\|^2 dV_\omega \le C_7$ and

$$\|f(x)\| = \|\sigma(x)\| = \|g(x)\|\, \|e_F(x)\|^m \|e_L(x)\|^m = \|e_F(x)\|^k = e^{-k\psi(x)}.$$

In our orthonormal basis (f_j), we can write $f = \sum \lambda_j f_j$ with $\sum |\lambda_j|^2 \le C_7$. Therefore

$$e^{-k\psi(x)} = \|f(x)\| \le \sum |\lambda_j| \sup \|f_j(x)\| \le \sqrt{C_7 N}\, e^{m w_m(x)},$$

$$w_m(x) \ge \frac{1}{m}\log(C_7 N)^{-1/2} \|f(x)\| \ge -\frac{1}{m}\Big(\log(C_7N)^{1/2} + k\,\psi(x)\Big)$$

where $N = \dim H^0(X, kF+mL) = O(m^n)$. By adding $\varphi + \frac{k}{m}\psi$, we get $v_{m,\Omega} \ge \varphi - C_8 m^{-1}\log m$. Thus $\lim_{m\to+\infty} v_{m,\Omega} = \varphi$ everywhere, $T_m = \frac{i}{\pi}\partial\overline{\partial} v_{m,\Omega}$ converges weakly to $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$, and

$$\nu(T_m, x) = \nu(v_{m,\Omega}, x) \le \nu(\varphi, x) = \nu(T, x).$$

Note that $\nu(v_{m,\Omega}, x) = \frac{1}{m} \min \mathrm{ord}_x(f_j)$ where $\mathrm{ord}_x(f_j)$ is the vanishing order of f_j at x, so our initial lower bound for $\nu(T_m, x)$ combined with the last inequality gives

$$\nu(T, x) - \frac{n}{m} \le \frac{1}{m} \min \mathrm{ord}_x(f_j) \le \nu(T, x). \tag{13.17}$$

Step 2: *Construction of the divisors D_s.*

Select sections $(g_1, \ldots, g_N) \in H^0(X, k_0 F)$ with k_0 so large that $k_0 F$ is very ample, and set

$$h_{\ell,m} = f_1^\ell g_1 + \ldots + f_N^\ell g_N \in H^0\big(X, (k_0 + \ell k)F + \ell m L\big).$$

For almost every N-tuple $(g_1, \ldots, g_N)$, Lemma 13.18 below and the weak continuity of $\partial\overline{\partial}$ show that

$$\Delta_{\ell,m} = \frac{1}{\ell m} \frac{i}{\pi} \partial\overline{\partial} \log |h_{\ell,m}| = \frac{1}{\ell m} \operatorname{div}(h_{\ell,m})$$

converges weakly to $T_m = \frac{i}{\pi} \partial\overline{\partial} v_{m,\Omega}$ as ℓ tends to $+\infty$, and that

$$\nu(T_m, x) \le \nu\Big(\frac{1}{\ell m} \Delta_{\ell,m}, x\Big) \le \nu(T, x) + \frac{1}{\ell m}.$$

This, together with the first step, implies the proposition for some subsequence $D_s = \Delta_{\ell(s),s}$, $\ell(s) \gg s \gg 1$. We even obtain the more explicit inequality

$$\nu(T, x) - \frac{n}{m} \le \nu\Big(\frac{1}{\ell m} \Delta_{\ell,m}, x\Big) \le \nu(T, x) + \frac{1}{\ell m}. \qquad \square$$

(13.18) Lemma. *Let Ω be an open subset in $\mathbb{C}^n$ and let $f_1, \ldots, f_N \in H^0(\Omega, \mathcal{O}_\Omega)$ be non zero functions. Let $G \subset H^0(\Omega, \mathcal{O}_\Omega)$ be a finite dimensional subspace whose elements generate all* 1*-jets at any point of Ω. Finally, set $v = \sup \log |f_j|$ and*

$$h_\ell = f_1^\ell g_1 + \ldots + f_N^\ell g_N, \qquad g_j \in G \smallsetminus \{0\}.$$

Then for all $(g_1, \ldots, g_N)$ in $(G \smallsetminus \{0\})^N$ except a set of measure 0*, the sequence $\frac{1}{\ell} \log |h_\ell|$ converges to v in $L^1_{\mathrm{loc}}(\Omega)$ and*

$$\nu(v, x) \le \nu\Big(\frac{1}{\ell} \log |h_\ell|\Big) \le \nu(v, x) + \frac{1}{\ell}, \qquad \forall x \in X, \quad \forall \ell \ge 1.$$

Proof. The sequence $\frac{1}{\ell} \log |h_\ell|$ is locally uniformly bounded above and we have

$$\lim_{\ell \to +\infty} \frac{1}{\ell} \log \big|h_\ell(z)\big| = v(z)$$

at every point z where all absolute values $|f_j(z)|$ are distinct and all $g_j(z)$ are nonzero. This is a set of full measure in Ω because the sets $\{|f_j|^2 = |f_l|^2,\ j \ne l\}$ and $\{g_j = 0\}$ are real analytic and thus of zero measure (without loss of generality, we may assume that Ω is connected and that the f_j's are not pairwise proportional). The well-known uniform integrability properties of plurisubharmonic functions then show that $\frac{1}{\ell} \log |h_\ell|$ converges to v in $L^1_{\mathrm{loc}}(\Omega)$. It is easy to see that $\nu(v, x)$ is the minimum of the vanishing orders $\mathrm{ord}_x(f_j)$, hence

$$\nu(\log |h_\ell|, x) = \mathrm{ord}_x(h_\ell) \ge \ell\, \nu(v, x).$$

In the opposite direction, consider the set $\mathcal{E}_\ell$ of all $(N+1)$-tuples

$$(x, g_1, \ldots, g_N) \in \Omega \times G^N$$

for which $\nu(\log|h_\ell|, x) \geq \ell\,\nu(v,x) + 2$. Then $\mathcal{E}_\ell$ is a constructible set in $\Omega \times G^N$: it has a locally finite stratification by analytic sets, since

$$\mathcal{E}_\ell = \bigcup_{s\geq 0}\Big(\bigcup_{j,\,|\alpha|=s}\{x\,;\,D^\alpha f_j(x)\neq 0\}\times G^N\Big)\cap\bigcap_{|\beta|\leq \ell s+1}\{(x,(g_j))\,;\,D^\beta h_\ell(x)=0\}.$$

The fiber $\mathcal{E}_\ell\cap(\{x\}\times G^N)$ over a point $x\in\Omega$ where $\nu(v,x)=\min \mathrm{ord}_x(f_j)=s$ is the vector space of N-tuples $(g_j)\in G^N$ satisfying the equations $D^\beta\big(\sum f_j^\ell g_j(x)\big)=0$, $|\beta|\leq \ell s+1$. However, if $\mathrm{ord}_x(f_j)=s$, the linear map

$$(0,\ldots,0,g_j,0,\ldots,0)\longmapsto \big(D^\beta(f_j^\ell g_j(x))\big)_{|\beta|\leq \ell s+1}$$

has rank $n+1$, because it factorizes into an injective map $J^1_x g_j \mapsto J^{\ell s+1}_x(f_j^\ell g_j)$. It follows that the fiber $\mathcal{E}_\ell\cap(\{x\}\times G^N)$ has codimension at least $n+1$. Therefore

$$\dim \mathcal{E}_\ell \leq \dim(\Omega\times G^N)-(n+1)=\dim G^N - 1$$

and the projection of $\mathcal{E}_\ell$ on G^N has measure zero by Sard's theorem. By definition of $\mathcal{E}_\ell$, any choice of $(g_1,\ldots,g_N)\in G^N\smallsetminus\bigcup_{\ell\geq 1}\mathrm{pr}(\mathcal{E}_\ell)$ produces functions h_ℓ such that $\nu(\log|h_\ell|,x)\leq \ell\,\nu(v,x)+1$ on Ω. □

14. Invariance of Plurigenera of Varieties of General Type

The goal of this section is to give a proof of the following fundamental result on the invariance of plurigenera, proved by Y.T. Siu [Siu98]. A generalized version has been obtained shortly afterwards by Y. Kawamata [Kaw99], using only algebraic tools.

(14.0) Theorem (Siu). *Let $X\to S$ be a smooth projective family of varieties of general type on a connected base S. Then the plurigenus $p_m(X_t)=h^0(X_t,mK_{X_t})$ of fibers is independent of t for all $m\geq 0$.*

Given a family $\gamma: X\to S$ of projective varieties, the proof of the invariance of the plurigenera of fibers $X_t=\gamma^{-1}(t)$ is easily reduced to the case when the base is the unit disk $\Delta\subset\mathbb{C}$ (in general, one can always connect two arbitrary points of S by a chain of small analytic disks, and one can take the pull-back of the family in restriction to each of those disks). We can therefore suppose that $\gamma: X\to\Delta$. In that case, we have a canonical isomorphism $K_{X_t}\simeq K_{X|X_t}$ on each fiber, given by $u\mapsto dt\wedge u$ [in this way, we will allow ourselves to identify K_{X_t} and $K_{X|X_t}$ in the sequel]. We know from Grauert's direct image theorem [Gra60] that the direct image sheaves $\gamma_\star\mathcal{O}(mK_X)$ are coherent, and moreover, the plurigenera $p_m(X_t)=h^0(X_t,(mK_X)_{|X_t})$ are upper semi-continuous functions of t. The jumps occur precisely if a section on some fiber X_{t_0} does not extend to nearby fibers. Proving the invariance of plurigenera is thus equivalent to proving that a section of $mK_{X|X_{t_0}}$ on a fiber X_{t_0} can be extended to a neighborhood of X_{t_0} in X. We can assume without loss of generality that $t_0=0$. The strategy of Siu's proof consists more or less in the use of suitable singular hermitian metrics on K_X and K_{X_0} (with "minimal singularities"), combined with L^2 extension theorems with respect to these metrics.

14.1. Metrics with minimal singularities

One of the main ideas in the proof of the invariance of plurigenera – although it is not completely explicit in [Siu98] – rests on the fact that the singularities of hermitian metrics on a line bundle L can reflect very accurately the base loci of the sequence of linear systems $|mL|$, precisely when metrics with minimal singularities are used. We follow here the approach of [DPS00].

(14.1.1) Definition. *Let L be a pseudo-effective line bundle on a compact complex manifold X. Consider two hermitian metrics h_1, h_2 on L with curvature $i\Theta_{h_j}(L) \geq 0$ in the sense of currents.*

(i) *We will write $h_1 \preccurlyeq h_2$, and say that h_1 is less singular than h_2, if there exists a constant $C > 0$ such that $h_1 \leq Ch_2$.*

(ii) *We will write $h_1 \sim h_2$, and say that h_1, h_2 are equivalent with respect to singularities, if there exists a constant $C > 0$ such that $C^{-1}h_2 \leq h_1 \leq Ch_2$.*

Of course $h_1 \preccurlyeq h_2$ if and only if the associated weights in suitable trivializations locally satisfy $\varphi_2 \leq \varphi_1 + C$. This implies in particular $\nu(\varphi_1, x) \leq \nu(\varphi_2, x)$ at each point. The above definition is motivated by the following observation.

(14.1.2) Theorem. *For every pseudo-effective line bundle L over a compact complex manifold X, there exists up to equivalence of singularities a unique class of hermitian metrics h with minimal singularities such that $i\Theta_h(L) \geq 0$.*

Proof. The proof is almost trivial. We fix once for all a smooth metric h_∞ (whose curvature is of random sign and signature), and we write singular metrics of L under the form $h = h_\infty e^{-2\psi}$. The condition $i\Theta_h(L) \geq 0$ is equivalent to $\frac{i}{\pi}\partial\overline{\partial}\psi \geq -u$ where $u = i\Theta_{h_\infty}(L)$. This condition implies that ψ is plurisubharmonic up to the addition of the weight φ_∞ of h_∞, and therefore locally bounded from above. Since we are concerned with metrics only up to equivalence of singularities, it is always possible to adjust ψ by a constant in such a way that $\sup_X \psi = 0$. We now set

$$h_{\min} = h_\infty e^{-2\psi_{\min}}, \qquad \psi_{\min}(x) = \sup_\psi \psi(x)$$

where the supremum is extended to all functions ψ such that $\sup_X \psi = 0$ and $\frac{i}{\pi}\partial\overline{\partial}\psi \geq -u$. By standard results on plurisubharmonic functions (see Lelong [Lel69]), $\psi_{\min}$ still satisfies $\frac{i}{\pi}\partial\overline{\partial}\psi_{\min} \geq -u$ (i.e. the weight $\varphi_\infty + \psi_{\min}$ de $h_{\min}$ is plurisubharmonic), and $h_{\min}$ is obviously the metric with minimal singularities that we were looking for. □

Now, given a section $\sigma \in H^0(X, mL)$, the expression $h(\xi) = |\xi^m/\sigma(x)|^{2/m}$ defines a singular metric on L, which therefore necessarily has at least as much singularity as $h_{\min}$ as, i.e. $\frac{1}{m}\log|\sigma|^2 \leq \varphi_{\min} + C$ locally. In particular, $|\sigma|^2 e^{-m\varphi_{\min}}$ is locally bounded, hence $\sigma \in H^0(X, mL \otimes \mathcal{I}(h_{\min}^{\otimes m}))$. For all $m > 0$ we therefore have an isomorphism

$$H^0(X, mL \otimes \mathcal{I}(h_{\min}^{\otimes m})) \xrightarrow{\ \simeq\ } H^0(X, mL).$$

By the well-known properties of Lelong numbers (see Skoda [Sko72a]), the union of all zero varieties of the ideals $\mathcal{I}(h_{\min}^{\otimes m})$ is equal to the Lelong sublevel set

$$E_+(h_{\min}) = \big\{x \in X\,;\ \nu(\varphi_{\min}, x) > 0\big\}. \tag{14.1.3}$$

We will call this set the *virtual base locus* of L. It is always contained in the "algebraic" base locus

$$B_{\|L\|} = \bigcap_{m>0} B_{|mL|}, \qquad B_{|mL|} = \bigcap_{\sigma \in H^0(X,mL)} \sigma^{-1}(0),$$

but there may be a strict inclusion. This is the case for instance if $L \in \mathrm{Pic}^0(X)$ is such that all positive multiples mL have no nonzero sections; in that case $E_+(h_{\min}) = \emptyset$ but $\bigcap_{m>0} B_{|mL|} = X$. Another general situation where $E_+(h_{\min})$ and $B_{\|L\|}$ can differ is given by the following result.

(14.1.4) Proposition. *Let L be a big nef line bundle. Then $h_{\min}$ has zero Lelong numbers everywhere, i.e. $E_+(h_{\min}) = \emptyset$.*

Proof. Recall that L is big if its Kodaira-Iitaka dimension $\kappa(L)$ is equal to $n = \dim X$. In that case, it is well known that one can write $m_0 L = A + E$ with A ample and E effective, for m_0 sufficiently large. Then $mL = ((m - m_0)L + A) + E$ is the sum of an ample divisor $A_m = (m - m_0)L + A$ plus a (fixed) effective divisor, so that there is a hermitian metric h_m on L for which $i\Theta_{h_m}(L) = \frac{1}{m} i\Theta(A_m) + \frac{1}{m}[E]$, with a suitable smooth positive form $i\Theta(A_m)$. This shows that the Lelong numbers of the weight of h_m are $O(1/m)$, hence in the limit those of $h_{\min}$ are zero.

If h is a singular hermitian metric such that $i\Theta_h(L) \geq 0$ and

$$H^0(X, mL \otimes \mathcal{I}(h^{\otimes m})) \simeq H^0(X, mL) \qquad \text{for all } m \geq 0, \tag{14.1.5}$$

we say that h is an *analytic Zariski decomposition* of L. We have just seen that such a decomposition always exists and that $h = h_{\min}$ is a solution. The concept of analytic Zariski decomposition is motivated by its algebraic counterpart (the existence of which generally fails) : one says that L admits an *algebraic Zariski decomposition* if there exists an integer m_0 such that $m_0 L \simeq \mathcal{O}(E + D)$ where E is an effective divisor and D a nef divisor, in such a way that $H^0(X, kD) \simeq H^0(X, km_0 L)$ for all $k \geq 0$. If $\mathcal{O}(*D)$ is generated by sections, there is a smooth metric with semipositive curvature on $\mathcal{O}(D)$, and this metric induces a singular hermitian metric h on L of curvature current $\frac{1}{m_0}(i\Theta(\mathcal{O}(D)) + [E])$. Its poles are defined by the effective $\mathbb{Q}$-divisor $\frac{1}{m_0} E$. For this metric, we of course have $\mathcal{I}(h^{\otimes k m_0}) = \mathcal{O}(-kE)$, hence (14.1.5) holds true at least when m is a multiple of m_0.

14.2. A uniform global generation property

The "uniform global generation property" shows in some sense that the curvature of the tensor product sheaf $L \otimes \mathcal{I}(h)$ is uniformly bounded below, for any singular hermitian metric h with nonnegative curvature on L.

(14.2.1) Proposition. *There exists an ample line bundle G on X such that for every pseudoeffective line bundle (L, h), the sheaf $\mathcal{O}(G+L)\otimes\mathcal{I}(h)$ is generated by its global sections. In fact, G can be chosen as follows: pick any very ample line bundle A, and take G such that $G-(K_X+nA)$ is ample, e.g. $G = K_X+(n+1)A$.*

Proof. Let φ be the weight of the metric h on a small neighborhood of a point $z_0 \in X$. Assume that we have a local section u of $\mathcal{O}(G+L)\otimes\mathcal{I}(h)$ on a coordinate open ball $B = B(z_0, \delta)$, such that

$$\int_B |u(z)|^2 e^{-2\varphi(z)}|z-z_0|^{-2(n+\varepsilon)}dV(z) < +\infty.$$

Then Skoda's division theorem [Sko72b] implies $u(z) = \sum(z_j - z_{j,0})v_j(z)$ with

$$\int_B |v_j(z)|^2 e^{-2\varphi(z)}|z-z_0|^{-2(n-1+\varepsilon)}dV(z) < +\infty,$$

in particular $u_{z_0} \in \mathcal{O}(G+L)\otimes\mathcal{I}(h)\otimes\mathfrak{m}_{X,z_0}$. Select a very ample line bundle A on X. We take a basis $\sigma = (\sigma_j)$ of sections of $H^0(X, G\otimes\mathfrak{m}_{X,z_0})$ and multiply the metric h of G by the factor $|\sigma|^{-2(n+\varepsilon)}$. The weight of the above metric has singularity $(n+\varepsilon)\log|z-z_0|^2$ at z_0, and its curvature is

$$i\Theta(G) + (n+\varepsilon)i\partial\overline{\partial}\log|\sigma|^2 \geq i\Theta(G) - (n+\varepsilon)\Theta(A). \tag{14.2.2}$$

Now, let f be a local section in $H^0(B, \mathcal{O}(G+L)\otimes\mathcal{I}(h))$ on $B = B(z_0, \delta)$, δ small. We solve the global $\overline{\partial}$ equation

$$\overline{\partial}u = \overline{\partial}(\theta f) \qquad \text{on } X$$

with a cut-off function θ supported near z_0 and with the weight associated with our above choice of metric on $G+L$. Thanks to Nadel's Theorem 6.11, the solution exists if the metric of $G+L-K_X$ has positive curvature. As $i\Theta_h(L) \geq 0$ in the sense of currents, (14.2.2) shows that a sufficient condition is $G - K_X - nA > 0$ (provided that ε is small enough). We then find a smooth solution u such that $u_{z_0} \in \mathcal{O}(G+L)\otimes\mathcal{I}(h)\otimes\mathfrak{m}_{X,z_0}$, hence

$$F := \theta f - u \in H^0(X, \mathcal{O}(G+L)\otimes\mathcal{I}(h))$$

is a global section differing from f by a germ in $\mathcal{O}(G+L)\otimes\mathcal{I}(h)\otimes\mathfrak{m}_{X,z_0}$. Nakayama's lemma implies that $H^0(X, \mathcal{O}(G+L)\otimes\mathcal{I}(h))$ generates the stalks of $\mathcal{O}(G+L)\otimes\mathcal{I}(h)$.

14.3. A special case of the Ohsawa-Takegoshi-Manivel L^2 extension theorem

We will need the special case below of the Ohsawa-Takegoshi L^2 extension theorem. (Notice that, in this way, the proof of the theorem on invariance of plurigenera requires 3 essentially different types of L^2 existence theorems !).

(14.3.1) Theorem. *Let $\gamma : X \to \Delta$ be a projective family of projective manifolds parametrized by the open unit disk $\Delta \subset \mathbb{C}$. Let $X_0 = \gamma^{-1}(0)$, $n = \dim_{\mathbb{C}} X_0$, and let $L \to X$ be a line bundle equipped with a hermitian metric written locally as $e^{-\chi}$,* *such that $i\partial\bar\partial\chi \ge \omega$ in the sense of currents, with a suitable positive $(1,1)$-form ω on X. Let $0 < r < 1$ and $\Delta_r = \{t \in \Delta;\ |t| < r\}$. Then there exists a positive constant A_r such that the following statement holds. For every holomorphic n-form f on X_0 with values in L such that*

$$\int_{X_0} |f|^2 e^{-\chi} dV < \infty,$$

there exists a holomorphic $(n+1)$-form $\tilde f$ on $\gamma^{-1}(\Delta_r)$ with values in L, such that $\tilde f|_{X_0} = f \wedge \gamma^*(dt)$ on X_0 and

$$\int_X |\tilde f|^2 e^{-\chi} dV \le A_r \int_{X_0} |f|^2 e^{-\chi} dV.$$

(14.3.2) Remark. It should be noticed that no hermitian metric on the tangent bundle of X_0 or X is required in order to define the integral of the square of the norm of holomorphic forms f and $\tilde f$ on X_0 and X ; in fact, it suffices to integrate the volume forms $i^{n^2} f \wedge \overline{f}$ and $i^{(n+1)^2} \tilde f \wedge \overline{\tilde f}$.

14.4. Construction of a hermitian metric with positive curvature on K_{X_0}

From now on, we suppose that $X \to \Delta$ is a family of projective manifolds of general type (i.e. that all fibers X_t are of general type). A technical point to be settled first is that the hermitian metric on K_X must be chosen in such a way that its curvature current dominates a smooth positive definite $(1,1)$-form, so that the Skoda division theorem and the Ohsawa-Takegoshi-Manivel L^2 extension theorem can be applied. This point is easily settled by means of a variation of Kodaira's technique which consists in expressing a sufficiently large multiple of a big line bundle as the sum of an effective divisor and of an ample line bundle.

(14.4.1) Lemma. *There exists a positive integer a such that $aK_X = D + F$, where D is an effective divisor on X which does not contain X_0, and where F is a positive line bundle on X.*

Proof. Let F be a positive line bundle on X, and let r_a be the generic rank of $H^0(X_t, aK_{X_t} - F)$, which is achieved for $t \in \Delta \smallsetminus S_a$ in the complement of a suitable locally finite subset $S_a \subset \Delta$. Fix $t_1 \in \Delta \smallsetminus \bigcup S_a$. Since X_{t_1} is of general type, we know that $h^0(X_{t_1}, aK_{X_{t_1}}) \ge c\,a^n$, hence $h^0(X_{t_1}, aK_{X_{t_1}} - F) \ge c'a^n$ for suitable $c, c' > 0$ and a large enough. By the choice of t_1, every non zero section of $H^0(X_{t_1}, aK_{X_{t_1}} - F)$ can be extended into a section s of $H^0(X, aK_X - F)$, hence $aK_X = D + F$ where D is the zero divisor of s. If necessary, we can eliminate any unwanted component X_0 in the decomposition of D by dividing s with a suitable power of t. □

The next step is to construct hermitian metrics on K_{X_0} and K_X, respectively, and to compare their multiplier ideal sheaves. By the results of section 14.1, there exists a metric with minimal singularities on X_0 (unique up to equivalence of singularities). We will denote by φ_0 the weight of this metric. Similarly, there exists a metric with minimal singularities on every relatively compact neighborhood of X_0 in X, and by shrinking the base Δ, we can suppose that this metric exists on the whole space X. We will denote by φ the *restriction to* X_0 of the weight of this metric. By definition, φ is at least as singular as φ_0 on X_0, hence by adding a constant, we may eventually assume that $\varphi \le \varphi_0$.

On the other hand, Lemma 14.4.1 shows that we can choose an integer $a \ge 2$ such that $aK_X = D+F$, where D is an effective divisor on X which does not contain X_0 as a component, and F is an ample line bundle on X. After a has been replaced with a sufficiently large multiple, we can make the following additional assumptions:

(14.4.2) $F - K_X > K_X + nA$ *for some very ample line bundle* A *on* X.

(14.4.3) *there is a basis of sections of* $H^0(X,F)_{|X_0}$ *providing an embedding of* X_0 *onto a subvariety of projective space.*

Let s_D be the canonical section of the line bundle $\mathcal{O}(D)$, so that the divisor of s_D is D. Let $u_1,\dots,u_N \in H^0(X,F)$ be sections such that

$$u_{1|X_0},\dots,u_{N|X_0}$$

form a basis of $H^0(X,F)_{|X_0}$. Since $s_D u_j \in H^0(X, aK_X)$ $(1 \le j \le N)$, we get a hermitian metric

$$e^{-\psi} = \left(\frac{1}{|s_D|^2\sum |u_j|^2}\right)^{\frac{1}{a}}$$

on the line bundle K_X. Moreover

$$\psi = \frac{1}{a}\left(\log|s_D|^2 + \chi\right)$$

where $\chi = \log(\sum |u_j|^2)$ defines a smooth hermitian metric with positive curvature on F. Therefore, $i\partial\bar\partial\psi$ is a positive definite current. Furthermore, the singularities of ψ on X are at least as large as those of the weight φ which is by definition the weight with minimal singularities on X. By adjusting again the weights with constants, we can state:

(14.4.4) Lemma. *The hermitian metric* $e^{-\psi}$ *has positive definite curvature and* $\psi \le \varphi \le \varphi_0$ *on* X_0.

The crucial argument is a comparison result for the multiplier ideal sheaves on X_0 defined by $\ell\varphi_0$ and $\ell\varphi$, respectively, when ℓ is large. In the sequel, the notation $\mathcal{I}((\ell+a-\varepsilon)\varphi+\varepsilon\psi)$ will stand for an ideal sheaf on X_0 (and not for an ideal sheaf on X, even when the weight is possibly defined on the whole space X).

(14.4.5) Proposition. *Select* $0 < \varepsilon < 1$ *so small that* $e^{-\varepsilon\psi}$ *is locally integrable on* X *(possibly after shrinking the base* Δ*) and* $e^{-\varepsilon\psi}_{|X_0}$ *is locally integrable on* X_0*. Then*

$$\mathcal{I}((\ell-\varepsilon)\varphi_0 + (a+\varepsilon)\psi) \subset \mathcal{I}((\ell-1+a-\varepsilon)\varphi + \varepsilon\psi)$$

for every integer $\ell \ge 1$.

Proof. We argue by induction on ℓ. For $\ell = 1$, Lemma 14.4.4 implies

$$(1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi \le (1-\varepsilon)\varphi_0 + a\varphi + \varepsilon\psi \le C + (a-\varepsilon)\varphi + \varepsilon\psi$$

since φ_0 is locally bounded above by a constant C. Thus we get

$$\mathcal{I}((1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi) \subset \mathcal{I}((a-\varepsilon)\varphi + \varepsilon\psi),$$

as desired.

Now, suppose that the induction step ℓ has been settled. Take an arbitrary germ of function f in the ideal sheaf $\mathcal{I}((\ell+1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi)$, defined on a small neighborhood U of a point $P \in X_0$. Fix a local non vanishing holomorphic section e of $(\ell+a)K_{X_0}$ on U. Then $s = fe$ is a section of

$$\mathcal{O}((\ell+a)K_{X_0}) \otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi)$$

on U. Observe that, given an arbitrary plurisubharmonic function ξ, we have

$$\mathcal{I}(\xi + \log|s_D|^2) = \mathcal{I}(\xi) \otimes \mathcal{O}(-D).$$

Writing $aK_{X_0} = (D+F)_{|X_0}$ thanks to Lemma 14.4.1 and $a\psi = \log|s_D|^2 + \chi$ by definition of ψ, we can reinterpret s as a section of

$$\begin{aligned}\mathcal{O}(\ell K_{X_0} + (D+F)_{|X_0}) &\otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0 + \varepsilon\psi + \log|s_D|^2 + \chi)\\ &= \mathcal{O}((\ell+2)K_{X_0} + E_{|X_0}) \otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0 + \varepsilon\psi),\end{aligned}$$

where $E = F - 2K_X$ (since χ is smooth, adding χ does not change the multiplier ideal sheaf). Let us observe that $(\ell+1-\varepsilon)\varphi_0 + \varepsilon\psi$ defines a hermitian metric with positive curvature on $(\ell+1)K_{X_0}$. By the above hypothesis (14.4.2) and Proposition 14.2.1 applied with $Y = X_0$ and $L = (\ell+1)K_{X_0}$, we conclude that

$$\mathcal{O}((\ell+2)K_{X_0} + E_{|X_0}) \otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0 + \varepsilon\psi)$$

is generated by global sections on X_0. We can therefore, without loss of generality, restrict ourselves to the case of germs f such that fe coincides on U with a global section

$$s \in H^0\left(X_0, \mathcal{O}((\ell+2)K_{X_0} + E_{|X_0}) \otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0 + \varepsilon\psi)\right).$$

By reversing the order of calculations, we find

$$\begin{aligned}H^0\,(X_0, \mathcal{O}((\ell+2)K_{X_0} &+ E_{|X_0}) \otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0 + \varepsilon\psi))\\ &= H^0\,(X_0, \mathcal{O}((\ell+a)K_{X_0}) \otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi))\\ &\subset H^0\,(X_0, \mathcal{O}((\ell+a)K_{X_0}) \otimes \mathcal{I}((\ell-\varepsilon)\varphi_0 + (a+\varepsilon)\psi))\\ &\subset H^0\,(X_0, \mathcal{O}((\ell+a)K_{X_0}) \otimes \mathcal{I}((\ell+a-1-\varepsilon)\varphi + \varepsilon\psi)),\end{aligned}$$

[the first inclusion is obtained by neglecting the term φ_0 in the weight, and the second one is a consequence of the induction hypothesis for step ℓ]. Now, the weight

$$(\ell+a-1-\varepsilon)\varphi + \varepsilon\psi$$

defines a hermitian metric with positive definite curvature on $L = (\ell+a-1)K_X$, and Theorem 14.3.1 implies that s can be extended into a global section $\tilde{s} \in H^0(X, (\ell+a)K_X)$ (maybe after shrinking Δ). The definition of φ implies $|\tilde{s}|^2 \le C\,e^{(\ell+a)\varphi}$, hence

$$|\tilde{s}|^2 e^{-(\ell+a)\varphi - \varepsilon\psi} \le C\,e^{-\varepsilon\psi}$$

is integrable on X. from there, we conclude that

$$fe = s = \tilde{s}_{|X_0} \in \mathcal{O}((\ell+a)K_{X_0}) \otimes \mathcal{I}((\ell+a)\varphi + \varepsilon\psi),$$

hence $f \in \mathcal{I}((\ell+a)\varphi + \varepsilon\psi)$. The step $\ell+1$ of the induction is proved.

14.5. Proof of the invariance of plurigenera.

Fix an integer $m > 0$ and an arbitrary section $s \in H^0(X_0, mK_{X_0})$. By definition of φ_0, we have $|s|^2 \le Ce^{m\varphi_0}$ on X_0. If s_D is the canonical section of $\mathcal{O}(-D)$ of divisor D, we conclude that $s^\ell s_D$ is locally L^2 with respect to the weight $e^{-\ell m\varphi_0-(a+\varepsilon)\psi}$, for $a\psi$ has the same singularities as $\log|s_D|^2$ and $e^{-\varepsilon\psi}$ is supposed to be locally integrable on X_0. The functions $s^\ell s_D$ are locally in the ideal sheaf $\mathcal{I}((\ell m-\varepsilon)\varphi_0 + (a+\varepsilon)\psi)$. By Proposition 14.4.5, they belong to $\mathcal{I}((\ell m-1-\varepsilon)\varphi+\varepsilon\psi)$, i.e.

$$\int_U |s^\ell s_D|^2 e^{-(\ell m-1-\varepsilon)\varphi-\varepsilon\psi} < +\infty$$

on each sufficiently small open set U such that $K_{X_0}|U$ and $\mathcal{O}(-D)|U$ are trivial. In an equivalent way, we can write

$$\int_U |s|^{2\ell} e^{-(\ell m-1-\varepsilon)\varphi+(a-\varepsilon)\psi} < +\infty.$$

Take ℓ large enough so that $a/(\ell-1) \le \varepsilon$. Then $e^{\varphi-\frac{a}{\ell-1}\psi} \le Ce^{-\varepsilon\psi}$ is integrable on U. By Hölder inequality with conjugate exponents ℓ, $\ell' = \ell/(\ell-1)$, we find

$$\begin{aligned} +\infty &> \left(\int_U |s|^{2\ell} e^{-(\ell m-1-\varepsilon)\varphi+(a-\varepsilon)\psi}\right)^{1/\ell} \left(\int_U e^{\varphi-\frac{a}{\ell-1}\psi}\right)^{(\ell-1)/\ell} \\ &\ge \int_U |s|^2 e^{-(m-\frac{1}{\ell}-\frac{\varepsilon}{\ell})\varphi+(\frac{a}{\ell}-\frac{\varepsilon}{\ell})\psi} e^{(1-\frac{1}{\ell})\varphi-\frac{a}{\ell}\psi} = \int_U |s|^2 e^{-(m-1-\delta)\varphi-\delta\psi} \end{aligned}$$

with $\delta = \varepsilon/\ell$. We can consider s as a section of $K_{X_0} + L_{|X_0}$ with $L = (m-1)K_X$. The weight $(m-1-\delta)\varphi - \delta\psi$ defines a hermitian metric on L with positive definite curvature, and s is globally L^2 with respect to this metric. By the Ohsawa-Takegoshi-Manivel extension theorem, we can extend s into a section $\tilde{s} \in H^0(X, K_X + L) = H^0(X, mK_X)$, possibly after shrinking Δ again. This achieves the proof of the main theorem. □

15. Subadditivity of Multiplier Ideals and Fujita's Approximate Zariski Decompostion

We first notice the following basic restriction formula, which is just a rephrasing of the Ohsawa-Takegoshi extension theorem.

(15.1) Restriction formula. *Let φ be a plurisubharmonic function on a complex manifold X, and let $Y \subset X$ be a submanifold. Then*

$$\mathcal{I}(\varphi_{|Y}) \subset \mathcal{I}(\varphi)_{|Y}.$$

Thus, in some sense, the singularities of φ can only get worse if we restrict to a submanifold (if the restriction of φ to some connected component of Y is identically $-\infty$, we agree that the corresponding multiplier ideal sheaf is zero). The proof is straightforward and just amounts to extending locally a germ of function f on Y near a point $y_0 \in Y$ to a function $\tilde{f}$ on a small Stein neighborhood of y_0 in X, which is possible by the Ohsawa-Takegoshi extension theorem. As a direct consequence, we get:

(15.2) Subadditivity Theorem.

(i) *Let X_1, X_2 be complex manifolds, $\pi_i : X_1 \times X_2 \to X_i$, $i = 1, 2$ the projections, and let φ_i be a plurisubharmonic function on X_i. Then*

$$\mathcal{I}(\varphi_1 \circ \pi_1 + \varphi_2 \circ \pi_2) = \pi_1^\star \mathcal{I}(\varphi_1) \cdot \pi_2^\star \mathcal{I}(\varphi_2).$$

(ii) *Let X be a complex manifold and let φ, ψ be plurisubharmonic functions on X. Then*

$$\mathcal{I}(\varphi + \psi) \subset \mathcal{I}(\varphi) \cdot \mathcal{I}(\psi)$$

Proof. (i) Let us fix two relatively compact Stein open subsets $U_1 \subset X_1$, $U_2 \subset X_2$. Then $\mathcal{H}^2(U_1 \times U_2, \varphi_1 \circ \pi_1 + \varphi_2 \circ \pi_2, \pi_1^\star dV_1 \otimes \pi_2^\star dV_2)$ is the Hilbert tensor product of $\mathcal{H}^2(U_1, \varphi_1, dV_1)$ and $\mathcal{H}^2(U_2, \varphi_2, dV_2)$, and admits $(f'_k \boxtimes f''_l)$ as a Hilbert basis, where (f'_k) and (f''_l) are respective Hilbert bases. Since $\mathcal{I}(\varphi_1 \circ \pi_1 + \varphi_2 \circ \pi_2)_{|U_1 \times U_2}$ is generated as an $\mathcal{O}_{U_1 \times U_2}$ module by the $(f'_k \boxtimes f''_l)$ (Proposition 5.7), we conclude that (i) holds true.

(ii) We apply (i) to $X_1 = X_2 = X$ and the restriction formula to $Y =$ diagonal of $X \times X$. Then

$$\begin{aligned}\mathcal{I}(\varphi + \psi) &= \mathcal{I}\big((\varphi \circ \pi_1 + \psi \circ \pi_2)_{|Y}\big) \subset \mathcal{I}\big(\varphi \circ \pi_1 + \psi \circ \pi_2\big)_{|Y} \\ &= \Big(\pi_1^\star \mathcal{I}(\varphi) \otimes \pi_2^\star \mathcal{I}(\psi)\Big)_{|Y} = \mathcal{I}(\varphi) \cdot \mathcal{I}(\psi).\end{aligned}$$

(15.3) Proposition. *Let $f : X \to Y$ be an arbitrary holomorphic map, and let φ be a plurisubharmonic function on Y. Then $\mathcal{I}(\varphi \circ f) \subset f^\star \mathcal{I}(\varphi)$.*

Proof. Let

$$\Gamma_f = \{(x, f(x)\,;\, x \in X\} \subset X \times Y$$

be the graph of f, and let $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ be the natural projections. Then we can view $\varphi \circ f$ as the restriction of $\varphi \circ \pi_Y$ to Γ_f, as π_X is a biholomorphism from Γ_f to X. Hence the restriction formula implies

$$\mathcal{I}(\varphi \circ f) = \mathcal{I}\big((\varphi \circ \pi_Y)_{|\Gamma_f}\big) \subset \mathcal{I}(\varphi \circ \pi_Y)_{|\Gamma_f} = \big(\pi_Y^\star \mathcal{I}(\varphi)\big)_{|\Gamma_f} = f^\star \mathcal{I}(\varphi). \qquad \square$$

As an application of subadditivity, we now reprove a result of Fujita [Fuj93], relating the growth of sections of multiples of a line bundle to the Chern numbers of its “largest nef part”. Fujita’s original proof is by contradiction, using the Hodge index theorem and intersection inequalities. The present method arose in the course of joint work with R. Lazarsfeld [Laz99].

Let X be a projective n-dimensional algebraic variety and L a line bundle over X. We define the *volume* of L to be

$$v(L) = \limsup_{k\to+\infty} \frac{n!}{k^n} h^0(X, kL) \in [0, +\infty[.$$

The line bundle is said to be *big* if $v(L) > 0$. If L is ample, we have $h^q(X, kL) = 0$ for $q \geq 1$ and $k \gg 1$ by the Kodaira-Serre vanishing theorem, hence

$$h^0(X, kL) \sim \chi(X, kL) \sim \frac{L^n}{n!} k^n$$

by the Riemann-Roch formula. Thus $v(L) = L^n$ ($= c_1(L)^n$) if L is ample. This is still true if L is nef (numerically effective), i.e. if $L \cdot C \geq 0$ for every effective curve C. In fact, one can show that $h^q(X, kL) = O(k^{n-q})$ in that case. The following well-known proposition characterizes big line bundles.

(15.4) Proposition. *The line bundle L is big if and only if there is a multiple $m_0 L$ such that $m_0 L = E + A$, where E is an effective divisor and A an ample divisor.*

Proof. If the condition is satisfied, the decomposition $km_0L = kE + kA$ gives rise to an injection $H^0(X, kA) \hookrightarrow H^0(X, km_0L)$, thus $v(L) \geq m_0^{-n} v(A) > 0$. Conversely, assume that L is big, and take A to be a very ample nonsingular divisor in X. The exact sequence

$$0 \longrightarrow \mathcal{O}_X(kL - A) \longrightarrow \mathcal{O}_X(kL) \longrightarrow \mathcal{O}_A(kL_{|A}) \longrightarrow 0$$

gives rise to a cohomology exact sequence

$$0 \to H^0(X, kL - A) \longrightarrow H^0(X, kL) \longrightarrow H^0(A, kL_{|A}),$$

and $h^0(A, kL_{|A}) = O(k^{n-1})$ since $\dim A = n - 1$. Now, the assumption that L is big implies that $h^0(X, kL) > ck^n$ for infinitely many k, hence $H^0(X, m_0L - A) \neq 0$ for some large integer m_0. If E is the divisor of a section in $H^0(X, m_0L - A)$, we find $m_0L - A = E$, as required. □

(15.5) Lemma. *Let G be an arbitrary line bundle. For every $\varepsilon > 0$, there exists a positive integer m and a sequence $\ell_\nu \uparrow +\infty$ such that*

$$h^0\big(X, \ell_\nu(mL - G)\big) \geq \frac{\ell_\nu^m m^n}{n!}\big(v(L) - \varepsilon\big),$$

in other words, $v(mL - G) \geq m^n(v(L) - \varepsilon)$ for m large enough.

Proof. Clearly, $v(mL - G) \geq v(mL - (G + E))$ for every effective divisor E. We can take E so large that $G + E$ is very ample, and we are thus reduced to the case where G is very ample by replacing G with $G + E$. By definition of $v(L)$, there exists a sequence $k_\nu \uparrow +\infty$ such that

$$h^0(X, k_\nu L) \geq \frac{k_\nu^n}{n!}\Big(v(L) - \frac{\varepsilon}{2}\Big).$$

We take $m \gg 1$ (to be precisely chosen later), and $\ell_\nu = \big[\frac{k_\nu}{m}\big]$, so that $k_\nu = \ell_\nu m + r_\nu$, $0 \leq r_\nu < m$. Then

$$\ell_\nu(mL - G) = k_\nu L - (r_\nu L + \ell_\nu G).$$

Fix a constant $a \in \mathbb{N}$ such that $aG - L$ is an effective divisor. Then $r_\nu L \le maG$ (with respect to the cone of effective divisors), hence

$$h^0\big(X, \ell_\nu(mL - G)\big) \ge h^0\big(X, k_\nu L - (\ell_\nu + am)G\big).$$

We select a smooth divisor D in the very ample linear system $|G|$. By looking at global sections associated with the exact sequences of sheaves

$$0 \to \mathcal{O}(-(j+1)D) \otimes \mathcal{O}(k_\nu L) \to \mathcal{O}(-jD) \otimes \mathcal{O}(k_\nu L) \to \mathcal{O}_D(k_\nu L - jD) \to 0,$$

$0 \le j < s$, we infer inductively that

$$\begin{aligned} h^0(X, k_\nu L - sD) &\ge h^0(X, k_\nu L) - \sum_{0 \le j < s} h^0\big(D, \mathcal{O}_D(k_\nu L - jD)\big) \\ &\ge h^0(X, k_\nu L) - s\, h^0\big(D, k_\nu L_{|D}\big) \\ &\ge \frac{k_\nu^n}{n!}\Big(v(L) - \frac{\varepsilon}{2}\Big) - s\, C k_\nu^{n-1} \end{aligned}$$

where C depends only on L and G. Hence, by putting $s = \ell_\nu + am$, we get

$$\begin{aligned} h^0\big(X, \ell_\nu(mL - G)\big) &\ge \frac{k_\nu^n}{n!}\Big(v(L) - \frac{\varepsilon}{2}\Big) - C(\ell_\nu + am)k_\nu^{n-1} \\ &\ge \frac{\ell_\nu^n m^n}{n!}\Big(v(L) - \frac{\varepsilon}{2}\Big) - C(\ell_\nu + am)(\ell_\nu + 1)^{n-1} m^{n-1} \end{aligned}$$

and the desired conclusion follows by taking $\ell_\nu \gg m \gg 1$. □

We are now ready to prove Fujita's decomposition theorem, as reproved in [DEL00].

(15.6) Theorem (Fujita). *Let L be a big line bundle. Then for every $\varepsilon > 0$, there exists a modification $\mu : \widetilde{X} \to X$ and a decomposition $\mu^\star L = E + A$, where E is an effective $\mathbb{Q}$-divisor and A an ample $\mathbb{Q}$-divisor, such that $A^n > v(L) - \varepsilon$.*

(15.7) Remark. Of course, if $\mu^\star L = E + A$ with E effective and A nef, we get an injection

$$H^0(\widetilde{X}, kA) \hookrightarrow H^0(\widetilde{X}, kE + kA) = H^0(\widetilde{X}, k\mu^\star L) = H^0(X, kL)$$

for every integer k which is a multiple of the denominator of E, hence $A^n \le v(L)$.

(15.8) Remark. Once Theorem 15.6 is proved, the same kind of argument easily shows that

$$v(L) = \lim_{k \to +\infty} \frac{n!}{k^n} h^0(X, kL),$$

because the formula is true for every ample line bundle A.

Proof of Theorem 15.6. It is enough to prove the theorem with A being a big and nef divisor. In fact, Proposition 15.4 then shows that we can write $A = E' + A'$ where E' is an effective $\mathbb{Q}$-divisor and A' an ample $\mathbb{Q}$-divisor, hence

$$E + A = E + \varepsilon E' + (1-\varepsilon)A + \varepsilon A'$$

where $A'' = (1-\varepsilon)A + \varepsilon A'$ is ample and the intersection number A''^n approaches A^n as closely as we want. Let G be as in Lemma 14.2.1 (uniform global generation). Lemma 15.5 implies that $v(mL-G) > m^n(v(L)-\varepsilon)$ for m large. By Theorem 14.1.2, there exists a hermitian metric h_m of weight φ_m on $mL - G$ such that

$$H^0\big(X, \ell(mL-G)\big) = H^0\big(X, \ell(mL-G) \otimes \mathcal{I}(\ell\varphi_m)\big)$$

for every $\ell \geq 0$. We take a smooth modification $\mu : \widetilde{X} \to X$ such that

$$\mu^\star \mathcal{I}(\varphi_m) = \mathcal{O}_{\widetilde{X}}(-E)$$

is an invertible ideal sheaf in $\mathcal{O}_{\widetilde{X}}$. This is possible by taking the blow-up of X with respect to the ideal $\mathcal{I}(\varphi_m)$ and by resolving singularities (Hironaka [Hir64]). Lemma 14.2.1 applied to $L' = mL - G$ implies that $\mathcal{O}(mL) \otimes \mathcal{I}(\varphi_m)$ is generated by its global sections, hence its pull-back $\mathcal{O}(m\,\mu^\star L - E)$ is also generated. This implies

$$m\,\mu^\star L = E + A$$

where E is an effective divisor and A is a nef (semi-ample) divisor in $\widetilde{X}$. We find

$$\begin{aligned} H^0(\widetilde{X}, \ell A) &= H^0\big(\widetilde{X}, \ell(m\,\mu^\star L - E)\big) \\ &\supset H^0\big(\widetilde{X}, \mu^\star\big(\mathcal{O}(\ell m L) \otimes \mathcal{I}(\varphi_m)^\ell\big)\big) \\ &\supset H^0\big(\widetilde{X}, \mu^\star\big(\mathcal{O}(\ell m L) \otimes \mathcal{I}(\ell\varphi_m)\big)\big), \end{aligned}$$

thanks to the subadditivity property of multiplier ideals. Moreover, the direct image $\mu_\star\mu^\star\mathcal{I}(\ell\varphi_m)$ coincides with the integral closure of $\mathcal{I}(\ell\varphi_m)$, hence with $\mathcal{I}(\ell\varphi_m)$, because a multiplier ideal sheaf is always integrally closed. From this we infer

$$\begin{aligned} H^0(\widetilde{X}, \ell A) &\supset H^0\big(X, \mathcal{O}(\ell m L) \otimes \mathcal{I}(\ell\varphi_m)\big) \\ &\supset H^0\big(X, \mathcal{O}(\ell(mL-G)) \otimes \mathcal{I}(\ell\varphi_m)\big) \\ &= H^0\big(X, \mathcal{O}(\ell(mL-G))\big). \end{aligned}$$

By Lemma 15.5, we find

$$h^0(\widetilde{X}, \ell A) \geq \frac{\ell^n}{n!} m^n \big(v(L) - \varepsilon\big)$$

for infinitely many ℓ, therefore $v(A) = A^n \geq m^n(v(L)-\varepsilon)$. Theorem 15.6 is proved, up to a minor change of notation $E \mapsto \frac{1}{m}E$, $A \mapsto \frac{1}{m}A$. □

We conclude by using Fujita's theorem to establish a geometric interpretation of the volume $v(L)$. Suppose as above that X is a smooth projective variety of dimension n, and that L is a big line bundle on X. Given a large integer $k \gg 0$, denote by $B_k \subset X$ the base-locus of the linear system $|kL|$. The *moving self-intersection number* $(kL)^{[n]}$ of $|kL|$ is defined by choosing n general divisors $D_1, \ldots, D_n \in |kL|$ and putting

$$(kL)^{[n]} = \#\Big(D_1 \cap \ldots \cap D_n \cap (X - B_k)\Big).$$

In other words, we simply count the number of intersection points away from the base locus of n general divisors in the linear system $|kL|$. This notion arises for example in Matsusaka's proof of his "big theorem". We show that the volume $v(L)$

of L measures the rate of growth with respect to k of these moving self-intersection numbers:

(15.9) Proposition. *One has*

$$v(L) = \limsup_{k\to\infty} \frac{(kL)^{[n]}}{k^n}.$$

Proof. We start by interpreting $(kL)^{[n]}$ geometrically. Let $\mu_k : X_k \longrightarrow X$ be a modification of $|kL|$ such that $\mu_k^\star|kL| = |V_k| + F_k$, where

$$P_k := \mu_k^\star(kL) - F_k$$

is generated by sections, and $H^0(X, \mathcal{O}_X(kL)) = V_k = H^0(X_k, \mathcal{O}_{X_k}(P_k))$, so that $B_k = \mu_k(F_k)$. Then evidently $(kL)^{[n]}$ counts the number of intersection points of n general divisors in P_k, and consequently

$$(kL)^{[n]} = (P_k)^n.$$

Since then P_k is big (and nef) for $k \gg 0$, we have $v(P_k) = (P_k)^n$. Also, $v(kL) \geq v(P_k)$ since P_k embeds in $\mu_k^\star(kL)$. Hence

$$v(kL) \geq (kL)^{[n]} \qquad \forall k \gg 0.$$

On the other hand, an easy argument in the spirit of Lemma (15.5) shows that $v(kL) = k^n \cdot v(L)$ (cf. also [ELN96], Lemma 3.4), and so we conclude that

$$v(L) \geq \frac{(kL)^{[n]}}{k^n}. \tag{15.10}$$

for every $k \gg 0$.

For the reverse inequality we use Fujita's theorem. Fix $\varepsilon > 0$, and consider the decomposition $\mu^\star L = A + E$ on $\mu : \widetilde{X} \longrightarrow X$ constructed in Fujita's theorem. Let k be any positive integer such that kA is integral and globally generated. By taking a common resolution we can assume that X_k dominates $\widetilde{X}$, and hence we can write

$$\mu_k^\star kL \sim A_k + E_k$$

with A_k globally generated and

$$(A_k)^n \geq k^n \cdot (v(L) - \varepsilon).$$

But then A_k embeds in P_k and both $\mathcal{O}(A_k)$ and $\mathcal{O}(P_k)$ are globally generated, consequently

$$(A_k)^n \leq (P_k)^n = (kL)^{[n]}.$$

Therefore

$$\frac{(kL)^{[n]}}{k^n} \geq v(L) - \varepsilon. \tag{15.11}$$

But (15.11) holds for any sufficiently large and divisible k, and in view of (15.10) the Proposition follows. □

16. Hard Lefschetz Theorem with Multiplier Ideal Sheaves

16.1. Main statement

The goal of this section is to prove the following surjectivity theorem, which can be seen as an extension of the hard Lefschetz theorem. We closely follow the exposition of [DPS00].

(16.1.1) Theorem. *Let (L, h) be a pseudo-effective line bundle on a compact Kähler manifold (X, ω), of dimension n, let $\Theta_h(L) \geq 0$ be its curvature current and $\mathcal{I}(h)$ the associated multiplier ideal sheaf. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism*

$$\Phi^q_{\omega,h} : H^0(X, \Omega^{n-q}_X \otimes L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h)).$$

The special case when L is nef is due to Takegoshi [Tak97]. An even more special case is when L is semi-positive, i.e. possesses a smooth metric with semi-positive curvature. In that case the multiple ideal sheaf $\mathcal{I}(h)$ coincides with $\mathcal{O}_X$ and we get the following consequence already observed by Mourougane [Mou99].

(16.1.2) Corollary. *Let (L, h) be a semi-positive line bundle on a compact Kähler manifold (X, ω) of dimension n. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism*

$$\Phi^q_{\omega} : H^0(X, \Omega^{n-q}_X \otimes L) \longrightarrow H^q(X, \Omega^n_X \otimes L).$$

The proof of Theorem (16.1.1) is based on the Bochner formula, combined with a use of harmonic forms with values in the hermitian line bundle (L, h). The method can be applied only after h has been made smooth at least in the complement of an analytic set. However, we have to accept singularities even in the regularized metrics because only a very small incompressible loss of positivity is acceptable in the Bochner estimate (by the results of [Dem92], singularities can only be removed at the expense of a fixed loss of positivity). Also, we need the multiplier ideal sheaves to be preserved by the smoothing process. This is possible thanks to a suitable "equisingular" regularization process.

16.2. Equisingular approximations of quasi plurisubharmonic functions

A quasi-plurisubharmonic (quasi-psh) function is by definition a function φ which is locally equal to the sum of a psh function and of a smooth function, or equivalently, a locally integrable function φ such that $i\partial\overline{\partial}\varphi$ is locally bounded below by $-C\omega$ where ω is a hermitian metric and C a constant. We say that φ has logarithmic poles if φ is locally bounded outside an analytic set A and has singularities of the form

$$\varphi(z) = c \log \sum_k |g_k|^2 + O(1)$$

with $c > 0$ and g_k holomorphic, on a neighborhood of every point of A. Our goal is to show the following

(16.2.1) Theorem. *Let $T = \alpha + i\partial\overline{\partial}\varphi$ be a closed $(1,1)$-current on a compact hermitian manifold (X,ω), where α is a smooth closed $(1,1)$-form and φ a quasi-psh function. Let γ be a continuous real $(1,1)$-form such that $T \ge \gamma$. Then one can write $\varphi = \lim_{\nu\to+\infty} \varphi_\nu$ where*

a) *φ_ν is smooth in the complement $X \smallsetminus Z_\nu$ of an analytic set $Z_\nu \subset X$;*

b) *(φ_ν) is a decreasing sequence, and $Z_\nu \subset Z_{\nu+1}$ for all ν;*

c) *$\int_X (e^{-2\varphi} - e^{-2\varphi_\nu})dV_\omega$ is finite for every ν and converges to 0 as $\nu \to +\infty$;*

d) *$\mathcal{I}(\varphi_\nu) = \mathcal{I}(\varphi)$ for all ν ("equisingularity");*

e) *$T_\nu = \alpha + i\partial\overline{\partial}\varphi_\nu$ satisfies $T_\nu \ge \gamma - \varepsilon_\nu\omega$, where $\lim_{\nu\to+\infty} \varepsilon_\nu = 0$.*

(16.2.2) Remark. It would be interesting to know whether the φ_ν can be taken to have logarithmic poles along Z_ν. Unfortunately, the proof given below destroys this property in the last step. Getting it to hold true seems to be more or less equivalent to proving the semi-continuity property

$$\lim_{\varepsilon\to 0_+} \mathcal{I}((1+\varepsilon)\varphi) = \mathcal{I}(\varphi).$$

Actually, this can be checked in dimensions 1 and 2, but is unknown in higher dimensions (and probably quite hard to establish).

Proof *of Theorem 16.2.1.* Clearly, by replacing T with $T - \alpha$ and γ with $\gamma - \alpha$, we may assume that $\alpha = 0$ and $T = i\partial\overline{\partial}\varphi \ge \gamma$. We divide the proof in four steps.

Step 1. Approximation by quasi-psh functions with logarithmic poles.

By [Dem92], there is a decreasing sequence (ψ_ν) of quasi-psh functions with logarithmic poles such that $\varphi = \lim \psi_\nu$ and $i\partial\overline{\partial}\psi_\nu \ge \gamma - \varepsilon_\nu\omega$. We need a little bit more information on those functions, hence we first recall the main techniques used for the construction of (ψ_ν). For $\varepsilon > 0$ given, fix a covering of X by open balls $B_j = \{|z^{(j)}| < r_j\}$ with coordinates $z^{(j)} = (z_1^{(j)}, \ldots, z_n^{(j)})$, such that

$$0 \le \gamma + c_j\, i\partial\overline{\partial}|z^{(j)}|^2 \le \varepsilon\omega \qquad \text{on} \quad B_j, \tag{16.2.3}$$

for some real number c_j. This is possible by selecting coordinates in which γ is diagonalized at the center of the ball, and by taking the radii $r_j > 0$ small enough (thanks to the fact that γ is continuous). We may assume that these coordinates come from a finite sample of coordinates patches covering X, on which we perform suitable linear coordinate changes (by invertible matrices lying in some compact subset of the complex linear group). By taking additional balls, we may also assume that $X = \bigcup B''_j$ where

$$B''_j \subset\subset B'_j \subset\subset B_j$$

are concentric balls $B'_j = \{|z^{(j)}| < r'_j = r_j/2\}$, $B''_j = \{|z^{(j)}| < r''_j = r_j/4\}$. We define

$$\psi_{\varepsilon,\nu,j} = \frac{1}{2\nu}\log\sum_{k\in\mathbb{N}} |f_{\nu,j,k}|^2 - c_j|z^{(j)}|^2 \qquad \text{on} \quad B_j, \tag{16.2.4}$$

where $(f_{\nu,j,k})_{k\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_{\nu,j}$ of holomorphic functions on B_j with finite L^2 norm

$$\|u\|^2 = \int_{B_j} |u|^2 e^{-2\nu(\varphi + c_j|z^{(j)}|^2)} d\lambda(z^{(j)}).$$

(The dependence of $\psi_{\varepsilon,\nu,j}$ on ε is through the choice of the open covering (B_j)). Observe that the choice of c_j in (16.2.3) guarantees that $\varphi + c_j|z^{(j)}|^2$ is plurisubharmonic on B_j, and notice also that

$$\sum_{k\in\mathbb{N}} |f_{\nu,j,k}(z)|^2 = \sup_{f\in\mathcal{H}_{\nu,j},\, \|f\|\le 1} |f(z)|^2 \tag{16.2.5}$$

is the square of the norm of the continuous linear form $\mathcal{H}_{\nu,j} \to \mathbb{C}$, $f \mapsto f(z)$. We claim that there exist constants C_i, $i = 1, 2, \ldots$ depending only on X and γ (thus independent of ε and ν), such that the following uniform estimates hold:

$$i\partial\overline{\partial}\psi_{\varepsilon,\nu,j} \ge -c_j\, i\partial\overline{\partial}|z^{(j)}|^2 \ge \gamma - \varepsilon\omega \qquad \text{on } B'_j \ \ (B'_j \subset\subset B_j), \tag{16.2.6}$$

$$\varphi(z) \le \psi_{\varepsilon,\nu,j}(z) \le \sup_{|\zeta - z|\le r} \varphi(\zeta) + \frac{n}{\nu}\log\frac{C_1}{r} + C_2 r^2 \qquad \forall z \in B'_j,\ \ r < r_j - r'_j, \tag{16.2.7}$$

$$|\psi_{\varepsilon,\nu,j} - \psi_{\varepsilon,\nu,k}| \le \frac{C_3}{\nu} + C_4\varepsilon\big(\min(r_j, r_k)\big)^2 \qquad \text{on } B'_j \cap B'_k. \tag{16.2.8}$$

Actually, the Hessian estimate (16.2.6) is obvious from (16.2.3) and (16.2.4). As in the proof of ([Dem92], Prop. 3.1), (16.2.7) results from the Ohsawa-Takegoshi L^2 extension theorem (left hand inequality) and from the mean value inequality (right hand inequality). Finally, as in ([Dem92], Lemma 3.6 and Lemma 4.6), (16.2.8) is a consequence of Hörmander's L^2 estimates. We briefly sketch the idea. Assume that the balls B_j are small enough, so that the coordinates $z^{(j)}$ are still defined on a neighborhood of all balls $\overline{B}_k$ which intersect B_j (these coordinates can be taken to be linear transforms of coordinates belonging to a fixed finite set of coordinate patches covering X, selected once for all). Fix a point $z_0 \in B'_j \cap B'_k$. By (16.2.4) and (16.2.5), we have

$$\psi_{\varepsilon,\nu,j}(z_0) = \frac{1}{\nu}\log|f(z_0)| - c_j|z^{(j)}|^2$$

for some holomorphic function f on B_j with $\|f\| = 1$. We consider the weight function

$$\Phi(z) = 2\nu(\varphi(z) + c_k|z^{(k)}|^2) + 2n\log|z^{(k)} - z_0^{(k)}|,$$

on both B_j and B_k. The trouble is that a priori we have to deal with different weights, hence a comparison of weights is needed. By the Taylor formula applied at z_0, we get

$$\Big|c_k|z^{(k)} - z_0^{(k)}|^2 - c_j|z^{(j)} - z_0^{(j)}|^2\Big| \le C\varepsilon\big(\min(r_j, r_k)\big)^2 \qquad \text{on } B_j \cap B_k$$

[the only nonzero term of degree 2 has type $(1,1)$ and its Hessian satisfies

$$-\varepsilon\omega \le i\partial\overline{\partial}(c_k|z^{(k)}|^2 - c_j|z^{(j)}|^2) \le \varepsilon\omega$$

by (16.2.3); we may suppose $r_j \ll \varepsilon$ so that the terms of order 3 and more are negligible]. By writing $|z^{(j)}|^2 = |z^{(j)} - z_0^{(j)}|^2 + |z_0^{(j)}|^2 + 2\,\mathrm{Re}\langle z^{(j)} - z_0^{(j)}, z_0^{(j)}\rangle$, we obtain

$$\begin{aligned} c_k|z^{(k)}|^2 - c_j|z^{(j)}|^2 = {} & 2c_k\,\mathrm{Re}\langle z^{(k)} - z_0^{(k)}, z_0^{(k)}\rangle - 2c_j\,\mathrm{Re}\langle z^{(j)} - z_0^{(j)}, z_0^{(j)}\rangle \\ & + c_k|z_0^{(k)}|^2 - c_j|z_0^{(j)}|^2 \pm C\varepsilon(\min(r_j, r_k))^2. \end{aligned}$$

We use a cut-off function θ equal to 1 in a neighborhood of z_0 and with support in $B_j \cap B_k$; as $z_0 \in B'_j \cap B'_k$, the function θ can be taken to have its derivatives uniformly bounded when z_0 varies. We solve the equation $\overline{\partial} u = \overline{\partial}(\theta f e^{\nu g})$ on B_k, where g is the holomorphic function

$$g(z) = c_k \langle z^{(k)} - z_0^{(k)}, z_0^{(k)} \rangle - c_j \langle z^{(j)} - z_0^{(j)}, z_0^{(j)} \rangle.$$

Thanks to Hörmander's L^2 estimates [Hör66], the L^2 solution for the weight Φ yields a holomorphic function $f' = \theta f e^{\nu g} - u$ on B_k such that $f'(z_0) = f(z_0)$ and

$$\int_{B_k} |f'|^2 e^{-2\nu(\varphi + c_k |z^{(k)}|^2)} d\lambda(z^{(k)}) \le C' \int_{B_j \cap B_k} |f|^2 |e^{\nu g}|^2 e^{-2\nu(\varphi + c_k |z^{(k)}|^2)} d\lambda(z^{(k)}) \le$$
$$C' \exp\Big(2\nu\big(c_k |z_0^{(k)}|^2 - c_j |z_0^{(j)}|^2 + C\varepsilon(\min(r_j, r_k))^2\big)\Big) \int_{B_j} |f|^2 e^{-2\nu(\varphi + c_j |z^{(j)}|^2)} d\lambda(z^{(j)}).$$

Let us take the supremum of $\frac{1}{\nu} \log |f(z_0)| = \frac{1}{\nu} \log |f'(z_0)|$ over all f with $\|f\| \le 1$. By the definition of $\psi_{\varepsilon,\nu,k}$ ((16.2.4) and (16.2.5)) and the bound on $\|f'\|$, we find

$$\psi_{\varepsilon,\nu,k}(z_0) \le \psi_{\nu,j}(z_0) + \frac{\log C'}{2\nu} + C\varepsilon(\min(r_j, r_k))^2,$$

whence (16.2.8) by symmetry. Assume that ν is so large that $C_3/\nu < C_4 \varepsilon (\inf_j r_j)^2$. We "glue" all functions $\psi_{\varepsilon,\nu,j}$ into a function $\psi_{\varepsilon,\nu}$ globally defined on X, and for this we set

$$\psi_{\varepsilon,\nu}(z) = \sup_{j,\ B'_j \ni z} \Big(\psi_{\varepsilon,\nu,j}(z) + 12\, C_4 \varepsilon (r_j'^2 - |z^{(j)}|^2)\Big) \qquad \text{on } X.$$

Every point of X belongs to some ball B''_k, and for such a point we get

$$12\, C_4 \varepsilon (r_k'^2 - |z^{(k)}|^2) \ge 12\, C_4 \varepsilon (r_k'^2 - r_k''^2) > 2 C_4 r_k^2 > \frac{C_3}{\nu} + C_4 \varepsilon (\min(r_j, r_k))^2.$$

This, together with (16.2.8), implies that in $\psi_{\varepsilon,\nu}(z)$ the supremum is never reached for indices j such that $z \in \partial B'_j$, hence $\psi_{\varepsilon,\nu}$ is well defined and continuous, and by standard properties of upper envelopes of (quasi)-plurisubharmonic functions we get

$$i\partial\overline{\partial}\psi_{\varepsilon,\nu} \ge \gamma - C_5 \varepsilon \omega \tag{16.2.9}$$

for $\nu \ge \nu_0(\varepsilon)$ large enough. By inequality (16.2.7) applied with $r = e^{-\sqrt{\nu}}$, we see that $\lim_{\nu \to +\infty} \psi_{\varepsilon,\nu}(z) = \varphi(z)$. At this point, the difficulty is to show that $\psi_{\varepsilon,\nu}$ is decreasing with ν – this may not be formally true, but we will see at Step 3 that this is essentially true. Another difficulty is that we must simultaneously let ε go to 0, forcing us to change the covering as we want the error to get smaller and smaller in (16.2.9).

Step 2. A comparison of integrals.
We claim that

$$I := \int_X \big(e^{-2\varphi} - e^{-2\max(\varphi, \frac{\ell}{\ell-1}\psi_{\nu,\varepsilon}) + a}\big) dV_\omega < +\infty \tag{16.2.10}$$

for every $\ell \in \,]1, \nu]$ and $a \in \mathbb{R}$. In fact

$$\begin{aligned} I &\le \int_{\{\varphi<\frac{\ell}{\ell-1}\psi_{\varepsilon,\nu}+a\}} e^{-2\varphi}dV_\omega = \int_{\{\varphi<\frac{\ell}{\ell-1}\psi_{\varepsilon,\nu}\}+a} e^{2(\ell-1)\varphi-2\ell\varphi}dV_\omega \\ &\le e^{2(\ell-1)a}\int_X e^{2\ell(\psi_{\varepsilon,\nu}-\varphi)}dV_\omega \le C\Big(\int_X e^{2\nu(\psi_{\varepsilon,\nu}-\varphi)}dV_\omega\Big)^{\frac{\ell}{\nu}} \end{aligned}$$

by Hölder's inequality. In order to show that these integrals are finite, it is enough, by the definition and properties of the functions $\psi_{\varepsilon,\nu}$ and $\psi_{\varepsilon,\nu,j}$, to prove that

$$\int_{B'_j} e^{2\nu\psi_{\varepsilon,\nu,j}-2\nu\varphi}d\lambda = \int_{B'_j}\Big(\sum_{k=0}^{+\infty}|f_{\nu,j,k}|^2\Big)e^{-2\nu\varphi}d\lambda < +\infty.$$

By the strong Noetherian property of coherent ideal sheaves (see e.g. [GR84]), we know that the sequence of ideal sheaves generated by the holomorphic functions $(f_{\nu,j,k}(z)\overline{f_{\nu,j,k}(\overline{w})})_{k\le k_0}$ on $B_j\times B_j$ is locally stationary as k_0 increases, hence independant of k_0 on $B'_j\times B'_j \subset\subset B_j\times B_j$ for k_0 large enough. As the sum of the series $\sum_k f_{\nu,j,k}(z)\overline{f_{\nu,j,k}(\overline{w})}$ is bounded by

$$\Big(\sum_k |f_{\nu,j,k}(z)|^2\sum_k|f_{\nu,j,k}(\overline{w})|^2\Big)^{1/2}$$

and thus uniformly covergent on every compact subset of $B_j\times B_j$, and as the space of sections of a coherent ideal sheaf is closed under the topology of uniform convergence on compact subsets, we infer from the Noetherian property that the holomorphic function $\sum_{k=0}^{+\infty} f_{\nu,j,k}(z)\overline{f_{\nu,j,k}(\overline{w})}$ is a section of the coherent ideal sheaf generated by $(f_{\nu,j,k}(z)\overline{f_{\nu,j,k}(\overline{w})})_{k\le k_0}$ over $B'_j\times B'_j$, for k_0 large enough. Hence, by restricting to the conjugate diagonal $w=\overline{z}$, we get

$$\sum_{k=0}^{+\infty}|f_{\nu,j,k}(z)|^2 \le C\sum_{k=0}^{k_0}|f_{\nu,j,k}(z)|^2 \qquad \text{on } B'_j.$$

This implies

$$\int_{B'_j}\Big(\sum_{k=0}^{+\infty}|f_{\nu,j,k}|^2\Big)e^{-2\varphi}d\lambda \le C\int_{B'_j}\Big(\sum_{k=0}^{k_0}|f_{\nu,j,k}|^2\Big)e^{-2\varphi}d\lambda = C(k_0+1).$$

Property (16.2.10) is proved.

Step 3. Subadditivity of the approximating sequence $\psi_{\varepsilon,\nu}$.

We want to compare $\psi_{\varepsilon,\nu_1+\nu_2}$ and ψ_{ε,ν_1}, ψ_{ε,ν_2} for every pair of indices ν_1, ν_2, first when the functions are associated with the same covering $X=\bigcup B_j$. Consider a function $f\in\mathcal{H}_{\nu_1+\nu_2,j}$ with

$$\int_{B_j}|f(z)|^2e^{-2(\nu_1+\nu_2)\varphi_j(z)}d\lambda(z)\le 1, \qquad \varphi_j(z)=\varphi(z)+c_j|z^{(j)}|^2.$$

We may view f as a function $\hat{f}(z,z)$ defined on the diagonal Δ of $B_j\times B_j$. Consider the Hilbert space of holomorphic functions u on $B_j\times B_j$ such that

$$\int_{B_j\times B_j}|u(z,w)|^2e^{-2\nu_1\varphi_j(z)-2\nu_2\varphi_j(w)}d\lambda(z)d\lambda(w)<+\infty.$$

By the Ohsawa-Takegoshi L^2 extension theorem [OT87], there exists a function $\widetilde{f}(z,w)$ on $B_j \times B_j$ such that $\widetilde{f}(z,z) = f(z)$ and

$$\int_{B_j\times B_j} |\widetilde{f}(z,w)|^2 e^{-2\nu_1\varphi_j(z)-2\nu_2\varphi_j(w)} d\lambda(z)d\lambda(w) \leq C_7 \int_{B_j} |f(z)|^2 e^{-2(\nu_1+\nu_2)\varphi_j(z)} d\lambda(z) = C_7,$$

where the constant C_7 only depends on the dimension n (it is actually independent of the radius r_j if say $0 < r_j \leq 1$). As the Hilbert space under consideration on $B_j \times B_j$ is the completed tensor product $\mathcal{H}_{\nu_1,j} \mathbin{\widehat{\otimes}} \mathcal{H}_{\nu_2,j}$, we infer that

$$\widetilde{f}(z,w) = \sum_{k_1,k_2} c_{k_1,k_2} f_{\nu_1,j,k_1}(z) f_{\nu_2,j,k_2}(w)$$

with $\sum_{k_1,k_2} |c_{k_1,k_2}|^2 \leq C_7$. By restricting to the diagonal, we obtain

$$|f(z)|^2 = |\widetilde{f}(z,z)|^2 \leq \sum_{k_1,k_2} |c_{k_1,k_2}|^2 \sum_{k_1} |f_{\nu_1,j,k_1}(z)|^2 \sum_{k_2} |f_{\nu_2,j,k_2}(z)|^2.$$

From (16.2.3) and (16.2.4), we get

$$\psi_{\varepsilon,\nu_1+\nu_2,j} \leq \frac{\log C_7}{\nu_1+\nu_2} + \frac{\nu_1}{\nu_1+\nu_2}\psi_{\varepsilon,\nu_1,j} + \frac{\nu_2}{\nu_1+\nu_2}\psi_{\varepsilon,\nu_2,j},$$

in particular

$$\psi_{\varepsilon,2^\nu,j} \leq \psi_{\varepsilon,2^{\nu-1},j} + \frac{C_8}{2^\nu},$$

and we see that $\psi_{\varepsilon,2^\nu} + C_8 2^{-\nu}$ is a decreasing sequence. By Step 2 and Lebesgue's monotone convergence theorem, we infer that for every $\varepsilon,\delta > 0$ and $a \leq a_0 \ll 0$ fixed, the integral

$$I_{\varepsilon,\delta,\nu} = \int_X \Big(e^{-2\varphi} - e^{-2\max(\varphi,(1+\delta)(\psi_{2^\nu,\varepsilon}+a))}\Big) dV_\omega$$

converges to 0 as ν tends to $+\infty$ (take $\ell = \frac{1}{\delta}+1$ and $2^\nu > \ell$ and a_0 such that $\delta\sup_X\varphi + a_0 \leq 0$; we do not have monotonicity strictly speaking but need only replace a by $a + C_8 2^{-\nu}$ to get it, thereby slightly enlarging the integral).

Step 4. Selection of a suitable upper envelope.
For the simplicity of notation, we assume here that $\sup_X \varphi = 0$ (possibly after subtracting a constant), hence we can take $a_0 = 0$ in the above. We may even further assume that all our functions $\psi_{\varepsilon,\nu}$ are nonpositive. By Step 3, for each $\delta = \varepsilon = 2^{-k}$, we can select an index $\nu = p(k)$ such that

$$I_{2^{-k},2^{-k},p(k)} = \int_X \Big(e^{-2\varphi} - e^{-2\max(\varphi,(1+2^{-k})\psi_{2^{-k},2^{p(k)}})}\Big) dV_\omega \leq 2^{-k} \tag{16.2.11}$$

By construction, we have an estimate $i\partial\overline{\partial}\psi_{2^{-k},2^{p(k)}} \geq \gamma - C_5 2^{-k}\omega$, and the functions $\psi_{2^{-k},2^{p(k)}}$ are quasi-psh with logarithmic poles. Our estimates (especially (16.2.7)) imply that $\lim_{k\to+\infty}\psi_{2^{-k},2^{p(k)}}(z) = \varphi(z)$ as soon as $2^{-p(k)}\log\big(1/\inf_j r_j(k)\big) \to 0$ (notice that the r_j's now depend on $\varepsilon = 2^{-k}$). We set

(16.2.12) $$\varphi_\nu(z) = \sup_{k\geq\nu}(1+2^{-k})\psi_{2^{-k},2^{p(k)}}(z).$$

By construction (φ_ν) is a decreasing sequence and satisfies the estimates

$$\varphi_\nu \geq \max\big(\varphi, (1+2^{-\nu})\psi_{2^{-\nu},2^{p(\nu)}}\big), \qquad i\partial\overline{\partial}\varphi_\nu \geq \gamma - C_5 2^{-\nu}\omega.$$

Inequality (16.2.11) implies that

$$\int_X (e^{-2\varphi} - e^{-2\varphi_\nu})dV_\omega \leq \sum_{k=\nu}^{+\infty} 2^{-k} = 2^{1-\nu}.$$

Finally, if Z_ν is the set of poles of $\psi_{2^{-\nu},2^{p(\nu)}}$, then $Z_\nu \subset Z_{\nu+1}$ and φ_ν is continuous on $X \smallsetminus Z_\nu$. The reason is that in a neighborhood of every point $z_0 \in X \smallsetminus Z_\nu$, the term $(1+2^{-k})\psi_{2^{-k},2^{p(k)}}$ contributes to φ_ν only when it is larger than $(1+2^{-\nu})\psi_{2^{-\nu},2^{p(\nu)}}$. Hence, by the almost-monotonicity, the relevant terms of the sup in (16.2.12) are squeezed between $(1+2^{-\nu})\psi_{2^{-\nu},2^{p(\nu)}}$ and $(1+2^{-k})(\psi_{2^{-\nu},2^{p(\nu)}}+C_8 2^{-\nu})$, and therefore there is uniform convergence in a neighborhood of z_0. Finally, condition c) implies that

$$\int_U |f|^2(e^{-2\varphi} - e^{-2\varphi_\nu})dV_\omega < +\infty$$

for every germ of holomorphic function $f \in \mathcal{O}(U)$ at a point $x \in X$. Therefore both integrals $\int_U |f|^2 e^{-2\varphi}dV_\omega$ and $\int_U |f|^2 e^{-2\varphi_\nu}dV_\omega$ are simultaneously convergent or divergent, i.e. $\mathcal{I}(\varphi) = \mathcal{I}(\varphi_\nu)$. Theorem 16.2.1 is proved, except that φ_ν is possibly just continuous instead of being smooth. This can be arranged by Richberg's regularization theorem [Ri68], at the expense of an arbitrary small loss in the Hessian form. □

(16.2.13) Remark. By a very slight variation of the proof, we can strengthen condition c) and obtain that for every $t > 0$

$$\int_X (e^{-2t\varphi} - e^{-2t\varphi_\nu})dV_\omega$$

is finite for ν large enough and converges to 0 as $\nu \to +\infty$. This implies that the sequence of multiplier ideals $\mathcal{I}(t\varphi_\nu)$ is a stationary decreasing sequence, with $\mathcal{I}(t\varphi_\nu) = \mathcal{I}(t\varphi)$ for ν large.

16.3. A Bochner type inequality

Let (L,h) be a smooth hermitian line bundle on a (non necessarily compact) Kähler manifold (Y,ω). We denote by $|\ \ | = |\ \ |_{\omega,h}$ the pointwise hermitian norm on $\Lambda^{p,q}T^\star_Y \otimes L$ associated with ω and h, and by $\|\ \ \| = \|\ \ \|_{\omega,h}$ the global L^2 norm

$$\|u\|^2 = \int_Y |u|^2 dV_\omega \qquad \text{where} \quad dV_\omega = \frac{\omega^n}{n!}$$

We consider the $\overline{\partial}$ operator acting on (p,q)-forms with values in L, its adjoint $\overline{\partial}^\star_h$ with respect to h and the complex Laplace-Beltrami operator $\Delta''_h = \overline{\partial}\overline{\partial}^\star_h + \overline{\partial}^\star_h\overline{\partial}$. Let v be a smooth $(n-q,0)$-form with compact support in Y. Then $u = \omega^q \wedge v$ satisfies

$$(16.3.1)\qquad \|\overline\partial u\|^2+\|\overline\partial{}^\star_h u\|^2=\|\overline\partial v\|^2+\int_Y\sum_{I,J}\Big(\sum_{j\in J}\lambda_j\Big)|u_{IJ}|^2$$

where $\lambda_1\le\ldots\le\lambda_n$ are the curvature eigenvalues of $\Theta_h(L)$ expressed in an orthonormal frame $(\partial/\partial z_1,\ldots,\partial/\partial z_n)$ (at some fixed point $x_0\in Y$), in such a way that

$$\omega_{x_0}=i\sum_{1\le j\le n}dz_j\wedge d\overline z_j,\qquad \Theta_h(L)_{x_0}=i\partial\overline\partial\varphi_{x_0}=i\sum_{1\le j\le n}\lambda_j dz_j\wedge d\overline z_j.$$

The proof of (16.3.1) proceeds by checking that

$$(16.3.2)\qquad (\overline\partial{}^\star_\varphi\,\overline\partial+\overline\partial\,\overline\partial{}^\star_\varphi)(v\wedge\omega^q)-(\overline\partial{}^\star_\varphi\,\overline\partial v)\wedge\omega^q=q\,i\partial\overline\partial\varphi\wedge\omega^{q-1}\wedge v,$$

taking the inner product with $u=\omega^q\wedge v$ and integrating by parts in the left hand side. In order to check (16.3.2), we use the identity $\overline\partial{}^\star_\varphi=e^\varphi\overline\partial{}^\star(e^{-\varphi}\bullet)=\overline\partial{}^\star+\nabla^{0,1}\varphi\,\lrcorner\,\bullet$. Let us work in a local trivialization of L such that $\varphi(x_0)=0$ and $\nabla\varphi(x_0)=0$. At x_0 we then find

$$\begin{aligned}(\overline\partial{}^\star_\varphi\,\overline\partial+\overline\partial\,\overline\partial{}^\star_\varphi)(\omega^q\wedge v)-\omega^q\wedge(\overline\partial{}^\star_\varphi\,\overline\partial v)=&\\ \big[(\overline\partial{}^\star\,\overline\partial+\overline\partial\,\overline\partial{}^\star)(\omega^q\wedge v)-\omega^q\wedge(\overline\partial{}^\star\,\overline\partial v)\big]&+\overline\partial(\nabla^{0,1}\varphi\,\lrcorner\,(\omega^q\wedge v)).\end{aligned}$$

However, the term $[\ldots]$ corresponds to the case of a trivial vector bundle and it is well known in that case that $[\Delta'',\omega^q\wedge\bullet]=0$, hence $[\ldots]=0$. On the other hand

$$\nabla^{0,1}\varphi\,\lrcorner\,(\omega^q\wedge v)=q(\nabla^{0,1}\varphi\,\lrcorner\,\omega)\wedge\omega^{q-1}\wedge v=-q\,i\partial\varphi\wedge\omega^{q-1}\wedge v,$$

and so

$$(\overline\partial{}^\star_\varphi\,\overline\partial+\overline\partial\,\overline\partial{}^\star_\varphi)(\omega^q\wedge v)-\omega^q\wedge(\overline\partial{}^\star_\varphi\,\overline\partial v)=q\,i\partial\overline\partial\varphi\wedge\omega^{q-1}\wedge v.$$

Our formula is thus proved when v is smooth and compactly supported. In general, we have:

(16.3.3) Proposition. *Let (Y,ω) be a* complete *Kähler manifold and (L,h) a smooth hermitian line bundle such that the curvature possesses a uniform lower bound $\Theta_h(L)\ge -C\omega$. For every measurable $(n-q,0)$-form v with L^2 coefficients and values in L such that $u=\omega^q\wedge v$ has differentials $\overline\partial u$, $\overline\partial{}^\star u$ also in L^2, we have*

$$\|\overline\partial u\|^2+\|\overline\partial{}^\star_h u\|^2=\|\overline\partial v\|^2+\int_Y\sum_{I,J}\Big(\sum_{j\in J}\lambda_j\Big)|u_{IJ}|^2$$

(*here, all differentials are computed in the sense of distributions*).

Proof. Since (Y,ω) is assumed to be complete, there exists a sequence of smooth forms v_ν with compact support in Y (obtained by truncating v and taking the convolution with a regularizing kernel) such that $v_\nu\to v$ in L^2 and such that $u_\nu=\omega^q\wedge v_\nu$ satisfies $u_\nu\to u$, $\overline\partial u_\nu\to\overline\partial u$, $\overline\partial{}^\star u_\nu\to\overline\partial{}^\star u$ in L^2. By the curvature assumption, the final integral in the right hand side of (16.3.1) must be under control (i.e. the integrand becomes nonnegative if we add a term $C\|u\|^2$ on both sides, $C\gg 0$). We thus get the equality by passing to the limit and using Lebesgue's monotone convergence theorem. □

16.4. Proof of Theorem (16.1.1)

To fix the ideas, we first indicate the proof in the much simpler case when (L,h) is hermitian semipositive, and then treat the general case.

(16.4.1) Special case. (L,h) is (smooth) hermitian semipositive.

Let $\{\beta\} \in H^q(X, \Omega_X^n \otimes L)$ be an arbitrary cohomology class. By standard L^2 Hodge theory, $\{\beta\}$ can be represented by a smooth harmonic $(0,q)$-form β with values in $\Omega_X^n \otimes L$. We can also view β as a (n,q)-form with values in L. The pointwise Lefschetz isomorphism produces a unique $(n-q,0)$-form α such that $\beta = \omega^q \wedge \alpha$. Proposition 16.3.3 then yields

$$\|\overline{\partial}\alpha\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j\in J} \lambda_j\Big)|\alpha_{IJ}|^2 = \|\overline{\partial}\beta\|^2 + \|\overline{\partial}_h^\star\beta\|^2 = 0,$$

and the curvature eigenvalues λ_j are nonnegative by our assumption. Hence $\overline{\partial}\alpha = 0$ and $\{\alpha\} \in H^0(X, \Omega_X^{n-q} \otimes L)$ is mapped to $\{\beta\}$ by $\Phi^q_{\omega,h} = \omega^q \wedge \bullet$.

(16.4.2) General case.

There are several difficulties. The first difficulty is that the metric h is no longer smooth and we cannot directly represent cohomology classes by harmonic forms. We circumvent this problem by smoothing the metric on an (analytic) Zariski open subset and by avoiding the remaining poles on the complement. However, some careful estimates have to be made in order to take the error terms into account.

Fix $\varepsilon = \varepsilon_\nu$ and let $h_\varepsilon = h_{\varepsilon_\nu}$ be an approximation of h, such that h_ε is smooth on $X \smallsetminus Z_\varepsilon$ (Z_ε being an analytic subset of X), $\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$, $h_\varepsilon \leq h$ and $\mathcal{I}(h_\varepsilon) = \mathcal{I}(h)$. This is possible by Theorem 16.2.1. Now, we can find a family

$$\omega_{\varepsilon,\delta} = \omega + \delta(i\partial\overline{\partial}\psi_\varepsilon + \omega), \qquad \delta > 0$$

of *complete Kähler* metrics on $X \smallsetminus Z_\varepsilon$, where ψ_ε is a quasi-psh function on X with $\psi_\varepsilon = -\infty$ on Z_ε, ψ_ε on $X \smallsetminus Z_\varepsilon$ and $i\partial\overline{\partial}\psi_\varepsilon + \omega \geq 0$ (see e.g. [Dem82], Théorème 1.5). By construction, $\omega_{\varepsilon,\delta} \geq \omega$ and $\lim_{\delta\to 0} \omega_{\varepsilon,\delta} = \omega$. We look at the L^2 Dolbeault complex $L^\bullet_{\varepsilon,\delta}$ of $(n,\bullet)$-forms on $X \smallsetminus Z_\varepsilon$, where the L^2 norms are induced by $\omega_{\varepsilon,\delta}$ on differential forms and by h_ε on elements in L. Specifically

$$L^q_{\varepsilon,\delta} = \Big\{u{:}X \smallsetminus Z_\varepsilon \to \Lambda^{n,q}T_X^\star \otimes L;\ \int_{X\smallsetminus Z_\varepsilon} (|u|^2_{\Lambda^{n,q}\omega_{\varepsilon,\delta}\otimes h_\varepsilon} + |\overline{\partial}u|^2_{\Lambda^{n,q+1}\omega_{\varepsilon,\delta}\otimes h_\varepsilon})dV_{\omega_{\varepsilon,\delta}} < \infty\Big\}.$$

Let $\mathcal{L}^q_{\varepsilon,\delta}$ be the corresponding sheaf of germs of locally L^2 sections on X (the local L^2 condition should hold on X, not only on $X \smallsetminus Z_\varepsilon$!). Then, for all $\varepsilon > 0$ and $\delta \geq 0$, $(\mathcal{L}^q_{\varepsilon,\delta}, \overline{\partial})$ is a resolution of the sheaf $\Omega_X^n \otimes L \otimes \mathcal{I}(h_\varepsilon) = \Omega_X^n \otimes L \otimes \mathcal{I}(h)$. This is because L^2 estimates hold locally on small Stein open sets, and the L^2 condition on $X \smallsetminus Z_\varepsilon$ forces holomorphic sections to extend across Z_ε ([Dem82b], Lemme 6.9).

Let $\{\beta\} \in H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h))$ be a cohomology class represented by a smooth form with values in $\Omega_X^n \otimes L \otimes \mathcal{I}(h)$ (one can use a Čech cocycle and convert it to an element in the C^∞ Dolbeault complex by means of a partition of unity, thanks to the usual De Rham-Weil isomorphism). Then

$$\|\beta\|^2_{\varepsilon,\delta} \leq \|\beta\|^2 = \int_X |\beta|^2_{\Lambda^{n,q}\omega\otimes h} dV_\omega < +\infty.$$

The reason is that $|\beta|^2_{\Lambda^{n,q}\omega\otimes h} dV_\omega$ decreases as ω increases. This is just an easy calculation, shown by comparing two metrics ω, ω' which are expressed in diagonal form in suitable coordinates; the norm $|\beta|^2_{\Lambda^{n,q}\omega\otimes h}$ turns out to decrease faster than the volume dV_ω increases; see e.g. [Dem82b], Lemme 3.2; a special case is $q = 0$, then $|\beta|^2_{\Lambda^{n,q}\omega\otimes h} dV_\omega = i^{n^2}\beta\wedge\overline{\beta}$ with the identification $L\otimes\overline{L}\simeq\mathbb{C}$ given by the metric h, hence the integrand is even independent of ω in that case.

By the proof of the De Rham-Weil isomorphism, the map $\alpha\mapsto\{\alpha\}$ from the cocycle space $Z^q(\mathcal{L}^\bullet_{\varepsilon,\delta})$ equipped with its L^2 topology, into $H^q(X,\Omega^n_X\otimes L\otimes\mathcal{I}(h))$ equipped with its finite vector space topology, is continuous. Also, Banach's open mapping theorem implies that the coboundary space $B^q(\mathcal{L}^\bullet_{\varepsilon,\delta})$ is closed in $Z^q(\mathcal{L}^\bullet_{\varepsilon,\delta})$. This is true for all $\delta\ge 0$ (the limit case $\delta = 0$ yields the strongest L^2 topology in bidegree (n,q)). Now, β is a $\overline{\partial}$-closed form in the Hilbert space defined by $\omega_{\varepsilon,\delta}$ on $X\smallsetminus Z_\varepsilon$, so there is a $\omega_{\varepsilon,\delta}$-harmonic form $u_{\varepsilon,\delta}$ in the same cohomology class as β, such that

$$\|u_{\varepsilon,\delta}\|_{\varepsilon,\delta}\le\|\beta\|_{\varepsilon,\delta}.$$

(16.4.3) Remark. The existence of a harmonic representative holds true only for $\delta>0$, because we need to have a complete Kähler metric on $X\smallsetminus Z_\varepsilon$. The trick of employing $\omega_{\varepsilon,\delta}$ instead of a fixed metric ω, however, is not needed when Z_ε is (or can be taken to be) empty. This is the case if (L,h) is such that $\mathcal{I}(h)=\mathcal{O}_X$ and L is nef. Indeed, in that case, from the very definition of nefness, it is easy to prove that we can take the φ_ν's to be everywhere smooth in Theorem 16.2.1. However, we will see in § 16.5 that multiplier ideal sheaves are needed even in case L is nef, when $\mathcal{I}(h)\ne\mathcal{O}_X$.

Let $v_{\varepsilon,\delta}$ be the unique $(n-q,0)$-form such that $u_{\varepsilon,\delta}=v_{\varepsilon,\delta}\wedge\omega^q_{\varepsilon,\delta}$ ($v_{\varepsilon,\delta}$ exists by the pointwise Lefschetz isomorphism). Then

$$\|v_{\varepsilon,\delta}\|_{\varepsilon,\delta}=\|u_{\varepsilon,\delta}\|_{\varepsilon,\delta}\le\|\beta\|_{\varepsilon,\delta}\le\|\beta\|.$$

As $\sum_{j\in J}\lambda_j\ge -q\varepsilon$ by the assumption on $\Theta_{h_\varepsilon}(L)$, the Bochner formula yields

$$\|\overline{\partial}v_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\le q\varepsilon\|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\le q\varepsilon\|\beta\|^2.$$

These uniform bounds imply that there are subsequences $u_{\varepsilon,\delta_\nu}$ and $v_{\varepsilon,\delta_\nu}$ with $\delta_\nu\to 0$, possessing weak-L^2 limits $u_\varepsilon=\lim_{\nu\to+\infty}u_{\varepsilon,\delta_\nu}$ and $v_\varepsilon=\lim_{\nu\to+\infty}v_{\varepsilon,\delta_\nu}$. The limit $u_\varepsilon=\lim_{\nu\to+\infty}u_{\varepsilon,\delta_\nu}$ is with respect to $L^2(\omega)=L^2(\omega_{\varepsilon,0})$. To check this, notice that in bidegree $(n-q,0)$, the space $L^2(\omega)$ has the weakest topology of all spaces $L^2(\omega_{\varepsilon,\delta})$; indeed, an easy calculation as in ([Dem82b], Lemme 3.2) yields

$$|f|^2_{\Lambda^{n-q,0}\omega\otimes h}dV_\omega\le|f|^2_{\Lambda^{n-q,0}\omega_{\varepsilon,\delta}\otimes h}dV_{\omega_{\varepsilon,\delta}}\qquad\text{if } f \text{ is of type } (n-q,0).$$

On the other hand, the limit $v_\varepsilon=\lim_{\nu\to+\infty}v_{\varepsilon,\delta_\nu}$ takes place in all spaces $L^2(\omega_{\varepsilon,\delta})$, $\delta>0$, since the topology gets stronger and stronger as $\delta\downarrow 0$ [possibly not in $L^2(\omega)$, though, because in bidegree (n,q) the topology of $L^2(\omega)$ might be strictly stronger than that of all spaces $L^2(\omega_{\varepsilon,\delta})$]. The above estimates yield

$$\begin{aligned}
&\|v_\varepsilon\|^2_{\varepsilon,0}=\int_X|v_\varepsilon|^2_{\Lambda^{n-q,0}\omega\otimes h_\varepsilon}dV_\omega\le\|\beta\|^2,\\
&\|\overline{\partial}v_\varepsilon\|^2_{\varepsilon,0}\le q\varepsilon\|\beta\|^2_{\varepsilon,0},\\
&u_\varepsilon=\omega^q\wedge v_\varepsilon\equiv\beta\qquad\text{in } H^q(X,\Omega^n_X\otimes L\otimes\mathcal{I}(h_\varepsilon)).
\end{aligned}$$

Again, by arguing in a given Hilbert space $L^2(h_{\varepsilon_0})$, we find L^2 convergent subsequences $u_\varepsilon \to u$, $v_\varepsilon \to v$ as $\varepsilon \to 0$, and in this way get $\overline{\partial} v = 0$ and

$$\|v\|^2 \le \|\beta\|^2,$$
$$u = \omega^q \wedge v \equiv \beta \qquad \text{in} \ \ H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h)).$$

Theorem 16.1.1 is proved. Notice that the equisingularity property $\mathcal{I}(h_\varepsilon) = \mathcal{I}(h)$ is crucial in the above proof, otherwise we could not infer that $u \equiv \beta$ from the fact that $u_\varepsilon \equiv \beta$. This is true only because all cohomology classes $\{u_\varepsilon\}$ lie in the same fixed cohomology group $H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h))$, whose topology is induced by the topology of $L^2(\omega)$ on $\overline{\partial}$-closed forms (e.g. through the De Rham-Weil isomorphism). □

16.5. A counterexample

In view of Corollary 16.1.2, one might wonder whether the morphism Φ^q_ω would not still be surjective when L is a nef vector bundle. We will show that this is unfortunately not so, even in the case of algebraic surfaces.

Let B be an elliptic curve and let V be the rank 2 vector bundle over B which is defined as the (unique) non split extension

$$0 \to \mathcal{O}_B \to V \to \mathcal{O}_B \to 0.$$

In particular, the bundle V is numerically flat, i.e. $c_1(V) = 0$, $c_2(V) = 0$. We consider the ruled surface $X = \mathbb{P}(V)$. On that surface there is a unique section $C = \mathbb{P}(\mathcal{O}_B) \subset X$ with $C^2 = 0$ and

$$\mathcal{O}_X(C) = \mathcal{O}_{\mathbb{P}(V)}(1)$$

is a nef line bundle. It is easy to see that

$$h^0(X, \mathcal{O}_{\mathbb{P}(V)}(m)) = h^0(B, S^m V) = 1$$

for all $m \in \mathbb{N}$ (otherwise we would have $mC = aC + M$ where aC is the fixed part of the linear system $|mC|$ and $M \neq 0$ the moving part, thus $M^2 \ge 0$ and $C \cdot M > 0$, contradiction). We claim that

$$h^0(X, \Omega^1_X(kC)) = 2$$

for all $k \ge 2$. This follows by tensoring the exact sequence

$$0 \to \Omega^1_{X|C} \to \Omega^1_X \to \pi^* \Omega^1_C \simeq \mathcal{O}_C \to 0$$

by $\mathcal{O}_X(kC)$ and observing that

$$\Omega^1_{X|C} = K_X = \mathcal{O}_X(-2C).$$

From this, we get

$$0 \to H^0(X, \mathcal{O}_X((k-2)C)) \to H^0(X, \Omega^1_X \mathcal{O}(kC)) \to H^0(X, \mathcal{O}_X(kC))$$

where $h^0(X, \mathcal{O}_X((k-2)C)) = h^0(X, \mathcal{O}_X(kC)) = 1$ for all $k \ge 2$. Moreover, the last arrow is surjective because we can multiply a section of $H^0(X, \mathcal{O}_X(kC))$ by a nonzero section in $H^0(X, \pi^* \Omega^1_B)$ to get a preimage. Our claim follows. We now consider the diagram

$$\begin{array}{ccc} H^0(X, \Omega^1_X(2C)) & \xrightarrow{\wedge\omega} & H^1(X, K_X(2C)) \\ \simeq \big\downarrow & & \big\downarrow \varphi \\ H^0(X, \Omega^1_X(3C)) & \xrightarrow[\psi]{\wedge\omega} & H^1(X, K_X(3C)). \end{array}$$

Since $K_X(2C) \simeq \mathcal{O}_X$ and $K_X(3C) \simeq \mathcal{O}_X(C)$, the cohomology sequence of

$$0 \to K_X(2C) \to K_X(3C) \to K_X(3C)|C \simeq \mathcal{O}_C \to 0$$

immediately implies $\varphi = 0$ (notice that $h^1(X, K_X(2C)) = h^1(X, K_X(3C)) = 1$, since $h^1(B, \mathcal{O}_B) = h^1(B, V) = 1$), and $h^2(X, K_X(2C)) = h^2(B, \mathcal{O}_B) = 0$). Therefore the diagram implies $\psi = 0$, and we get:

(16.5.1) Proposition. *$L = \mathcal{O}_{\mathbb{P}(V)}(3)$ is a counterample to* (16.1.2) *in the nef case.* □

By Corollary (16.1.2), we infer that $\mathcal{O}_X(3)$ cannot be hermitian semi-positive and we thus again obtain – by a quite different method – the result of [DPS94], example 1.7.

(16.5.2) Corollary. *Let B be an elliptic curve, V the vector bundle given by the unique non-split extension*

$$0 \to \mathcal{O}_B \to V \to \mathcal{O}_B \to 0.$$

Let $X = \mathbb{P}(V)$. Then $L = \mathcal{O}_X(1)$ is nef but not hermitian semi-positive (*nor does any multiple, e.g. the anticanonical line bundle $-K_X = \mathcal{O}_X(-2)$ is nef but not semi-positive*)*.*

17. Nef and Pseudo-Effective Cones in Kähler Geometry

We now introduce important concepts of positivity for cohomology classes of type $(1,1)$ in the general Kähler context.

(17.1) Definition. *Let X be a compact Kähler manifold.*

(i) *The Kähler cone is the set $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an* *<u>open convex cone</u>.*

(ii) *The pseudo-effective cone is the set $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{T\}$ of closed positive currents of type* $(1,1)$*. This is a <u>closed convex cone</u>.*

$\mathcal{K}$

$\mathcal{E}$

$\overline{\mathcal{K}}$ = nef cone in $H^{1,1}(X,\mathbb{R})$

$\mathcal{E}$ = pseudo-effective cone in $H^{1,1}(X,\mathbb{R})$

The openness of $\mathcal{K}$ is clear by definition, and the closedness of $\mathcal{E}$ follows from the fact that bounded sets of currents are weakly compact (as follows from the similar weak compactness property for bounded sets of positive measures). It is then clear that $\overline{\mathcal{K}} \subset \mathcal{E}$.

In spite of the fact that cohomology groups can be defined either in terms of forms or currents, it turns out that the cones $\overline{\mathcal{K}}$ and $\mathcal{E}$ are in general different. To see this, it is enough to observe that a Kähler class $\{\alpha\}$ satisfies $\int_Y \alpha^p > 0$ for every p-dimensional analytic set. On the other hand, if X is the surface obtained by blowing-up $\mathbb{P}^2$ in one point, then the exceptional divisopr $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

In case X is projective, it is interesting to consider also the algebraic analogues of our "transcendental cones" $\mathcal{K}$ and $\mathcal{E}$, which consist of suitable integral divisor classes. Since the cohomology classes of such divisors live in $H^2(X,\mathbb{Z})$, we are led to introduce the Neron-Severi lattice and the associated Neron-Severi space

$$\begin{aligned}\mathrm{NS}(X) &:= H^{1,1}(X,\mathbb{R}) \cap \big(H^2(X,\mathbb{Z})/\{\text{torsion}\}\big),\\ \mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}.\end{aligned}$$

All classes of real divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}$, lie by definition in $\mathrm{NS}_{\mathbb{R}}(X)$. Notice that the integral lattice $H^2(X,\mathbb{Z})/\{\text{torsion}\}$ need not hit at all the subspace $H^{1,1}(X,\mathbb{R}) \subset H^2(X,\mathbb{R})$ in the Hodge decomposition, hence in general the Picard number

$$\rho(X) = \operatorname{rank}_{\mathbb{Z}} \mathrm{NS}(X) = \dim_{\mathbb{R}} \mathrm{NS}_{\mathbb{R}}(X)$$

satisfies $\rho(X) \le h^{1,1} = \dim_{\mathbb{R}} H^{1,1}(X,\mathbb{R})$, but the equality can be strict (actually, it is well known that a generic complex torus $X = \mathbb{C}^n/\Lambda$ satisfies $\rho(X) = 0$ and $h^{1,1} = n^2$). In order to deal with the case of algebraic varieties we introduce

$$\mathcal{K}_{\mathrm{NS}} = \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X), \qquad \mathcal{E}_{\mathrm{NS}} = \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X).$$

$\mathrm{NS}_{\mathbb{R}}(X)$

$\mathcal{K}_{\mathrm{NS}}$

$\mathcal{E}_{\mathrm{NS}}$

A very important fact is that the "Neron-Severi part" of any of the open or closed transcendental cones $\mathcal{K}$, $\mathcal{E}$, $\overline{\mathcal{K}}$, $\mathcal{E}^\circ$ is algebraic, i.e. can be characterized in simple algebraic terms.

(17.2) Theorem. *Let X be a projective manifold. Then*

(i) *$\mathcal{E}_{\mathrm{NS}}$ is the closure of the cone generated by classes of effective divisors, i.e. divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}_+$.*

(ii) *$\mathcal{K}_{\mathrm{NS}}$ is the open cone generated by classes of ample (or very ample) divisors A (Recall that a divisor A is said to be very ample if the linear system $H^0(X, \mathcal{O}(A))$ provides an embedding of X in projective space).*

(iii) *The interior $\mathcal{E}^\circ_{\mathrm{NS}}$ is the cone generated by classes of big divisors, namely divisors D such that $h^0(X, \mathcal{O}(kD)) \ge c\,k^{\dim X}$ for k large.*

(iv) *The closed cone $\overline{\mathcal{K}}_{\mathrm{NS}}$ consists of the closure of the cone generated by nef divisors D (or nef line bundles L), namely effective integral divisors D such that $D\cdot C \ge 0$ for every curve C.*

Sketch of proof. These results were already observed (maybe in a slightly different terminology) in [Dem90]. If we denote by $\mathcal{K}_{\mathrm{alg}}$ the open cone generated by ample divisors, resp. by $\mathcal{E}_{\mathrm{alg}}$ the closure of the cone generated by effective divisors, it is obvious that

$$\mathcal{K}_{\mathrm{alg}} \subset \mathcal{K}_{\mathrm{NS}}, \qquad \mathcal{E}_{\mathrm{alg}} \subset \mathcal{E}_{\mathrm{NS}}.$$

As was to be expected, the interesting part lies in the converse inclusions. The inclusion $K_{\mathrm{NS}} \subset \mathcal{K}_{\mathrm{alg}}$ is more or less equivalent to the Kodaira embedding theorem : if a rational class $\{\alpha\}$ is in $\mathcal{K}$, then some multiple of $\{\alpha\}$ is the first Chern class of a hermitian line bundle L whose curvature form is Kähler. Therefore L is ample and $\{\alpha\} \in \mathcal{K}_{\mathrm{alg}}$; property (ii) follows.

Similarly, if we take a rational class $\{\alpha\} \in \mathcal{E}^\circ_{\mathrm{NS}}$, then it is still in $\mathcal{E}$ by subtracting a small multiple $\varepsilon\omega$ of a Kähler class, hence some multiple of $\{\alpha\}$ is the first Chern class of a hermitian line bundle (L, h) with curvature form

$$T = \Theta_h(L) := -\frac{i}{2\pi} i\partial\overline{\partial} \log h \geq \varepsilon\omega.$$

Let us apply Theorem 6.14 to a metric of the form $h^k e^{-m\psi}$ on $L^{\otimes k}$, where ψ has logarithmic poles at given points x_j in X. It is then easily shown that $L^{\otimes k}$ admits sections which have given m-jets at the point x_j, provided that $k \geq Cm$, $C \gg 1$, and the x_j are chosen outside the Lelong sublevel sets of $\log h$. From this we get $h^0(X, L^{\otimes k}) \geq m^n/n! \geq ck^n$, hence the linear system kL can be represented by a big divisor. This implies (iii) and also that $\mathcal{E}^\circ_{\rm NS} \subset \mathcal{E}_{\rm alg}$. Therefore $\mathcal{E}_{\rm NS} \subset \mathcal{E}_{\rm alg}$ by passing to the closure ; (i) follows. The statement (iv) about nef divisors follows e.g. from [Kle66], [Har70], since every nef divisor is a limit of a sequence of ample rational divisors. □

As a natural extrapolation of the algebraic situation, we say that $\overline{\mathcal{K}}$ is the cone of *nef* $(1,1)$-cohomology classes (even though these classes are not necessarily integral). Property 17.2 (i) also explains the terminology used for the pseudo-effective cone.

18. Numerical Characterization of the Kähler Cone

We describe here the main results obtained in [DP03]. The upshot is that the Kähler cone depends only on the intersection product of the cohomology ring, the Hodge structure and the homology classes of analytic cycles. More precisely, we have :

(18.1) Theorem. *Let X be a compact Kähler manifold. Let $\mathcal{P}$ be the set of real* $(1,1)$ *cohomology classes $\{\alpha\}$ which are numerically positive on analytic cycles, i.e. such that $\int_Y \alpha^p > 0$ for every irreducible analytic set Y in X, $p = \dim Y$. Then the Kähler cone $\mathcal{K}$ of X is one of the connected components of $\mathcal{P}$.*

(18.2) Special case. *If X is projective algebraic, then $\mathcal{K} = \mathcal{P}$.*

These results (which are new even in the projective case) can be seen as a generalization of the well-known Nakai-Moishezon criterion. Recall that the Nakai-Moishezon criterion provides a necessary and sufficient criterion for a line bundle to be ample: *a line bundle $L \to X$ on a projective algebraic manifold X is ample if and only if*

$$L^p \cdot Y = \int_Y c_1(L)^p > 0,$$

for every algebraic subset $Y \subset X$, $p = \dim Y$.

It turns out that the numerical conditions $\int_Y \alpha^p > 0$ also characterize arbitrary transcendental Kähler classes when X is projective : this is precisely the meaning of the special case 18.2.

(18.3) Example. The following example shows that the cone $\mathcal{P}$ need not be connected (and also that the components of $\mathcal{P}$ need not be convex, either). Let us consider for instance a complex torus $X = \mathbb{C}^n/\Lambda$. It is well-known that a generic torus X does not possess any analytic subset except finite subsets and X itself. In that case, the numerical positivity is expressed by the single condition $\int_X \alpha^n > 0$. However, on a torus, $(1,1)$-classes are in one-to-one correspondence with constant hermitian forms α on $\mathbb{C}^n$. Thus, for X generic, $\mathcal{P}$ is the set of hermitian forms on $\mathbb{C}^n$ such that $\det(\alpha) > 0$, and Theorem 18.1 just expresses the elementary result of linear algebra saying that the set $\mathcal{K}$ of positive definite forms is one of the connected components of the open set $\mathcal{P} = \{\det(\alpha) > 0\}$ of hermitian forms of positive determinant (the other components, of course, are the sets of forms of signature (p, q), $p + q = n$, q even. They are not convex when $p > 0$ and $q > 0$).

Sketch of proof of Theorems 18.1 and 18.2. By definition a Kähler current is a closed positive current T of type $(1,1)$ such that $T \geq \varepsilon\omega$ for some smooth Kähler metric ω and $\varepsilon > 0$ small enough. The crucial steps of the proof of Theorem 18.1 are contained in the following statements.

(18.4) Proposition (Paun [Pau98a, 98b])**.** *Let X be a compact complex manifold* (*or more generally a compact complex space*)*. Then*

(i) *The cohomology class of a closed positive* $(1,1)$*-current* $\{T\}$ *is nef if and only if the restriction* $\{T\}_{|Z}$ *is nef for every irreducible component Z in any of the Lelong sublevel sets* $E_c(T)$*.*

(ii) *The cohomology class of a Kähler current* $\{T\}$ *is a Kähler class* (*i.e. the class of a smooth Kähler form*) *if and only if the restriction* $\{T\}_{|Z}$ *is a Kähler class for every irreducible component Z in any of the Lelong sublevel sets* $E_c(T)$*.*

The proof of Proposition 18.4 is not extremely hard if we take for granted the fact that Kähler currents can be approximated by Kähler currents with logarithmic poles, a fact which was first proved in [Dem92] (see also section 9 below). Thus in (ii), we may assume that $T = \alpha + \mathrm{i}\,\partial\overline{\partial}\varphi$ is a current with analytic singularities, where φ is a quasi-psh function with logarithmic poles on some analytic set Z, and φ smooth on $X \smallsetminus Z$. Now, we proceed by an induction on dimension (to do this, we have to consider analytic spaces rather than with complex manifolds, but it turns out that this makes no difference for the proof). Hence, by the induction hypothesis, there exists a smooth potential ψ on Z such that $\alpha_{|Z} + \mathrm{i}\,\partial\overline{\partial}\psi > 0$ along Z. It is well known that one can then find a potential $\widetilde{\psi}$ on X such that $\alpha + \mathrm{i}\,\partial\overline{\partial}\widetilde{\psi} > 0$ in a neighborhood V of Z (but possibly non positive elsewhere). Essentially, it is enough to take an arbitrary extension of ψ to X and to add a large multiple of the square of the distance to Z, at least near smooth points; otherwise, we stratify Z by its successive singularity loci, and proceed again by induction on the dimension of these loci. Finally, we use a a standard gluing procedure : the current $T = \alpha + \mathrm{i}\,\max_\varepsilon(\varphi, \widetilde{\psi} - C)$, $C \gg 1$, will be equal to $\alpha + \mathrm{i}\,\partial\overline{\partial}\varphi > 0$ on $X \smallsetminus V$, and to a smooth Kähler form on V. □

The next (and more substantial step) consists of the following result which is reminiscent of the Grauert-Riemenschneider conjecture ([Siu84], [Dem85]).

(18.5) Theorem ([DP03])**.** *Let X be a compact Kähler manifold and let $\{\alpha\}$ be a nef class* (*i.e.* $\{\alpha\} \in \overline{\mathcal{K}}$)*. Assume that $\int_X \alpha^n > 0$. Then $\{\alpha\}$ contains a Kähler current T, in other words $\{\alpha\} \in \mathcal{E}^\circ$.*

Step 1. The basic argument is to prove that for every irreducible analytic set $Y \subset X$ of codimension p, the class $\{\alpha\}^p$ contains a closed positive (p,p)-current Θ such that $\Theta \geq \delta[Y]$ for some $\delta > 0$. For this, we use two ingredients. The first is a "local" calculation which completes Lemma 6.17 by analyzing the behavior of Monge-Ampre masses concentrated near an analytic set. The second is based on the exact solution of Monge-Ampre equations by means of the Calabi-Yau theorem [Yau78].

(18.6) Lemma. *Let X be a compact complex manifold X equipped with a Kähler metric $\omega = i\sum_{1\le j,k\le n}\omega_{jk}(z)dz_j\wedge d\overline{z}_k$ and let $Y\subset X$ be an analytic subset of X. Then there exist globally defined quasi-plurisubharmonic potentials ψ and $(\psi_\varepsilon)_{\varepsilon\in]0,1]}$ on X, satisfying the following properties.*

(i) *The function ψ is smooth on $X\smallsetminus Y$, satisfies $i\partial\overline{\partial}\psi\ge -A\omega$ for some $A>0$, and ψ has logarithmic poles along Y, i.e., locally near Y*

$$\psi(z)\sim\log\sum_k|g_k(z)|+O(1)$$

where (g_k) is a local system of generators of the ideal sheaf $\mathcal{I}_Y$ of Y in X.

(ii) *We have $\psi=\lim_{\varepsilon\to 0}\downarrow\psi_\varepsilon$ where the ψ_ε are C^∞ and possess a uniform Hessian estimate*

$$i\partial\overline{\partial}\psi_\varepsilon\ge -A\omega\qquad\text{on } X.$$

(iii) *Consider the family of hermitian metrics*

$$\omega_\varepsilon := \omega+\frac{1}{2A}i\partial\overline{\partial}\psi_\varepsilon\ge\frac{1}{2}\omega.$$

For any point $x_0\in Y$ and any neighborhood U of x_0, the volume element of ω_ε has a uniform lower bound

$$\int_{U\cap V_\varepsilon}\omega_\varepsilon^n\ge\delta(U)>0,$$

where $V_\varepsilon=\{z\in X\,;\ \psi(z)<\log\varepsilon\}$ is the "tubular neighborhood" of radius ε around Y.

(iv) *For every integer $p\ge 0$, the family of positive currents ω_ε^p is bounded in mass. Moreover, if Y contains an irreducible component Y' of codimension p, there is a uniform lower bound*

$$\int_{U\cap V_\varepsilon}\omega_\varepsilon^p\wedge\omega^{n-p}\ge\delta_p(U)>0$$

in any neighborhood U of a regular point $x_0\in Y'$. In particular, any weak limit Θ of ω_ε^p as ε tends to 0 satisfies $\Theta\ge\delta'[Y']$ for some $\delta'>0$.

Proof. Properties (i) and (ii) were already proved in Lemma 6.17. We continue our calculations with the same functions ψ and ψ_ε.By construction, the real $(1,1)$-form $\omega_\varepsilon := \omega+\frac{1}{2A}i\partial\overline{\partial}\psi_\varepsilon$ satisfies $\omega_\varepsilon\ge\frac{1}{2}\omega$, hence it is Kähler and its eigenvalues with respect to ω are at least equal to $1/2$.

Assume now that we are in a neighborhood U of a regular point $x_0\in Y$ where Y has codimension p. Then $\gamma_{j,k}=\theta_j\partial g_{j,k}$ at x_0, hence the rank of the system of $(1,0)$-forms $(\gamma_{j,k})_{k\ge 1}$ is at least equal to p in a neighborhood of x_0. Fix a holomorphic local coordinate system $(z_1,\ldots,z_n)$ such that $Y=\{z_1=\ldots=z_p=0\}$ near x_0, and let $S\subset T_X$ be the holomorphic subbundle generated by $\partial/\partial z_1,\ldots,\partial/\partial z_p$. This choice ensures that the rank of the system of $(1,0)$-forms $(\gamma_{j,k|S})$ is everywhere equal to p. By (1,3) and the minimax principle applied to the p-dimensional subspace $S_z \subset T_{X,z}$, we see that the p-largest eigenvalues of ω_ε are bounded below by $c\varepsilon^2/(e^{2\psi}+\varepsilon^2)^2$.

However, we can even restrict the form defined in (3.6) to the $(p-1)$-dimensional subspace $S\cap\operatorname{Ker}\tau$ where $\tau(\xi) := \sum_{j,k}\theta_j\overline{g_{j,k}}\gamma_{j,k}(\xi)$, to see that the $(p-1)$-largest eigenvalues of ω_ε are bounded below by $c/(e^{2\psi}+\varepsilon^2)$, $c>0$. The p-th eigenvalue is then bounded by $c\varepsilon^2/(e^{2\psi}+\varepsilon^2)^2$ and the remaining $(n-p)$-ones by $1/2$. From this we infer

$$\omega_\varepsilon^n \ge c\,\frac{\varepsilon^2}{(e^{2\psi}+\varepsilon^2)^{p+1}}\omega^n \qquad \text{near } x_0,$$
$$\omega_\varepsilon^p \ge c\,\frac{\varepsilon^2}{(e^{2\psi}+\varepsilon^2)^{p+1}}\Big(i\sum_{1\le\ell\le p}\gamma_{j,k_\ell}\wedge\overline{\gamma_{j,k_\ell}}\Big)^p$$

where $(\gamma_{j,k_\ell})_{1\le\ell\le p}$ is a suitable p-tuple extracted from the $(\gamma_{j,k})$, such that $\bigcap_\ell\operatorname{Ker}\gamma_{j,k_\ell}$ is a smooth complex (but not necessarily holomorphic) subbundle of codimension p of T_X; by the definition of the forms $\gamma_{j,k}$, this subbundle must coincide with T_Y along Y. From this, properties (iii) and (iv) follow easily; actually, up to constants, we have $e^{2\psi}+\varepsilon^2 \sim |z_1|^2+\ldots+|z_p|^2+\varepsilon^2$ and

$$i\sum_{1\le\ell\le p}\gamma_{j,k_\ell}\wedge\overline{\gamma_{j,k_\ell}} \ge c\,i\partial\overline{\partial}(|z_1|^2+\ldots+|z_p|^2) - O(\varepsilon)i\partial\overline{\partial}|z|^2 \qquad \text{on } U\cap V_\varepsilon,$$

hence, by a straightforward calculation,

$$\omega_\varepsilon^p\wedge\omega^{n-p} \ge c\big(i\partial\overline{\partial}\log(|z_1|^2+\ldots+|z_p|^2+\varepsilon^2)\big)^p\wedge\big(i\partial\overline{\partial}(|z_{p+1}|^2+\ldots+|z_n|^2)\big)^{n-p}$$

on $U\cap V_\varepsilon$; notice also that $\omega_\varepsilon^n \ge 2^{-(n-p)}\omega_\varepsilon^p\wedge\omega^{n-p}$, so any lower bound for the volume of $\omega_\varepsilon^p\wedge\omega^{n-p}$ will also produce a bound for the volume of ω_ε^n. As it is well known, the (p,p)-form

$$\Big(\frac{i}{2\pi}\partial\overline{\partial}\log(|z_1|^2+\ldots+|z_p|^2+\varepsilon^2)\Big)^p \qquad \text{on } \mathbb{C}^n$$

can be viewed as the pull-back to $\mathbb{C}^n=\mathbb{C}^p\times\mathbb{C}^{n-p}$ of the Fubini-Study volume form of the complex p-dimensional projective space of dimension p containing $\mathbb{C}^p$ as an affine Zariski open set, rescaled by the dilation ratio ε. Hence it converges weakly to the current of integration on the p-codimensional subspace $z_1=\ldots=z_p=0$. Moreover the volume contained in any compact tubular cylinder

$$\{|z'|\le C\varepsilon\}\times K'' \subset \mathbb{C}^p\times\mathbb{C}^{n-p}$$

depends only on C and K (as one sees after rescaling by ε). The fact that ω_ε^p is uniformly bounded in mass can be seen easily from the fact that

$$\int_X \omega_\varepsilon^p\wedge\omega^{n-p} = \int_X \omega^n,$$

as ω and ω_ε are in the same Kähler class. Let Θ be any weak limit of ω_ε^p. By what we have just seen, Θ carries non zero mass on every p-codimensional component Y' of Y, for instance near every regular point. However, standard results of the theory of currents (support theorem and Skoda's extension result) imply that $\mathbf{1}_{Y'}\Theta$ is a closed positive current and that $\mathbf{1}_{Y'}\Theta = \lambda[Y']$ is a nonnegative multiple of the current of integration on Y'. The fact that the mass of Θ on Y' is positive yields $\lambda > 0$. Lemma 18.6 is proved. □

(18.7) Lemma ([Yau78]). *Let (X, ω) be a compact Kähler manifold and $n = \dim X$. Then for any smooth volume form $f > 0$ such that $\int_X f = \int_X \omega^n$, there exist a Kähler metric $\widetilde{\omega} = \omega + i\partial\overline{\partial}\varphi$ in the same Kähler class as ω, such that $\widetilde{\omega}^n = f$.* □

We exploit this by observing that $\alpha + \varepsilon\omega$ is a Kähler class, and by solving the Monge-Ampère equation

$$(\alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon)^n = C_\varepsilon \omega_\varepsilon^n \tag{18.7a}$$

where (ω_ε) is the family of Kähler metrics on X produced by Lemma 18.6 (iii), such that their volume is concentrated in an ε-tubular neighborhood of Y.

$$C_\varepsilon = \frac{\int_X \alpha_\varepsilon^n}{\int_X \omega_\varepsilon^n} = \frac{\int_X (\alpha + \varepsilon\omega)^n}{\int_X \omega^n} \geq C_0 = \frac{\int_X \alpha^n}{\int_X \omega^n} > 0.$$

Let us denote by

$$\lambda_1(z) \leq \ldots \leq \lambda_n(z)$$

the eigenvalues of $\alpha_\varepsilon(z)$ with respect to $\omega_\varepsilon(z)$, at every point $z \in X$ (these functions are continuous with respect to z, and of course depend also on ε). The equation (18.7a) is equivalent to the fact that

$$\lambda_1(z) \ldots \lambda_n(z) = C_\varepsilon \tag{18.7b}$$

is constant, and the most important observation for us is that the constant C_ε is bounded away from 0, thanks to our assumption $\int_X \alpha^n > 0$.

Fix a regular point $x_0 \in Y$ and a small neighborhood U (meeting only the irreducible component of x_0 in Y). By Lemma 18.6, we have a uniform lower bound

$$\int_{U \cap V_\varepsilon} \omega_\varepsilon^p \wedge \omega^{n-p} \geq \delta_p(U) > 0. \tag{18.7c}$$

Now, by looking at the p smallest (resp. $(n-p)$ largest) eigenvalues λ_j of α_ε with respect to ω_ε, we find

$$\alpha_\varepsilon^p \geq \lambda_1 \ldots \lambda_p \, \omega_\varepsilon^p, \tag{18.7d}$$

$$\alpha_\varepsilon^{n-p} \wedge \omega_\varepsilon^p \geq \frac{1}{n!} \lambda_{p+1} \ldots \lambda_n \, \omega_\varepsilon^n, \tag{18.7e}$$

The last inequality (18.7e) implies

$$\int_X \lambda_{p+1} \ldots \lambda_n \, \omega_\varepsilon^n \leq n! \int_X \alpha_\varepsilon^{n-p} \wedge \omega_\varepsilon^p = n! \int_X (\alpha + \varepsilon\omega)^{n-p} \wedge \omega^p \leq M$$

for some constant $M > 0$ (we assume $\varepsilon \leq 1$, say). In particular, for every $\delta > 0$, the subset $E_\delta \subset X$ of points z such that $\lambda_{p+1}(z) \ldots \lambda_n(z) > M/\delta$ satisfies $\int_{E_\delta} \omega_\varepsilon^n \leq \delta$, hence

$$\int_{E_\delta} \omega_\varepsilon^p \wedge \omega^{n-p} \leq 2^{n-p} \int_{E_\delta} \omega_\varepsilon^n \leq 2^{n-p}\delta. \tag{18.7f}$$

The combination of (18.7c) and (18.7f) yields

$$\int_{(U\cap V_\varepsilon)\smallsetminus E_\delta}\omega_\varepsilon^p\wedge\omega^{n-p}\ge\delta_p(U)-2^{n-p}\delta.$$

On the other hand (18.7b) and (18.7d) imply

$$\alpha_\varepsilon^p\ge\frac{C_\varepsilon}{\lambda_{p+1}\ldots\lambda_n}\omega_\varepsilon^p\ge\frac{C_\varepsilon}{M/\delta}\omega_\varepsilon^p\qquad\text{on }(U\cap V_\varepsilon)\smallsetminus E_\delta.$$

From this we infer

$$\text{(18.7g)}\quad\int_{U\cap V_\varepsilon}\alpha_\varepsilon^p\wedge\omega^{n-p}\ge\frac{C_\varepsilon}{M/\delta}\int_{(U\cap V_\varepsilon)\smallsetminus E_\delta}\omega_\varepsilon^p\wedge\omega^{n-p}\ge\frac{C_\varepsilon}{M/\delta}(\delta_p(U)-2^{n-p}\delta)>0$$

provided that δ is taken small enough, e.g. $\delta=2^{-(n-p+1)}\delta_p(U)$. The family of (p,p)-forms α_ε^p is uniformly bounded in mass since

$$\int_X\alpha_\varepsilon^p\wedge\omega^{n-p}=\int_X(\alpha+\varepsilon\omega)^p\wedge\omega^{n-p}\le\text{Const}.$$

Inequality (18.7g) implies that any weak limit Θ of (α_ε^p) carries a positive mass on $U\cap Y$. By Skoda's extension theorem [Sko82], $\mathbf{1}_Y\Theta$ is a closed positive current with support in Y, hence $\mathbf{1}_Y\Theta=\sum c_j[Y_j]$ is a combination of the various components Y_j of Y with coefficients $c_j>0$. Our construction shows that Θ belongs to the cohomology class $\{\alpha\}^p$. Step 1 of Theorem 18.5 is proved.

Step 2. The second and final step consists in using a "diagonal trick": for this, we apply Step 1 to

$$\widetilde X=X\times X,\qquad\widetilde Y=\text{diagonal }\Delta\subset\widetilde X,\qquad\widetilde\alpha=\mathrm{pr}_1^*\,\alpha+\mathrm{pr}_2^*\,\alpha.$$

It is then clear that $\widetilde\alpha$ is nef on $\widetilde X$ and that

$$\int_{\widetilde X}(\widetilde\alpha)^{2n}=\binom{2n}{n}\Big(\int_X\alpha^n\Big)^2>0.$$

It follows by Step 1 that the class $\{\widetilde\alpha\}^n$ contains a Kähler current Θ of bidegree (n,n) such that $\Theta\ge\delta[\Delta]$ for some $\delta>0$. Therefore the push-forward

$$T:=(\mathrm{pr}_1)_*(\Theta\wedge\mathrm{pr}_2^*\,\omega)$$

is a positive $(1,1)$-current such that

$$T\ge\delta(\mathrm{pr}_1)_*([\Delta]\wedge\mathrm{pr}_2^*\,\omega)=\delta\omega.$$

It follows that T is a Kähler current. On the other hand, T is numerically equivalent to $(\mathrm{pr}_1)_*(\widetilde\alpha^n\wedge\mathrm{pr}_2^*\,\omega)$, which is the form given in coordinates by

$$x\mapsto\int_{y\in X}\big(\alpha(x)+\alpha(y)\big)^n\wedge\omega(y)=C\alpha(x)$$

where $C=n\int_X\alpha(y)^{n-1}\wedge\omega(y)$. Hence $T\equiv C\alpha$, which implies that $\{\alpha\}$ contains a Kähler current. Theorem 18.5 is proved. □

End of Proof of Theorems 18.1 and 18.2. Clearly the open cone $\mathcal K$ is contained in $\mathcal P$, hence in order to show that $\mathcal K$ is one of the connected components of $\mathcal P$, we need only show that $\mathcal K$ is closed in $\mathcal P$, i.e. that $\overline{\mathcal K}\cap\mathcal P\subset\mathcal K$. Pick a class $\{\alpha\}\in\overline{\mathcal K}\cap\mathcal P$. In particular $\{\alpha\}$ is nef and satisfies $\int_X\alpha^n>0$. By Theorem 18.5 we conclude that

$\{\alpha\}$ contains a Kähler current T. However, an induction on dimension using the assumption $\int_Y \alpha^p$ for all analytic subsets Y (we also use resolution of singularities for Y at this step) shows that the restriction $\{\alpha\}_{|Y}$ is the class of a Kähler current on Y. We conclude that $\{\alpha\}$ is a Kähler class by 18.4 (ii), therefore $\{\alpha\} \in \mathcal{K}$, as desired. □

The projective case 18.2 is a consequence of the following variant of Theorem 18.1.

(18.8) Corollary. *Let X be a compact Kähler manifold. A $(1,1)$ cohomology class $\{\alpha\}$ on X is Kähler if and only if there exists a Kähler metric ω on X such that $\int_Y \alpha^k \wedge \omega^{p-k} > 0$ for all irreducible analytic sets Y and all $k = 1, 2, \ldots, p = \dim Y$.*

Proof. The assumption clearly implies that

$$\int_Y (\alpha + t\omega)^p > 0$$

for all $t \in \mathbb{R}_+$, hence the half-line $\alpha + (\mathbb{R}_+)\omega$ is entirely contained in the cone $\mathcal{P}$ of numerically positive classes. Since $\alpha + t_0\omega$ is Kähler for t_0 large, we conclude that the half-line in entirely contained in the connected component $\mathcal{K}$, and therefore $\alpha \in \mathcal{K}$. □

In the projective case, we can take $\omega = c_1(H)$ for a given very ample divisor H, and the condition $\int_Y \alpha^k \wedge \omega^{p-k} > 0$ is equivalent to

$$\int_{Y \cap H_1 \cap \ldots \cap H_{p-k}} \alpha^k > 0$$

for a suitable complete intersection $Y \cap H_1 \cap \ldots \cap H_{p-k}$, $H_j \in |H|$. This shows that algebraic cycles are sufficient to test the Kähler property, and the special case 18.2 follows. On the other hand, we can pass to the limit in 18.8 by replacing α by $\alpha + \varepsilon\omega$, and in this way we get also a characterization of nef classes.

(18.9) Corollary. *Let X be a compact Kähler manifold. A $(1,1)$ cohomology class $\{\alpha\}$ on X is nef if and only if there exists a Kähler metric ω on X such that $\int_Y \alpha^k \wedge \omega^{p-k} \geq 0$ for all irreducible analytic sets Y and all $k = 1, 2, \ldots, p = \dim Y$.*

By a formal convexity argument, one can derive from 18.8 or 18.9 the following interesting consequence about the dual of the cone $\mathcal{K}$. We will not give the proof here, because it is just a simple tricky argument which does not require any new analysis.

(18.10) Theorem. *Let X be a compact Kähler manifold. A $(1,1)$ cohomology class $\{\alpha\}$ on X is nef if and only for every irreducible analytic set Y in X, $p = \dim X$ and every Kähler metric ω on X we have $\int_Y \alpha \wedge \omega^{p-1} \geq 0$. In other words, the dual of the nef cone $\overline{\mathcal{K}}$ is the closed convex cone in $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by cohomology classes of currents of the form $[Y] \wedge \omega^{p-1}$ in $H^{n-1,n-1}(X, \mathbb{R})$, where Y runs over the collection of irreducible analytic subsets of X and $\{\omega\}$ over the set of Kähler classes of X.*

Our main Theorem 18.1 has an important application to the deformation theory of compact Kähler manifolds.

(18.11) Theorem. *Let $\pi : \mathcal{X} \to S$ be a deformation of compact Kähler manifolds over an irreducible base S. Then there exists a countable union $S' = \bigcup S_\nu$ of analytic subsets $S_\nu \subsetneq S$, such that the Kähler cones $\mathcal{K}_t \subset H^{1,1}(X_t, \mathbb{C})$ of the fibers $X_t = \pi^{-1}(t)$ are invariant over $S \smallsetminus S'$ under parallel transport with respect to the $(1,1)$-projection $\nabla^{1,1}$ of the Gauss-Manin connection ∇ in the decomposition of*

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & 0 \\ * & \nabla^{1,1} & * \\ 0 & * & \nabla^{0,2} \end{pmatrix}$$

on the Hodge bundle $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

We moreover conjecture that for an arbitrary deformation $\mathcal{X} \to S$ of compact complex manifolds, the Kähler property is open with respect to the countable Zariski topology on the base S of the deformation.

Let us recall the general fact that all fibers X_t of a deformation over a connected base S are diffeomorphic, since $\mathcal{X} \to S$ is a locally trivial differentiable bundle. This implies that the cohomology bundle

$$S \ni t \mapsto H^k(X_t, \mathbb{C})$$

is locally constant over the base S. The corresponding (flat) connection of this bundle is called the Gauss-Manin connection, and will be denoted here by ∇. As is well known, the Hodge filtration

$$F^p(H^k(X_t, \mathbb{C})) = \bigoplus_{r+s=k, r\geq p} H^{r,s}(X_t, \mathbb{C})$$

defines a *holomorphic* subbundle of $H^k(X_t, \mathbb{C})$ (with respect to its locally constant structure). On the other hand, the Dolbeault groups are given by

$$H^{p,q}(X_t, \mathbb{C}) = F^p(H^k(X_t, \mathbb{C})) \cap \overline{F^{k-p}(H^k(X_t, \mathbb{C}))}, \qquad k = p + q,$$

and they form *real analytic* subbundles of $H^k(X_t, \mathbb{C})$. We are interested especially in the decomposition

$$H^2(X_t, \mathbb{C}) = H^{2,0}(X_t, \mathbb{C}) \oplus H^{1,1}(X_t, \mathbb{C}) \oplus H^{0,2}(X_t, \mathbb{C})$$

and the induced decomposition of the Gauss-Manin connection acting on H^2

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & * \\ * & \nabla^{1,1} & * \\ * & * & \nabla^{0,2} \end{pmatrix}.$$

Here the stars indicate suitable bundle morphisms – actually with the lower left and upper right stars being zero by Griffiths' transversality property, but we do not really care here. The notation $\nabla^{p,q}$ stands for the induced (real analytic, not necessarily flat) connection on the subbundle $t \mapsto H^{p,q}(X_t, \mathbb{C})$.

Sketch of Proof of Theorem 18.11. The result is local on the base, hence we may assume that S is contractible. Then the family is differentiably trivial, the Hodge bundle $t \mapsto H^2(X_t, \mathbb{C})$ is the trivial bundle and $t \mapsto H^2(X_t, \mathbb{Z})$ is a trivial lattice. We use the existence of a relative cycle space $C^p(\mathcal{X}/S) \subset C^p(\mathcal{X})$ which consists of all cycles contained in the fibres of $\pi : X \to S$. It is equipped with a canonical holomorphic projection

$$\pi_p : C^p(\mathcal{X}/S) \to S.$$

We then define the S_ν's to be the images in S of those connected components of $C^p(\mathcal{X}/S)$ which do not project onto S. By the fact that the projection is proper on each component, we infer that S_ν is an analytic subset of S. The definition of the S_ν's imply that the cohomology classes induced by the analytic cycles $\{[Z]\}$, $Z \subset X_t$, remain exactly the same for all $t \in S \smallsetminus S'$. This result implies in its turn that the conditions defining the numerically positive cones $\mathcal{P}_t$ remain the same, except for the fact that the spaces $H^{1,1}_{\mathbb{R}}(X_t) \subset H^2(X_t, \mathbb{R})$ vary along with the Hodge decomposition. At this point, a standard calculation implies that the $\mathcal{P}_t$ are invariant by parallel transport under $\nabla^{1,1}$. This is done as follows.

Since S is irreducible and S' is a countable union of analytic sets, it follows that $S \smallsetminus S'$ is arcwise connected by piecewise smooth analytic arcs. Let

$$\gamma : [0,1] \to S \smallsetminus S', \qquad u \mapsto t = \gamma(u)$$

be such a smooth arc, and let $\alpha(u) \in H^{1,1}(X_{\gamma(u)}, \mathbb{R})$ be a family of real $(1,1)$-cohomology classes which are constant by parallel transport under $\nabla^{1,1}$. This is equivalent to assuming that

$$\nabla(\alpha(u)) \in H^{2,0}(X_{\gamma(u)}, \mathbb{C}) \oplus H^{0,2}(X_{\gamma(u)}, \mathbb{C})$$

for all u. Suppose that $\alpha(0)$ is a numerically positive class in $X_{\gamma(0)}$. We then have

$$\alpha(0)^p \cdot \{[Z]\} = \int_Z \alpha(0)^p > 0$$

for all p-dimensional analytic cycles Z in $X_{\gamma(0)}$. Let us denote by

$$\zeta_Z(t) \in H^{2q}(X_t, \mathbb{Z}), \qquad q = \dim X_t - p,$$

the family of cohomology classes equal to $\{[Z]\}$ at $t = \gamma(0)$, such that $\nabla \zeta_Z(t) = 0$ (i.e. constant with respect to the Gauss-Manin connection). By the above discussion, $\zeta_Z(t)$ is of type (q,q) for all $t \in S$, and when $Z \subset X_{\gamma(0)}$ varies, $\zeta_Z(t)$ generates all classes of analytic cycles in X_t if $t \in S \smallsetminus S'$. Since ζ_Z is ∇-parallel and $\nabla\alpha(u)$ has no component of type $(1,1)$, we find

$$\frac{d}{du}(\alpha(u)^p \cdot \zeta_Z(\gamma(u)) = p\alpha(u)^{p-1} \cdot \nabla\alpha(u) \cdot \zeta_Z(\gamma(u)) = 0.$$

We infer from this that $\alpha(u)$ is a numerically positive class for all $u \in [0,1]$. This argument shows that the set $\mathcal{P}_t$ of numerically positive classes in $H^{1,1}(X_t, \mathbb{R})$ is invariant by parallel transport under $\nabla^{1,1}$ over $S \smallsetminus S'$.

By a standard result of Kodaira-Spencer [KS60] relying on elliptic PDE theory, every Kähler class in X_{t_0} can be deformed to a nearby Kähler class in nearby fibres X_t. This implies that the connected component of $\mathcal{P}_t$ which corresponds to the Kähler cone $\mathcal{K}_t$ must remain the same. The theorem is proved. □

As a by-product of our techniques, especially the regularization theorem for currents, we also get the following result for which we refer to [DP03].

(18.12) Theorem. *A compact complex manifold carries a Kähler current if and only if it is bimeromorphic to a Kähler manifold* (*or equivalently, dominated by a Kähler manifold*)*.*

This class of manifolds is called the *Fujiki class* $\mathcal{C}$. If we compare this result with the solution of the Grauert-Riemenschneider conjecture, it is tempting to make the following conjecture which would somehow encompass both results.

(18.13) Conjecture. *Let X be a compact complex manifold of dimension n. Assume that X possesses a nef cohomology class $\{\alpha\}$ of type $(1,1)$ such that $\int_X \alpha^n > 0$. Then X is in the Fujiki class $\mathcal{C}$. [Also, $\{\alpha\}$ would contain a Kähler current, as it follows from Theorem 18.5 if Conjecture 18.13 is proved].*

We want to mention here that most of the above results were already known in the cases of complex surfaces (i.e. in dimension 2), thanks to the work of N. Buchdahl [Buc99, 00] and A. Lamari [Lam99a, 99b].

Shortly after the original [DP03] manuscript appeared in April 2001, Daniel Huybrechts [Huy01] informed us Theorem 18.1 can be used to calculate the Kähler cone of a very general hyperkähler manifold: the Kähler cone is then equal to a suitable connected component of the positive cone defined by the Beauville-Bogomolov quadratic form. In the case of an arbitrary hyperkähler manifold, S.Boucksom [Bou02] later showed that a $(1,1)$ class $\{\alpha\}$ is Kähler if and only if it lies in the positive part of the Beauville-Bogomolov quadratic cone and moreover $\int_C \alpha > 0$ for all <u>*rational curves*</u> $C \subset X$ (see also [Huy99]).

19. Cones of Curves

In a dual way, we consider in $H^{n-1,n-1}_{\mathbb{R}}(X)$ the cone $\mathcal{N}$ generated by classes of positive currents T of type $(n-1,n-1)$ (i.e., of bidimension $(1,1)$). In the projective case, we also consider the intersection

By extension, we will say that $\overline{\mathcal{K}}$ is the cone of *nef* $(1,1)$-cohomology classes (even though they are not necessarily integral). We now turn ourselves to cones in cohomology of bidegree $(n-1,n-1)$.

(19.1) Definition. *Let X be a compact Kähler manifold.*

(i) *We define $\mathcal{N}$ to be the* (*closed*) *convex cone in $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by classes of positive currents T of type $(n-1,n-1)$* (*i.e., of bidimension* $(1,1)$)*.*

(ii) *We define the cone $\mathcal{M} \subset H^{n-1,n-1}_{\mathbb{R}}(X)$ of <u>movable classes</u> to be the closure of the convex cone generated by classes of currents of the form*

$$\mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})$$

where $\mu : \widetilde{X} \to X$ is an arbitrary modification (*one could just restrict oneself to compositions of blow-ups with smooth centers*)*, and the $\widetilde{\omega}_j$ are Kähler forms on $\widetilde{X}$. Clearly $\mathcal{M} \subset \mathcal{N}$.*

(iii) *Correspondingly, we introduce the intersections*

$$\mathcal{N}_{\mathrm{NS}} = \mathcal{N} \cap N_1(X), \qquad \mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X),$$

in the space of integral bidimension $(1,1)$*-classes*

$$N_1(X) := (H^{n-1,n-1}_{\mathbb{R}}(X) \cap H^{2n-2}(X,\mathbb{Z})/\mathrm{tors}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

(iv) *If* X *is projective, we define* $\mathrm{NE}(X)$ *to be the convex cone generated by all effective curves. Clearly* $\overline{\mathrm{NE}(X)} \subset \mathcal{N}_{\mathrm{NS}}$.

(v) *If* X *is projective, we say that* C *is a strongly movable curve if*

$$C = \mu_\star(\widetilde{A}_1 \cap \ldots \cap \widetilde{A}_{n-1})$$

for suitable very ample divisors $\widetilde{A}_j$ *on* $\widetilde{X}$*, where* $\mu : \widetilde{X} \to X$ *is a modification. We let* $\mathrm{SME}(X)$ *to be the convex cone generated by all strongly movable (effective) curves. Clearly* $\overline{\mathrm{SME}(X)} \subset \mathcal{M}_{\mathrm{NS}}$.

(vi) *We say that* C *is a movable curve if* $C = C_{t_0}$ *is a member of an analytic family* $(C_t)_{t\in S}$ *such that* $\bigcup_{t\in S} C_t = X$ *and, as such, is a reduced irreducible* 1*-cycle. We let* $\mathrm{ME}(X)$ *to be the convex cone generated by all movable (effective) curves.*

The upshot of this definition lies in the following easy observation.

(19.2) Proposition. *Let* X *be a compact Kähler manifold. Consider the Poincaré duality pairing*

$$H^{1,1}(X,\mathbb{R}) \times H^{n-1,n-1}_{\mathbb{R}}(X) \longrightarrow \mathbb{R}, \qquad (\alpha,\beta) \longmapsto \int_X \alpha \wedge \beta.$$

Then the duality pairing takes nonnegative values

(i) *for all pairs* $(\alpha,\beta) \in \overline{\mathcal{K}} \times \mathcal{N}$*;*

(ii) *for all pairs* $(\alpha,\beta) \in \mathcal{E} \times \mathcal{M}$.

(iii) *for all pairs* (α,β) *where* $\alpha \in \mathcal{E}$ *and* $\beta = [C_t] \in \mathrm{ME}(X)$ *is the class of a movable curve.*

Proof. (i) is obvious. In order to prove (ii), we may assume that $\beta = \mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})$ for some modification $\mu : \widetilde{X} \to X$, where $\alpha = \{T\}$ is the class of a positive $(1,1)$-current on X and $\widetilde{\omega}_j$ are Kähler forms on $\widetilde{X}$. Then

$$\int_X \alpha \wedge \beta = \int_X T \wedge \mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1}) = \int_X \mu^* T \wedge \widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1} \geq 0.$$

Here, we have used the fact that a closed positive $(1,1)$-current T always has a pull-back $\mu^\star T$, which follows from the fact that if $T = i\partial\overline{\partial}\varphi$ locally for some plurisubharmonic function in X, we can set $\mu^\star T = i\partial\overline{\partial}(\varphi\circ\mu)$. For (iii), we suppose $\alpha = \{T\}$ and $\beta = \{[C_t]\}$. Then we take an open covering (U_j) on X such that $T = i\partial\overline{\partial}\varphi_j$ with suitable plurisubharmonic functions φ_j on U_j. If we select a smooth partition of unity $\sum \theta_j = 1$ subordinate to (U_j), we then get

$$\int_X \alpha \wedge \beta = \int_{C_t} T_{|C_t} = \sum_j \int_{C_t \cap U_j} \theta_j i\partial\overline{\partial}\varphi_{j|C_t} \geq 0.$$

For this to make sense, it should be noticed that $T_{|C_t}$ is a well defined closed positive $(1,1)$-current (i.e. measure) on C_t for almost every $t \in S$, in the sense of Lebesgue measure. This is true only because (C_t) covers X, thus $\varphi_{j|C_t}$ is not identically $-\infty$ for almost every $t \in S$. The equality in the last formula is then shown by a regularization argument for T, writing $T = \lim T_k$ with $T_k = \alpha + i\partial\overline{\partial}\psi_k$ and a decreasing sequence of smooth almost plurisubharmonic potentials $\psi_k \downarrow \psi$ such that the Levi forms have a uniform lower bound $i\partial\overline{\partial}\psi_k \geq -C\omega$ (such a sequence exists by [Dem92]). Then, writing $\alpha = i\partial\overline{\partial}v_j$ for some smooth potential v_j on U_j, we have $T = i\partial\overline{\partial}\varphi_j$ on U_j with $\varphi_j = v_j + \psi$, and this is the decreasing limit of the smooth approximations $\varphi_{j,k} = v_j + \psi_k$ on U_j. Hence $T_{k|C_t} \to T_{|C_t}$ for the weak topology of measures on C_t. □

If $\mathcal{C}$ is a convex cone in a finite dimensional vector space E, we denote by $\mathcal{C}^\vee$ the dual cone, i.e. the set of linear forms $u \in E^\star$ which take nonnegative values on all elements of $\mathcal{C}$. By the Hahn-Banach theorem, we always have $\mathcal{C}^{\vee\vee} = \overline{\mathcal{C}}$.

Proposition 19.2 leads to the natural question whether the cones $(\mathcal{K}, \mathcal{N})$ and $(\mathcal{E}, \mathcal{M})$ are dual under Poincaré duality. This question is addressed in the next section. Before doing so, we observe that the algebraic and transcendental cones of $(n-1, n-1)$ cohomology classes are related by the following equalities.

(19.3) Theorem. *Let X be a projective manifold. Then*

(i) $\overline{\mathrm{NE}(X)} = \mathcal{N}_{\mathrm{NS}}$.

(ii) $\overline{\mathrm{SME}(X)} = \overline{\mathrm{ME}(X)} = \mathcal{M}_{\mathrm{NS}}$.

Proof. (i) It is a standard result of algebraic geometry (see e.g. [Har70]), that the cone of effective cone $\mathrm{NE}(X)$ is dual to the cone $\overline{\mathcal{K}_{\mathrm{NS}}}$ of nef divisors, hence

$$\mathcal{N}_{\mathrm{NS}} \supset \overline{\mathrm{NE}(X)} = \mathcal{K}^\vee.$$

On the other hand, (19.3) (i) implies that $\mathcal{N}_{\mathrm{NS}} \subset \mathcal{K}^\vee$, so we must have equality and (i) follows.

Similarly, (ii) requires a duality statement which will be established only in the next sections, so we postpone the proof. □

20. Main Duality Results

It is very well-known that the cone $\overline{\mathcal{K}_{\mathrm{NS}}}$ of nef divisors is dual to the cone $\mathcal{N}_{\mathrm{NS}}$ of effective curves if X is projective. The transcendental case can be stated as follows.

(20.1) Theorem (Demailly-Paun, 2001)**.** *If X is Kähler, the cones $\overline{\mathcal{K}} \subset H^{1,1}(X, \mathbb{R})$ and $\mathcal{N} \subset H_{\mathbb{R}}^{n-1,n-1}(X)$ are dual by Poincaré duality, and $\mathcal{N}$ is the closed convex cone generated by classes $[Y] \wedge \omega^{p-1}$ where $Y \subset X$ ranges over p-dimensional analytic subsets, $p = 1, 2, \ldots, n$, and ω ranges over Kähler forms.*

Proof. Indeed, Prop. 19.4 shows that the dual cone $\mathcal{K}^\vee$ contains $\mathcal{N}$ which itself contains the cone $\mathcal{N}'$ of all classes of the form $\{[Y] \wedge \omega^{p-1}\}$. The main result of [DP03] conversely shows that the dual of $(\mathcal{N}')^\vee$ is equal to $\overline{\mathcal{K}}$, so we must have

$$\mathcal{K}^\vee = \overline{\mathcal{N}'} = \mathcal{N}. \qquad \square$$

The other important duality result is the following characterization of pseudo-effective classes, proved in [BDPP03] (the "only if" part already follows from 19.4 (iii)).

(20.2) Theorem. *If X is projective, then a class $\alpha \in \mathrm{NS}_{\mathbb{R}}(X)$ is pseudo-effective if* (*and only if*) *it is in the dual cone of the cone* SME(X) *of strongly movable curves.*

In other words, a line bundle L is pseudo-effective if (and only if) $L \cdot C \geq 0$ for all *movable curves*, i.e., $L \cdot C \geq 0$ for every very generic curve C (not contained in a countable union of algebraic subvarieties). In fact, by definition of SME(X), it is enough to consider only those curves C which are images of generic complete intersection of very ample divisors on some variety $\widetilde{X}$, under a modification $\mu : \widetilde{X} \to X$.

By a standard blowing-up argument, it also follows that a line bundle L on a normal Moishezon variety is pseudo-effective if and only if $L \cdot C \geq 0$ for every movable curve C.

The Kähler analogue should be :

(20.3) Conjecture. *For an arbitrary compact Kähler manifold X, the cones $\mathcal{E}$ and $\mathcal{M}$ are dual.*

$\mathcal{K}$ $\mathcal{M}$

$\mathcal{K}_{\mathrm{NS}}$ duality $\mathcal{M}_{\mathrm{NS}}$

$\mathcal{E}$ $\mathcal{N}$

$\mathcal{E}_{\mathrm{NS}}$ $\mathcal{N}_{\mathrm{NS}}$

$\mathrm{NS}_{\mathbb{R}}(X)$ $H^{1,1}(X,\mathbb{R})$ $H^{n-1,n-1}_{\mathbb{R}}(X)$ $N_1(X)$

The relation between the various cones of movable curves and currents in (19.5) is now a rather direct consequence of Theorem 20.2. In fact, using ideas hinted in [DPS96], we can say a little bit more. Given an irreducible curve $C \subset X$, we consider its normal "bundle" $N_C = \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C)$, where $\mathcal{I}$ is the ideal sheaf of C. If C is a general member of a covering family (C_t), then N_C is nef. Now [DPS96] says that the dual cone of the pseudo-effective cone of X contains the closed cone spanned by curves with nef normal bundle, which in turn contains the cone of movable curves. In this way we get :

(20.4) Theorem. *Let X be a projective manifold. Then the following cones coincide.*

(i) *the cone* $\mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X)$;

(ii) *the closed cone* $\overline{\mathrm{SME}(X)}$ *of strongly movable curves* ;

(iii) *the closed cone* $\overline{\mathrm{ME}(X)}$ *of movable curves* ;

(iv) *the closed cone* $\overline{\mathrm{ME}_{\mathrm{nef}}(X)}$ *of curves with nef normal bundle.*

Proof. We have already seen that

$$\mathrm{SME}(X) \subset \mathrm{ME}(X) \subset \mathrm{ME}_{\mathrm{nef}}(X) \subset (\mathcal{E}_{\mathrm{NS}})^\vee$$

and

$$\mathrm{SME}(X) \subset \mathrm{ME}(X) \subset \mathcal{M}_{\mathrm{NS}} \subset (\mathcal{E}_{\mathrm{NS}})^\vee$$

by 19.4 (iii). Now Theorem 20.2 implies $(\mathcal{M}_{\mathrm{NS}})^\vee = \overline{\mathrm{SME}(X)}$, and 20.4 follows. □

(20.5) Corollary. *Let X be a projective manifold and L a line bundle on X.*

(i) *L is pseudo-effective if and only if $L \cdot C \geq 0$ for all curves C with nef normal sheaf N_C.*

(ii) *If L is big, then $L \cdot C > 0$ for all curves C with nef normal sheaf N_C.*

20.5 (i) strenghtens results from [PSS99]. It is however not yet clear whether $\mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X)$ is equal to the closed cone of curves with *ample* normal bundle (although we certainly expect this to be true).

The most important special case of Theorem 20.2 is

(20.6) Theorem. *If X is a projective manifold and is not uniruled, then K_X is pseudo-effective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$.*

Proof. If $K_X \notin \mathcal{E}_{\mathrm{NS}}$, Theorem 19.2 shows that there is a moving curve C_t such that $K_X \cdot C_t < 0$. The “bend-and-break” lemma then implies that there is family Γ_t of rational curves with $K_X \cdot \Gamma_t < 0$, so X is uniruled. □

A stronger result is expected to be true, namely :

(20.7) Conjecture (special case of the “abundance conjecture”)**.** *If K_X is pseudo-effective, then $\kappa(X) \geq 0$.*

21. Approximation of psh functions by logarithms of holomorphic functions

The fundamental tool is the Ohsawa-Takegoshi extension theorem in the following form ([OT87], see also [Dem00]).

(21.1) Theorem. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and let $Y \subset X$ be a nonsingular complex submanifold defined by a section s of some hermitian vector bundle E with* bounded *curvature tensor on Ω. Assume that s is everywhere transverse to the zero section and that $|s| \le e^{-1}$ on Ω. Then there is a constant $C > 0$ (depending only on E), with the following property: for every psh function φ on Ω, every holomorphic function f on Y with $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} e^{-\varphi} dV_Y < +\infty$, there exists an extension F of f to Ω such that*

$$\int_\Omega \frac{|F|^2}{|s|^{2r}(-\log|s|)^2} e^{-\varphi} dV_\Omega \le C \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} e^{-\varphi} dV_Y.$$

Here we simply take Y to be a point $\{z_0\}$. In this case, the theorem says that we can find $F \in \mathcal{O}(\Omega)$ with a prescribed value $F(z_0)$, such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \le C|F(z_0)|^2.$$

We now show that every psh function on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ can be approximated very accurately by psh functions with analytic singularities. The main idea is taken from [Dem92].

(21.2) Theorem. *Let φ be a plurisubharmonic function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi} d\lambda < +\infty$ and let $\varphi_m = \frac{1}{2m}\log\sum|\sigma_\ell|^2$ where (σ_ℓ) is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that*

a) $\varphi(z) - \dfrac{C_1}{m} \le \varphi_m(z) \le \sup\limits_{|\zeta - z| < r} \varphi(\zeta) + \dfrac{1}{m}\log\dfrac{C_2}{r^n}$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$ and

b) $\nu(\varphi, z) - \dfrac{n}{m} \le \nu(\varphi_m, z) \le \nu(\varphi, z)$ *for every $z \in \Omega$.*

Proof. Note that $\sum|\sigma_\ell(z)|^2$ is the square of the norm of the evaluation linear form $f \mapsto f(z)$ on $\mathcal{H}_\Omega(m\varphi)$. As φ is locally bounded above, the L^2 topology is actually stronger than the topology of uniform convergence on compact subsets of Ω. It follows that the series $\sum|\sigma_\ell|^2$ converges uniformly on Ω and that its sum is real analytic. Moreover we have

$$\varphi_m(z) = \sup_{f \in B(1)} \frac{1}{m}\log|f(z)|$$

where $B(1)$ is the unit ball of $\mathcal{H}_\Omega(m\varphi)$. For $r < d(z, \partial\Omega)$, the mean value inequality applied to the psh function $|f|^2$ implies

$$\begin{aligned}|f(z)|^2 &\leq \frac{1}{\pi^n r^{2n}/n!}\int_{|\zeta-z|<r}|f(\zeta)|^2 d\lambda(\zeta)\\ &\leq \frac{1}{\pi^n r^{2n}/n!}\exp\Big(2m\sup_{|\zeta-z|<r}\varphi(\zeta)\Big)\int_\Omega |f|^2 e^{-2m\varphi}d\lambda.\end{aligned}$$

If we take the supremum over all $f \in B(1)$ we get

$$\varphi_m(z) \leq \sup_{|\zeta-z|<r}\varphi(\zeta) + \frac{1}{2m}\log\frac{1}{\pi^n r^{2n}/n!}$$

and the second inequality in a) is proved. Conversely, the Ohsawa-Takegoshi extension theorem applied to the 0-dimensional subvariety $\{z\} \subset \Omega$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on Ω such that $f(z) = a$ and

$$\int_\Omega |f|^2 e^{-2m\varphi}d\lambda \leq C_3|a|^2 e^{-2m\varphi(z)},$$

where C_3 only depends on n and $\operatorname{diam}\Omega$. We fix a such that the right hand side is 1. This gives the other inequality

$$\varphi_m(z) \geq \frac{1}{m}\log|a| = \varphi(z) - \frac{\log C_3}{2m}.$$

The above inequality implies $\nu(\varphi_m, z) \leq \nu(\varphi, z)$. In the opposite direction, we find

$$\sup_{|x-z|<r}\varphi_m(x) \leq \sup_{|\zeta-z|<2r}\varphi(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n}.$$

Divide by $\log r$ and take the limit as r tends to 0. The quotient by $\log r$ of the supremum of a psh function over $B(x, r)$ tends to the Lelong number at x. Thus we obtain

$$\nu(\varphi_m, x) \geq \nu(\varphi, x) - \frac{n}{m}.$$

□

Theorem 21.2 implies in a straighforward manner a deep result of [Siu74] on the analyticity of the Lelong number sublevel sets.

(21.3) Corollary. *Let φ be a plurisubharmonic function on a complex manifold X. Then, for every $c > 0$, the Lelong number sublevel set*

$$E_c(\varphi) = \big\{z \in X\,;\ \nu(\varphi, z) \geq c\big\}$$

is an analytic subset of X.

Proof. Since analyticity is a local property, it is enough to consider the case of a psh function φ on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. The inequalities obtained in 21.2 b) imply that

$$E_c(\varphi) = \bigcap_{m \geq m_0} E_{c-n/m}(\varphi_m).$$

Now, it is clear that $E_c(\varphi_m)$ is the analytic set defined by the equations $\sigma_\ell^{(\alpha)}(z) = 0$ for all multi-indices α such that $|\alpha| < mc$. Thus $E_c(\varphi)$ is analytic as a (countable) intersection of analytic sets. □

(21.4) Regularization theorem for currents. *Let X be a compact complex manifold equipped with a hermitian metric ω. Let $T = \alpha + i\partial\overline{\partial}\psi$ be a closed $(1,1)$-current on X, where α is smooth and ψ is a quasi-plurisubharmonic function. Assume that $T \geq \gamma$ for some real $(1,1)$-form γ on X with real coefficients. Then there exists a sequence $T_k = \alpha + i\partial\overline{\partial}\psi_k$ of closed $(1,1)$-currents such that*

(i) *ψ_k (and thus T_k) is smooth on the complement $X \smallsetminus Z_k$ of an analytic set Z_k, and the Z_k's form an increasing sequence*

$$Z_0 \subset Z_1 \subset \ldots \subset Z_k \subset \ldots \subset X.$$

(ii) *There is a uniform estimate $T_k \geq \gamma - \delta_k\omega$ with $\lim \downarrow \delta_k = 0$ as k tends to $+\infty$.*

(iii) *The sequence (ψ_k) is non increasing, and we have $\lim \downarrow \psi_k = \psi$. As a consequence, T_k converges weakly to T as k tends to $+\infty$.*

(iv) *Near Z_k, the potential ψ_k has logarithmic poles, namely, for every $x_0 \in Z_k$, there is a neighborhood U of x_0 such that $\psi_k(z) = \lambda_k \log \sum_\ell |g_{k,\ell}|^2 + O(1)$ for suitable holomorphic functions $(g_{k,\ell})$ on U and $\lambda_k > 0$. Moreover, there is a (global) proper modification $\mu_k : \widetilde{X}_k \to X$ of X, obtained as a sequence of blow-ups with smooth centers, such that $\psi_k \circ \mu_k$ can be written locally on $\widetilde{X}_k$ as*

$$\psi_k \circ \mu_k(w) = \lambda_k\big(\sum n_\ell \log |\widetilde{g}_\ell|^2 + f(w)\big)$$

where $(\widetilde{g}_\ell = 0)$ are local generators of suitable (global) divisors D_ℓ on $\widetilde{X}_k$ such that $\sum D_\ell$ has normal crossings, n_ℓ are positive integers, and the f's are smooth functions on $\widetilde{X}_k$.

Proof Sketch of proof. We briefly indicate the main ideas, since the proof can only be reconstructed by patching together arguments which appeared in different places (although the core the proof is entirely in [Dem92]). After replacing T with $T - \alpha$, we can assume that $\alpha = 0$ and $T = i\partial\overline{\partial}\psi \geq \gamma$. Given a small $\varepsilon > 0$, we select a covering of X by open balls B_j together with holomorphic coordinates $(z^{(j)})$ and real numbers β_j such that

$$0 \leq \gamma - \beta_j\, i\partial\overline{\partial}|z^{(j)}|^2 \leq \varepsilon\, i\partial\overline{\partial}|z^{(j)}|^2 \qquad \text{on } B_j$$

(this can be achieved just by continuity of γ, after diagonalizing γ at the center of the balls). We now take a partition of unity (θ_j) subordinate to (B_j) such that $\sum \theta_j^2 = 1$, and define

$$\psi_k(z) = \frac{1}{2k} \log \sum_j \theta_j^2 e^{2k\beta_j|z^{(j)}|^2} \sum_{\ell\in\mathbb{N}} |g_{j,k,\ell}|^2$$

where $(g_{j,k,\ell})$ is a Hilbert basis of the Hilbert space of holomorphic functions f on B_j such that

$$\int_{B_j} |f|^2 e^{-2k(\psi - \beta_j|z^{(j)}|^2)} < +\infty.$$

Notice that by the Hessian estimate $i\partial\overline{\partial}\psi \geq \gamma \geq \beta_j\, i\partial\overline{\partial}|z^{(j)}|^2$, the weight involved in the L^2 norm is plurisubharmonic. It then follows from the proof of Proposition 3.7 in [Dem92] that all properties (i)–(iv) hold true, except possibly the fact that the sequence ψ_k can be chosen to be non increasing, and the existence of the modification in (iv). However, the multiplier ideal sheaves of the weights $k(\psi - \beta_j|z^{(j)}|^2)$ are generated by the $(g_{j,k,\ell})_\ell$ on B_j, and these sheaves glue together into a global coherent multiplier ideal sheaf $\mathcal{I}(k\psi)$ on X (see [DEL99]); the modification μ_k is then obtained by blowing-up the ideal sheaf $\mathcal{I}(k\psi)$ so that $\mu_k^*\mathcal{I}(k\psi)$ is an invertible ideal sheaf associated with a normal crossing divisor (Hironaka [Hir64]). The fact that ψ_k can be chosen to be non increasing follows from a quantitative version of the "subadditivity of multiplier ideal sheaves" which is proved in Step 3 of section 16.2 (originally appeared as Theorem 2.2.1 in [DPS01], see also ([DEL99]). □

22. Zariski Decomposition and Movable Intersections

Let X be compact Kähler and let $\alpha \in \mathcal{E}^\circ$ be in the *interior* of the pseudo–effective cone. In analogy with the algebraic context such a class α is called "big", and it can then be represented by a *Kähler current* T, i.e. a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.

(22.1) Theorem (Demailly [Dem92], [Bou02, 3.1.24]**.** *If T is a Kähler current, then one can write $T = \lim T_m$ for a sequence of Kähler currents T_m which have logarithmic poles with coefficients in $\frac{1}{m}\mathbb{Z}$, i.e. there are modifications $\mu_m : X_m \to X$ such that*

$$\mu_m^\star T_m = [E_m] + \beta_m$$

where E_m is an effective $\mathbb{Q}$-divisor on X_m with coefficients in $\frac{1}{m}\mathbb{Z}$ (*the "fixed part"*) *and β_m is a closed semi-positive form* (*the "movable part"*).

Proof. This is a direct consequence of the results of section 21. Locally we can write $T = i\partial\overline{\partial}\varphi$ for some strictly plurisubharmonic potential φ. By the Bergman kernel trick and the Ohsawa-Takegoshi L^2 extension theorem, we get local approximations

$$\varphi = \lim \varphi_m, \qquad \varphi_m(z) = \frac{1}{2m} \log \sum_\ell |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the set of holomorphic functions which are L^2 with respect to the weight $e^{-2m\varphi}$. This Hilbert basis is also a family of local generators of the globally defined multiplier ideal sheaf $\mathcal{I}(mT) = \mathcal{I}(m\varphi)$. Then $\mu_m : X_m \to X$ is obtained by blowing-up this ideal sheaf, so that

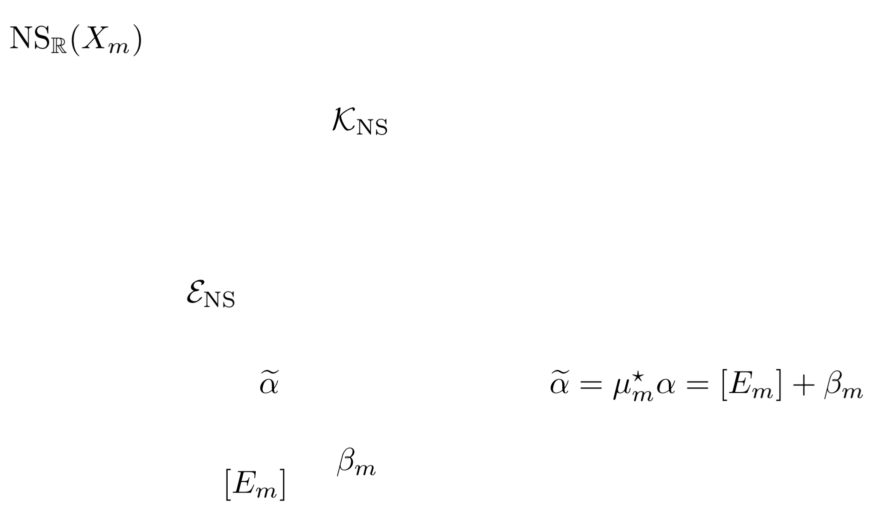
$$\mu_m^\star \mathcal{I}(mT) = \mathcal{O}(-mE_m).$$

We should notice that by approximating $T - \frac{1}{m}\omega$ instead of T, we can replace β_m by $\beta_m + \frac{1}{m}\mu^\star\omega$ which is a big class on X_m ; by playing with the multiplicities of the components of the exceptional divisor, we could even achieve that β_m is a Kähler class on X_m, but this will not be needed here. □

The more familiar algebraic analogue would be to take $\alpha = c_1(L)$ with a big line bundle L and to blow-up the base locus of $|mL|$, $m \gg 1$, to get a $\mathbb{Q}$-divisor decomposition

$$\mu_m^\star L \sim E_m + D_m, \qquad E_m \text{ effective}, \quad D_m \text{ free}.$$

Such a blow-up is usually referred to as a “log resolution” of the linear system $|mL|$, and we say that $E_m + D_m$ is an approximate Zariski decomposition of L. We will also use this terminology for Kähler currents with logarithmic poles.



(22.2) Definition. *We define the* <u>*volume*</u>*, or* <u>*movable self-intersection*</u> *of a big class* $\alpha \in \mathcal{E}^\circ$ *to be*

$$\mathrm{Vol}(\alpha) = \sup_{T\in\alpha} \int_{\widetilde{X}} \beta^n > 0$$

where the supremum is taken over all Kähler currents $T \in \alpha$ *with logarithmic poles, and* $\mu^\star T = [E] + \beta$ *with respect to some modification* $\mu : \widetilde{X} \to X$.

By Fujita [Fuj94] and Demailly-Ein-Lazarsfeld [DEL00], if L is a big line bundle, we have

$$\mathrm{Vol}(c_1(L)) = \lim_{m\to+\infty} D_m^n = \lim_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL),$$

and in these terms, we get the following statement.

(22.3) Proposition. *Let* L *be a big line bundle on the projective manifold* X*. Let* $\epsilon > 0$*. Then there exists a modification* $\mu : X_\epsilon \to X$ *and a decomposition* $\mu^*(L) = E + \beta$ *with* E *an effective* $\mathbb{Q}$*-divisor and* β *a big and nef* $\mathbb{Q}$*-divisor such that*

$$\mathrm{Vol}(L) - \varepsilon \le \mathrm{Vol}(\beta) \le \mathrm{Vol}(L).$$

It is very useful to observe that the supremum in Definition 22.2 is actually achieved by a collection of currents whose singularities satisfy a filtering property. Namely, if $T_1 = \alpha + i\partial\overline{\partial}\varphi_1$ and $T_2 = \alpha + i\partial\overline{\partial}\varphi_2$ are two Kähler currents with logarithmic poles in the class of α, then

$$T = \alpha + i\partial\overline{\partial}\varphi, \qquad \varphi = \max(\varphi_1, \varphi_2) \tag{22.4}$$

is again a Kähler current with weaker singularities than T_1 and T_2. One could define as well

$$(22.4') \qquad T = \alpha + i\partial\overline{\partial}\varphi, \qquad \varphi = \frac{1}{2m}\log(e^{2m\varphi_1} + e^{2m\varphi_2}),$$

where $m = \mathrm{lcm}(m_1, m_2)$ is the lowest common multiple of the denominators occuring in T_1, T_2. Now, take a simultaneous log-resolution $\mu_m : X_m \to X$ for which the singularities of T_1 and T_2 are resolved as $\mathbb{Q}$-divisors E_1 and E_2. Then clearly the associated divisor in the decomposition $\mu_m^\star T = [E] + \beta$ is given by $E = \min(E_1, E_2)$. By doing so, the volume $\int_{X_m} \beta^n$ gets increased, as we shall see in the proof of Theorem 22.5 below.

(22.5) Theorem (Boucksom [Bou02])**.** *Let X be a compact Kähler manifold. We denote here by $H^{k,k}_{\geq 0}(X)$ the cone of cohomology classes of type (k,k) which have non-negative intersection with all closed semi-positive smooth forms of bidegree $(n-k, n-k)$.*

(i) *For each integer $k = 1, 2, \ldots, n$, there exists a canonical "movable intersection product"*

$$\mathcal{E} \times \cdots \times \mathcal{E} \to H^{k,k}_{\geq 0}(X), \quad (\alpha_1, \ldots, \alpha_k) \mapsto \langle \alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1} \cdot \alpha_k \rangle$$

such that $\mathrm{Vol}(\alpha) = \langle \alpha^n \rangle$ *whenever α is a big class.*

(ii) *The product is increasing, homogeneous of degree 1 and superadditive in each argument, i.e.*

$$\langle \alpha_1 \cdots (\alpha'_j + \alpha''_j) \cdots \alpha_k \rangle \geq \langle \alpha_1 \cdots \alpha'_j \cdots \alpha_k \rangle + \langle \alpha_1 \cdots \alpha''_j \cdots \alpha_k \rangle.$$

It coincides with the ordinary intersection product when the $\alpha_j \in \overline{\mathcal{K}}$ are nef classes.

(iii) *The movable intersection product satisfies the Teissier-Hovanskii inequalities*

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_n \rangle \geq (\langle \alpha_1^n \rangle)^{1/n} \ldots (\langle \alpha_n^n \rangle)^{1/n} \qquad (\textit{with } \langle \alpha_j^n \rangle = \mathrm{Vol}(\alpha_j)\,).$$

(iv) *For $k = 1$, the above "product" reduces to a (non linear) projection operator*

$$\mathcal{E} \to \mathcal{E}_1, \qquad \alpha \to \langle \alpha \rangle$$

onto a certain convex subcone $\mathcal{E}_1$ of $\mathcal{E}$ such that $\overline{\mathcal{K}} \subset \mathcal{E}_1 \subset \mathcal{E}$. Moreover, there is a "divisorial Zariski decomposition"

$$\alpha = \{N(\alpha)\} + \langle \alpha \rangle$$

where $N(\alpha)$ is a uniquely defined effective divisor which is called the "negative divisorial part" of α. The map $\alpha \mapsto N(\alpha)$ is homogeneous and subadditive, and $N(\alpha) = 0$ if and only if $\alpha \in \mathcal{E}_1$.

(v) *The components of $N(\alpha)$ always consist of divisors whose cohomology classes are linearly independent, especially $N(\alpha)$ has at most $\rho = \mathrm{rank}_{\mathbb{Z}}\,\mathrm{NS}(X)$ components.*

Proof. We essentially repeat the arguments developped in [Bou02], with some simplifications arising from the fact that X is supposed to be Kähler from the start.

(i) First assume that all classes α_j are big, i.e. $\alpha_j \in \mathcal{E}^\circ$. Fix a smooth closed $(n-k, n-k)$ *semi-positive* form u on X. We select Kähler currents $T_j \in \alpha_j$ with logarithmic poles, and a simultaneous log-resolution $\mu : \widetilde{X} \to X$ such that

$$\mu^\star T_j = [E_j] + \beta_j.$$

We consider the direct image current $\mu_\star(\beta_1 \wedge \ldots \wedge \beta_k)$ (which is a closed positive current of bidegree (k,k) on X) and the corresponding integrals

$$\int_{\widetilde{X}} \beta_1 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \geq 0.$$

If we change the representative T_j with another current T'_j, we may always take a simultaneous log-resolution such that $\mu^\star T'_j = [E'_j] + \beta'_j$, and by using (22.4′) we can always assume that $E'_j \leq E_j$. Then $D_j = E_j - E'_j$ is an effective divisor and we find $[E_j] + \beta_j \equiv [E'_j] + \beta'_j$, hence $\beta'_j \equiv \beta_j + [D_j]$. A substitution in the integral implies

$$\begin{aligned}\int_{\widetilde{X}} &\beta'_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \\ &= \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u + \int_{\widetilde{X}} [D_1] \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \\ &\geq \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u.\end{aligned}$$

Similarly, we can replace successively all forms β_j by the β'_j, and by doing so, we find

$$\int_{\widetilde{X}} \beta'_1 \wedge \beta'_2 \wedge \ldots \wedge \beta'_k \wedge \mu^\star u \geq \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u.$$

We claim that the closed positive currents $\mu_\star(\beta_1 \wedge \ldots \wedge \beta_k)$ are uniformly bounded in mass. In fact, if ω is a Kähler metric in X, there exists a constant $C_j \geq 0$ such that $C_j\{\omega\} - \alpha_j$ is a Kähler class. Hence $C_j\omega - T_j \equiv \gamma_j$ for some Kähler form γ_j on X. By pulling back with μ, we find $C_j\mu^\star\omega - ([E_j] + \beta_j) \equiv \mu^\star\gamma_j$, hence

$$\beta_j \equiv C_j\mu^\star\omega - ([E_j] + \mu^\star\gamma_j).$$

By performing again a substitution in the integrals, we find

$$\int_{\widetilde{X}} \beta_1 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \leq C_1 \ldots C_k \int_{\widetilde{X}} \mu^\star\omega^k \wedge \mu^\star u = C_1 \ldots C_k \int_X \omega^k \wedge u$$

and this is true especially for $u = \omega^{n-k}$. We can now arrange that for each of the integrals associated with a countable dense family of forms u, the supremum is achieved by a sequence of currents $(\mu_m)_\star(\beta_{1,m} \wedge \ldots \wedge \beta_{k,m})$ obtained as direct images by a suitable sequence of modifications $\mu_m : \widetilde{X}_m \to X$. By extracting a subsequence, we can achieve that this sequence is weakly convergent and we set

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{m \to +\infty} \uparrow \{(\mu_m)_\star(\beta_{1,m} \wedge \beta_{2,m} \wedge \ldots \wedge \beta_{k,m})\}$$

(the monotonicity is not in terms of the currents themselves, but in terms of the integrals obtained when we evaluate against a smooth closed semi-positive form u). By evaluating against a basis of positive classes $\{u\} \in H^{n-k,n-k}(X)$, we infer by Poincaré duality that the class of $\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle$ is uniquely defined (although, in general, the representing current is not unique).

(ii) It is indeed clear from the definition that the movable intersection product is homogeneous, increasing and superadditive in each argument, at least when the α_j's are in $\mathcal{E}^\circ$. However, we can extend the product to the closed cone $\mathcal{E}$ by monotonicity, by setting

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{\delta \downarrow 0} \downarrow \langle (\alpha_1 + \delta\omega) \cdot (\alpha_2 + \delta\omega) \cdots (\alpha_k + \delta\omega) \rangle$$

for arbitrary classes $\alpha_j \in \mathcal{E}$ (again, monotonicity occurs only where we evaluate against closed semi-positive forms u). By weak compactness, the movable intersection product can always be represented by a closed positive current of bidegree (k,k).

(iii) The Teissier-Hovanskii inequalities are a direct consequence of the fact that they hold true for nef classes, so we just have to apply them to the classes $\beta_{j,m}$ on $\widetilde{X}_m$ and pass to the limit.

(iv) When $k = 1$ and $\alpha \in \mathcal{E}^0$, we have

$$\alpha = \lim_{m \to +\infty} \{(\mu_m)_\star T_m\} = \lim_{m \to +\infty} (\mu_m)_\star [E_m] + \{(\mu_m)_\star \beta_m\}$$

and $\langle \alpha \rangle = \lim_{m \to +\infty} \{(\mu_m)_\star \beta_m\}$ by definition. However, the images $F_m = (\mu_m)_\star F_m$ are effective $\mathbb{Q}$-divisors in X, and the filtering property implies that F_m is a decreasing sequence. It must therefore converge to a (uniquely defined) limit $F = \lim F_m := N(\alpha)$ which is an effective $\mathbb{R}$-divisor, and we get the asserted decomposition in the limit.

Since $N(\alpha) = \alpha - \langle \alpha \rangle$ we easily see that $N(\alpha)$ is subadditive and that $N(\alpha) = 0$ if α is the class of a smooth semi-positive form. When α is no longer a big class, we define

$$\langle \alpha \rangle = \lim_{\delta \downarrow 0} \downarrow \langle \alpha + \delta\omega \rangle, \qquad N(\alpha) = \lim_{\delta \downarrow 0} \uparrow N(\alpha + \delta\omega)$$

(the subadditivity of N implies $N(\alpha + (\delta+\varepsilon)\omega) \le N(\alpha + \delta\omega)$). The divisorial Zariski decomposition follows except maybe for the fact that $N(\alpha)$ might be a convergent countable sum of divisors. However, this will be ruled out when (v) is proved. As $N(\bullet)$ is subadditive and homogeneous, the set $\mathcal{E}_1 = \{\alpha \in \mathcal{E}\ ;\ N(\alpha) = 0\}$ is a closed convex conne, and we find that $\alpha \mapsto \langle \alpha \rangle$ is a projection of $\mathcal{E}$ onto $\mathcal{E}_1$ (according to [Bou02], $\mathcal{E}_1$ consists of those pseudo-effective classes which are "nef in codimension 1").

(v) Let $\alpha \in \mathcal{E}^\circ$, and assume that $N(\alpha)$ contains linearly dependent components F_j. Then already all currents $T \in \alpha$ should be such that $\mu^\star T = [E] + \beta$ where $F = \mu_\star E$ contains those linearly dependent components. Write $F = \sum \lambda_j F_j$, $\lambda_j > 0$ and assume that

$$\sum_{j \in J} c_j F_j \equiv 0$$

for a certain non trivial linear combination. Then some of the coefficients c_j must be negative (and some other positive). Then E is numerically equivalent to

$$E' \equiv E + t\mu^\star\Big(\sum \lambda_j F_j\Big),$$

and by choosing $t > 0$ appropriate, we obtain an effective divisor E' which has a zero coefficient on one of the components $\mu^\star F_{j_0}$. By replacing E with $\min(E, E')$ via (22.4′), we eliminate the component $\mu^\star F_{j_0}$. This is a contradiction since $N(\alpha)$ was supposed to contain F_{j_0}. □

(22.6) Definition. *For a class $\alpha \in H^{1,1}(X,\mathbb{R})$, we define the* <u>*numerical dimension*</u> *$\nu(\alpha)$ to be $\nu(\alpha) = -\infty$ if α is not pseudo-effective, and*

$$\nu(\alpha) = \max\{p \in \mathbb{N}\,;\ \langle\alpha^p\rangle \neq 0\}, \qquad \nu(\alpha) \in \{0, 1, \ldots, n\}$$

if α is pseudo-effective.

By the results of [DP03], a class is big ($\alpha \in \mathcal{E}^\circ$) if and only if $\nu(\alpha) = n$. Classes of numerical dimension 0 can be described much more precisely, again following Boucksom [Bou02].

(22.7) Theorem. *Let X be a compact Kähler manifold. Then the subset $\mathcal{D}_0$ of irreducible divisors D in X such that $\nu(D) = 0$ is countable, and these divisors are rigid as well as their multiples. If $\alpha \in \mathcal{E}$ is a pseudo-effective class of numerical dimension 0, then α is numerically equivalent to an effective $\mathbb{R}$-divisor $D = \sum_{j\in J} \lambda_j D_j$, for some finite subset $(D_j)_{j\in J} \subset \mathcal{D}_0$ such that the cohomology classes $\{D_j\}$ are linearly independent and some $\lambda_j > 0$. If such a linear combination is of numerical dimension 0, then so is any other linear combination of the same divisors.*

Proof. It is immediate from the definition that a pseudo-effective class is of numerical dimension 0 if and only if $\langle\alpha\rangle = 0$, in other words if $\alpha = N(\alpha)$. Thus $\alpha \equiv \sum \lambda_j D_j$ as described in 22.7, and since $\lambda_j\langle D_j\rangle \leq \langle\alpha\rangle$, the divisors D_j must themselves have numerical dimension 0. There is at most one such divisor D in any given cohomology class in $NS(X) \cap \mathcal{E} \subset H^2(X,\mathbb{Z})$, otherwise two such divisors $D \equiv D'$ would yield a blow-up $\mu : \widetilde{X} \to X$ resolving the intersection, and by taking $\min(\mu^\star D, \mu^\star D')$ via (22.4′), we would find $\mu^\star D \equiv E + \beta$, $\beta \neq 0$, so that $\{D\}$ would not be of numerical dimension 0. This implies that there are at most countably many divisors of numerical dimension 0, and that these divisors are rigid as well as their multiples. □

The above general concept of numerical dimension leads to a very natural formulation of the abundance conjecture for non-minimal (Kähler) varieties.

(22.8) Generalized abundance conjecture. *For an arbitrary compact Kähler manifold X, the Kodaira dimension should be equal to the numerical dimension :*

$$\kappa(X) = \nu(X) := \nu(c_1(K_X)).$$

This appears to be a fairly strong statement. In fact, it is not difficult to show that the generalized abundance conjecture would contain the $C_{n,m}$ conjectures.

(22.9) Remark. Using the Iitaka fibration, it is immediate to see that $\kappa(X) \leq \nu(X)$.

(22.10) Remark. It is known that abundance holds in case $\nu(X) = -\infty$ (if K_X is not pseudo-effective, no multiple of K_X can have sections), or in case $\nu(X) = n$. The latter follows from the solution of the Grauert-Riemenschneider conjecture in the form proven in [Dem85] (see also [DP03]).

In the remaining cases, the most tractable situation is probably the case when $\nu(X) = 0$. In fact Theorem 22.7 then gives $K_X \equiv \sum \lambda_j D_j$ for some effective divisor with numerically independent components, $\nu(D_j) = 0$. It follows that the λ_j are rational and therefore

$$(*)\qquad K_X \sim \sum \lambda_j D_j + F \qquad \text{where } \lambda_j \in \mathbb{Q}^+,\ \nu(D_j) = 0 \text{ and } F \in \mathrm{Pic}^0(X).$$

Especially, if we assume additionally that $q(X) = h^{0,1}(X)$ is zero, then mK_X is linearly equivalent to an integral divisor for some multiple m, and it follows immediately that $\kappa(X) = 0$. The case of a general projective (or compact Kähler) manifold with $\nu(X) = 0$ and positive irregularity $q(X) > 0$ would be interesting to understand.

23. The Orthogonality Estimate

The goal of this section is to show that, in an appropriate sense, approximate Zariski decompositions are almost orthogonal.

(23.1) Theorem. *Let X be a projective manifold, and let $\alpha = \{T\} \in \mathcal{E}^\circ_{\mathrm{NS}}$ be a big class represented by a Kähler current T. Consider an approximate Zariski decomposition*

$$\mu_m^\star T_m = [E_m] + [D_m]$$

Then

$$(D_m^{n-1}\cdot E_m)^2 \le 20\,(C\omega)^n\big(\mathrm{Vol}(\alpha) - D_m^n\big)$$

where $\omega = c_1(H)$ is a Kähler form and $C \ge 0$ is a constant such that $\pm\alpha$ is dominated by $C\omega$ (i.e., $C\omega \pm \alpha$ is nef).

Proof. For every $t \in [0,1]$, we have

$$\mathrm{Vol}(\alpha) = \mathrm{Vol}(E_m + D_m) \ge \mathrm{Vol}(tE_m + D_m).$$

Now, by our choice of C, we can write E_m as a difference of two nef divisors

$$E_m = \mu^\star\alpha - D_m = \mu_m^\star(\alpha + C\omega) - (D_m + C\mu_m^\star\omega).$$

(23.2) Lemma. *For all nef $\mathbb{R}$-divisors A, B we have*

$$\mathrm{Vol}(A - B) \ge A^n - nA^{n-1}\cdot B$$

as soon as the right hand side is positive.

Proof. In case A and B are integral (Cartier) divisors, this is a consequence of the holomorphic Morse inequalities, [Dem01, 8.4]; one can also argue by an elementary estimate of to $H^0(X, mA - B_1 - \ldots - B_m)$ via the Riemann-Roch formula (assuming A and B very ample, $B_1, \ldots, B_m \in |B|$ generic). If A and B are $\mathbb{Q}$-Cartier, we conclude by the homogeneity of the volume. The general case of $\mathbb{R}$-divisors follows by approximation using the upper semi-continuity of the volume [Bou02, 3.1.26]. □

(23.3) Remark. We hope that Lemma 23.2 also holds true on an arbitrary Kähler manifold for arbitrary nef (non necessarily integral) classes. This would follow from a generalization of holomorphic Morse inequalities to non integral classes. However the proof of such a result seems technically much more involved than in the case of integral classes.

(23.4) Lemma. *Let $\beta_1, \ldots, \beta_n$ and $\beta'_1, \ldots, \beta'_n$ be nef classes on a compact Kähler manifold $\widetilde{X}$ such that each difference $\beta'_j - \beta_j$ is pseudo-effective. Then the n-th intersection products satisfy*

$$\beta_1 \cdots \beta_n \leq \beta'_1 \cdots \beta'_n.$$

Proof. We can proceed step by step and replace just one β_j by $\beta' j \equiv \beta_j + T_j$ where T_j is a closed positive $(1,1)$-current and the other classes $\beta'_k = \beta_k$, $k \neq j$ are limits of Kähler forms. The inequality is then obvious. □

End of proof of Theorem 23.1. In order to exploit the lower bound of the volume, we write

$$tE_m + D_m = A - B, \qquad A = D_m + t\mu_m^\star(\alpha + C\omega), \quad B = t(D_m + C\mu_m^\star\omega).$$

By our choice of the constant C, both A and B are nef. Lemma 23.2 and the binomial formula imply

$$\begin{aligned}\mathrm{Vol}(tE_m{+}D_m) &\geq A^n - nA^{n-1}\cdot B \\ &= D_m^n + nt\, D_m^{n-1}\cdot \mu_m^\star(\alpha + C\omega) + \sum_{k=2}^{n} t^k \binom{n}{k} D_m^{n-k}\cdot \mu_m^\star(\alpha+C\omega)^k \\ &\quad - nt\, D_m^{n-1}\cdot (D_m + C\mu_m^\star\omega) \\ &\quad - nt^2 \sum_{k=1}^{n-1} t^{k-1}\binom{n-1}{k} D_m^{n-1-k}\cdot \mu_m^\star(\alpha + C\omega)^k \cdot (D_m + C\mu_m^\star\omega).\end{aligned}$$

Now, we use the obvious inequalities

$$D_m \leq \mu_m^\star(C\omega), \qquad \mu_m^\star(\alpha + C\omega) \leq 2\mu_m^\star(C\omega), \qquad D_m + C\mu_m^\star\omega \leq 2\mu_m^\star(C\omega)$$

in which all members are nef (and where the inequality $\leq$ means that the difference of classes is pseudo-effective). We use Lemma 23.4 to bound the last summation in the estimate of the volume, and in this way we get

$$\mathrm{Vol}(tE_m + D_m) \geq D_m^n + ntD_m^{n-1}\cdot E_m - nt^2 \sum_{k=1}^{n-1} 2^{k+1} t^{k-1}\binom{n-1}{k}(C\omega)^n.$$

We will always take t smaller than $1/10n$ so that the last summation is bounded by $4(n-1)(1+1/5n)^{n-2} < 4ne^{1/5} < 5n$. This implies

$$\mathrm{Vol}(tE_m + D_m) \geq D_m^n + nt\, D_m^{n-1}\cdot E_m - 5n^2t^2(C\omega)^n.$$

Now, the choice $t = \frac{1}{10n}(D_m^{n-1}\cdot E_m)((C\omega)^n)^{-1}$ gives by substituting

$$\frac{1}{20}\frac{(D_m^{n-1}\cdot E_m)^2}{(C\omega)^n} \le \mathrm{Vol}(E_m+D_m) - D_m^n \le \mathrm{Vol}(\alpha) - D_m^n$$

(and we have indeed $t \le \frac{1}{10n}$ by Lemma 23.4), whence Theorem 23.1. Of course, the constant 20 is certainly not optimal. □

(23.5) Corollary. *If $\alpha \in \mathcal{E}_{\mathrm{NS}}$, then the divisorial Zariski decomposition $\alpha = N(\alpha) + \langle\alpha\rangle$ is such that*

$$\langle\alpha^{n-1}\rangle \cdot N(\alpha) = 0.$$

Proof. By replacing α by $\alpha + \delta c_1(H)$, one sees that it is sufficient to consider the case where α is big. Then the orthogonality estimate implies

$$(\mu_m)_\star(D_m^{n-1})\cdot(\mu_m)_\star E_m = D_m^{n-1}\cdot(\mu_m)^\star(\mu_m)_\star E_m \le D_m^{n-1}\cdot E_m \le C(\mathrm{Vol}(\alpha) - D_m^n)^{1/2}.$$

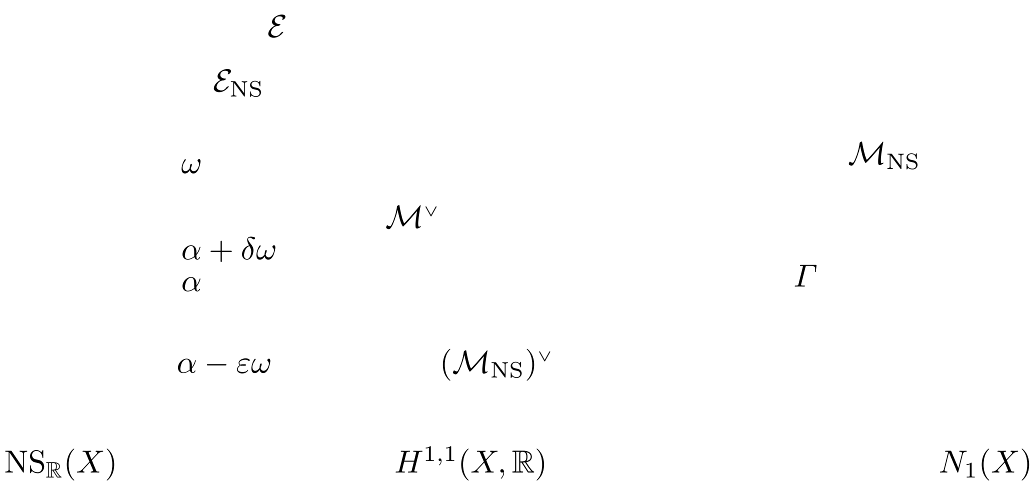
Since $\langle\alpha^{n-1}\rangle = \lim(\mu_m)_\star(D_m^{n-1})$, $N(\alpha) = \lim(\mu_m)_\star E_m$ and $\lim D_m^n = \mathrm{Vol}(\alpha)$, we get the desired conclusion in the limit. □

24. Proof of the Main Duality Theorem

The proof is reproduced from [BDPP03]. We want to show that $\mathcal{E}_{\mathrm{NS}}$ and $\mathrm{SME}(X)$ are dual (Theorem 20.2). By 19.4 (iii) we have in any case

$$\mathcal{E}_{\mathrm{NS}} \subset (\mathrm{SME}(X))^\vee.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$ on the boundary of $\mathcal{E}_{\mathrm{NS}}$ which is in the interior of $\mathrm{SME}(X)^\vee$.



Let $\omega = c_1(H)$ be an ample class. Since $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$, the class $\alpha + \delta\omega$ is big for every $\delta > 0$, and since $\alpha \in ((\mathrm{SME}(X))^\vee)^\circ$ we still have $\alpha - \varepsilon\omega \in (\mathrm{SME}(X))^\vee$ for $\varepsilon > 0$ small. Therefore

$$\alpha \cdot \Gamma \geq \varepsilon \omega \cdot \Gamma \tag{24.1}$$

for every movable curve Γ. We are going to contradict (24.1). Since $\alpha + \delta\omega$ is big, we have an approximate Zariski decomposition

$$\mu_\delta^\star(\alpha + \delta\omega) = E_\delta + D_\delta.$$

We pick $\Gamma = (\mu_\delta)_\star(D_\delta^{n-1})$. By the Hovanskii-Teissier concavity inequality

$$\omega \cdot \Gamma \geq (\omega^n)^{1/n}(D_\delta^n)^{(n-1)/n}.$$

On the other hand

$$\begin{aligned}\alpha \cdot \Gamma &= \alpha \cdot (\mu_\delta)_\star(D_\delta^{n-1}) \\ &= \mu_\delta^\star\alpha \cdot D_\delta^{n-1} \leq \mu_\delta^\star(\alpha + \delta\omega) \cdot D_\delta^{n-1} \\ &= (E_\delta + D_\delta) \cdot D_\delta^{n-1} = D_\delta^n + D_\delta^{n-1} \cdot E_\delta.\end{aligned}$$

By the orthogonality estimate, we find

$$\begin{aligned}\frac{\alpha \cdot \Gamma}{\omega \cdot \Gamma} &\leq \frac{D_\delta^n + \big(20(C\omega)^n(\mathrm{Vol}(\alpha + \delta\omega) - D_\delta^n)\big)^{1/2}}{(\omega^n)^{1/n}(D_\delta^n)^{(n-1)/n}} \\ &\leq C'(D_\delta^n)^{1/n} + C''\frac{(\mathrm{Vol}(\alpha + \delta\omega) - D_\delta^n)^{1/2}}{(D_\delta^n)^{(n-1)/n}}.\end{aligned}$$

However, since $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$, the class α cannot be big so

$$\lim_{\delta \to 0} D_\delta^n = \mathrm{Vol}(\alpha) = 0.$$

We can also take D_δ to approximate $\mathrm{Vol}(\alpha + \delta\omega)$ in such a way that $(\mathrm{Vol}(\alpha + \delta\omega) - D_\delta^n)^{1/2}$ tends to 0 much faster than D_δ^n. Notice that $D_\delta^n \geq \delta^n\omega^n$, so in fact it is enough to take

$$\mathrm{Vol}(\alpha + \delta\omega) - D_\delta^n \leq \delta^{2n}.$$

This is the desired contradiction by (24.1). □

(24.2) Remark. If holomorphic Morse inequalities were known also in the Kähler case, we would infer by the same proof that "α not pseudo-effective" implies the existence of a blow-up $\mu : \widetilde{X} \to X$ and a Kähler metric $\widetilde{\omega}$ on $\widetilde{X}$ such that $\alpha \cdot \mu_\star(\widetilde{\omega})^{n-1} < 0$. In the special case when $\alpha = K_X$ is not pseudo-effective, we would expect the Kähler manifold X to be covered by rational curves. The main trouble is that characteristic p techniques are no longer available. On the other hand it is tempting to approach the question via techniques of symplectic geometry :

(24.3) Question. *Let (M, ω) be a compact real symplectic manifold. Fix an almost complex structure J compatible with ω, and for this structure, assume that $c_1(M) \cdot \omega^{n-1} > 0$. Does it follow that M is covered by rational J-pseudoholomorphic curves ?*

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(June 18, 2007, printed on October 29, 2017)

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