Pseudo-effective line bundles on compact Kähler manifolds

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Abstract. The goal of this work is to pursue the study of pseudo-effective line bundles and vector bundles. Our first result is a generalization of the Hard Lefschetz theorem for cohomology with values in a pseudo-effective line bundle. The Lefschetz map is shown to be surjective when (and, in general, only when) the pseudo-effective line bundle is twisted by its multiplier ideal sheaf. This result has several geometric applications, e.g. to the study of compact Kähler manifolds with pseudo-effective canonical or anti-canonical line bundles. Another concern is to understand pseudo-effectivity in more algebraic terms. In this direction, we introduce the concept of an "almost" nef line bundle, and mean by this that the degree of the bundle is nonnegative on sufficiently generic curves. It can be shown that pseudo-effective line bundles are almost nef, and our hope is that the converse also holds true. This can be checked in some cases, e.g. for the canonical bundle of a projective 3-fold. From this, we derive some geometric properties of the Albanese map of compact Kähler 3-folds.

Contents

§0. Introduction

A line bundle L on a projective manifold X is *pseudo-effective* if $c_1(L)$ is in the closed cone in $H^{1,1}_{\mathbb{R}}(X)$ generated by the effective divisors. If X is only supposed to be Kähler, this definition is no longer very meaningful, instead we require that $c_1(L)$ is in the closure of the cone generated by the classes of closed positive $(1,1)$-currents. In case X is projective this is equivalent to the previous definition. Pseudo-effective line bundles on Kähler manifolds were first introduced in [De90]. The aim of this paper is to study pseudo-effective line bundles in general (as well as the concept of pseudo-effective vector bundle) and in particular varieties whose canonical or anticanonical bundles are pseudo-effective. Pseudo-effective line bundles can also be characterized in a differential-geometric way: they carry singular Hermitian metrics h, locally of the form $e^{-2\varphi}$ with φ integrable, such that the curvature current

$$\Theta_h(L) = i\partial\overline{\partial}\varphi$$

is positive. In this context the multiplier ideal sheaf $\mathcal{I}(h)$ plays an important role; by definition it is the ideal sheaf of local holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally integrable. If h is a smooth metric (with semi-positive curvature), then $\mathcal{I}(h) = \mathcal{O}_X$, but the converse is not true. Our first main result in § 2 is the following hard Lefschetz theorem

0.1. Theorem. *Let (L, h) be a pseudo-effective line bundle on a compact Kähler manifold (X, ω) of dimension n, let $\Theta_h(L) \geqslant 0$ be its curvature current and $\mathcal{I}(h)$ the associated multiplier ideal sheaf. Then, for every nonnegative integer q, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism*

$$H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \xrightarrow{\omega^q \wedge \bullet} H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h)).$$

The classical hard Lefschetz theorem is the case when L is trivial or unitary flat; then L has a metric h of zero curvature, whence $\mathcal{I}(h) = \mathcal{O}_X$. One might ask whether it is possible to omit the multiplier ideal sheaf when L is nef, i.e. when $c_1(L)$ is contained in the closure of the Kähler cone. This is however not the case, as shown by some example (see § 2.5). Therefore the above Lefschetz theorem provides a tool to distinguish between nef and Hermitian semi-positive line bundles at the cohomological level.

We then give two applications of the hard Lefschetz theorem. The first is concerned with compact Kähler manifolds X whose anticanonical bundle $-K_X$ is pseudo-effective, carrying a singular metric with semi-positive curvature and $\mathcal{I}(h) = \mathcal{O}_X$. Then we show that the Albanese map of X is a surjective submersion. We will come back to this type of problems later. The second application deals with Kähler manifolds with K_X pseudo-effective and should be considered as a contribution to Mori theory on Kähler manifolds. We show that if K_X has a singular metric h whose singularities are not "too bad" (e.g. if $\mathcal{I}(h) = \mathcal{O}_X$)), then either $\chi(X, \mathcal{O}_X) = 0$ (which provides a non-zero holomorphic form of odd degree) or

$$H^0(X, \Omega_X^q \otimes \mathcal{O}_X(mK_X)) \neq 0$$

for a fixed number q and infinitely many m. One might hope that this last condition enforces $\kappa(X) \geq 0$; at least we are able to show that it implies the existence of a non-constant meromorphic function on X. Using results of threefold classification, we are then able to show $\kappa(X) \geq 0$ for a compact Kähler threefold with K_X pseudo-effective having a metric of semi-positive curvature with "mild" singularities. In particular this holds if K_X is Hermitian semi-positive. Of course, the algebraic case ("Abundance Conjecture") is known since a some time by deep results of Mori theory. We also prove, however not as an application of the hard Lefschetz theorem and therefore postponed to § 5, that a compact Kähler threefold (isolated singularities are allowed but X must be $\mathbb{Q}$-factorial) with K_X pseudo-effective but not nef admits a curve C with $K_X \cdot C < 0$. In case X is smooth this implies the existence of a Mori contraction. Of course this is only new in the non-algebraic setting.

We next address (in § 3) the question whether pseudo-effective line bundles can be characterized in more algebraic terms in case the underlying manifold is projective. We say that a line bundle L is *almost nef*, if there is a family $A_i \subset X$, $i \in \mathbb{N}$, of proper algebraic subvarieties such that $L \cdot C \geqslant 0$ for all irreducible curves $C \not\subset \bigcup_i A_i$. The Zariski closure of the union of all curves C with $L \cdot C < 0$ will be called the *non-nef locus* of L. In this setting, pseudo-effective line bundles L turn out to be almost nef, but the converse seems to be a very hard problem (if at all true). The equivalence between pseudo-effectivity and almost nefness is however always true on surfaces; and, by using Mori theory, it is true for $L = K_X$ on every threefold.

In § 4 we study compact Kähler manifolds X with $-K_X$ pseudo-effective, resp. almost nef. First we study morphism $\varphi : X \longrightarrow Y$ and restrict to projective varieties X and Y. In general $-K_Y$ will not be pseudo-effective, resp. almost nef; the reason is that the non-nef locus of $-K_X$ might project onto Y. Ruling this out, we obtain

0.2. Theorem. *Let X and Y be normal projective $\mathbb{Q}$-Gorenstein varieties. Let $\varphi : X \to Y$ be a surjective map with connected fibers.*

(a) *Suppose that X and Y are smooth, that φ is a submersion, that $-K_X$ is pseudo-effective and that the zero locus of the multiplier ideal of a minimal metric on $-K_X$ does not project onto Y. Then $-K_Y$ is pseudo-effective.*

(b) *Let $-K_X$ be almost nef with non-nef locus not projecting onto Y. Then $-K_Y$ is generically nef.*

We say that a ($\mathbb{Q}$-) line bundle L on a normal n-dimensional projective variety X is *generically nef* if

$$L \cdot H_1 \cdots H_{n-1} \geqslant 0$$

for all ample divisors H_i on X. This is a much weaker notion than almost nefness.

0.3. Corollary. *Let X be a normal projective $\mathbb{Q}$-Gorenstein variety. Assume $-K_X$ almost nef with non-nef locus B.*

(a) *If $\varphi : X \to Y$ is a surjective morphism to a normal projective $\mathbb{Q}$-Gorenstein variety Y with $\varphi(B) \neq Y$, then $\kappa(Y) \leqslant 0$.*

(b) *The Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is surjective, if $\alpha(B) \neq \alpha(X)$.*

We shall call $-K_X$ *properly pseudo-effective* (*properly almost nef*) with respect to α, if it is pseudo-effective (almost nef), with $\alpha(B) \neq \alpha(X)$. Concerning the fundamental group, we show

0.4. Theorem. *Let X be a terminal 3-fold with $-K_X$ properly pseudo-effective (properly almost nef) with respect to the Albanese map. Then $\pi_1(X)$ is almost Abelian.*

If $-K_X$ is Hermitian semi-positive, then $\pi_1(X)$ is almost Abelian by [DPS96b] (this is true in any dimension); in case $-K_X$ nef, we know by (Paun [Pau96]) that $\pi_1(X)$ has polynomial growth.

If X is merely supposed to be Kähler, we have to restrict to the case when $-K_X$ is nef. Here, using the algebraic case settled in [PS97], our main result is

0.5. Theorem. *Let X be a compact Kähler 3-fold with $-K_X$ nef. Then the Albanese map is a surjective submersion. Moreover $\pi_1(X)$ is almost Abelian.*

The last section is concerned with pseudo-effective and almost nef vector bundles. Given a projective manifold X and a vector bundle E over X, we say that E is *pseudo-effective*, if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective and the union of all curves C with $\mathcal{O}(1)\cdot C < 0$ (i.e. the non-nef locus of the almost nef line bundle $\mathcal{O}(1)$) is contained in a union of subvarieties which does not project onto X. The definition of almost nefness is analogous to the rank 1 case. Basic properties of pseudo-effective and almost nef bundles are collected in

0.6. Theorem. *Let X be a projective manifold and E a vector bundle on X.*

(a) *If E is pseudo-effective (resp. almost nef) and Γ^a is any tensor representation, then $\Gamma^a E$ is again pseudo-effective (almost nef). In particular $S^m E$ and $\Lambda^q E$ are pseudo-effective (almost nef).*

(b) *If E is almost nef and if $s \in H^0(E^*)$ is a non-zero section, then s has no zeroes at all.*

(c) *If either E is pseudo-effective or almost nef with non-nef locus S having codimension at least 2, and if $\det E^*$ is almost nef, then E is numerically flat i.e. both E and E^* are nef, and then E has a filtration by Hermitian flat bundles.*

We then discuss projective manifolds X with pseudo-effective or almost nef tangent bundles T_X. Important examples are provided by almost homogeneous spaces, i.e. the automorphism group acts with an open orbit. It is easily seen that necessarily $\kappa(X) \le 0$. Moreover if $\kappa(X) = 0$, then $K_X \equiv 0$. Now in case $K_X \equiv 0$, and if T_X is pseudo-effective or if T_X is almost nef with non-nef locus of codimension at least 2, then X is Abelian after possibly finite étale cover. As a consequence, a Calabi-Yau manifold can never have a pseudo-effective tangent or cotangent bundle and the union of the curves C with $T_X|C$ is not nef (resp. $\Omega^1_X|C$ is not nef) is not contained in a countable union of analytic sets of codimension at least 2.

In general the Albanese map of X are surjective submersion and we give a precise description of the Albanese map in case $\dim X = 3$. In case neither X nor any finite étale cover of X has a holomorphic 1-form, X is expected to be simply connected. This is true if X is almost homogeneous, if $\dim X \le 3$ or if T_X is nef ([DPS94]).

The research on this paper started in fall 1996 with the important participation of Michael Schneider. After his tragic death in august 1997, the paper was finished – after some delay – by the two first-named authors who therefore carry full scientific responsibility. We would like to thank the referee for his very careful reading of the manuscript.

§1. Nef and pseudo-effective line bundles

We first recall (in the Kähler context) the basic concepts of numerical effectivity and pseudo-effectivity. The proofs as well as more details can be found in [De90, De92, DPS94, DPS96a]. Given a holomorphic line bundle on a complex manifold X and a Hermitian metric h on L, we denote by $\Theta_h(L) = iD_h^2$ the curvature of the Chern connection D_h associated with h. This is a real $(1,1)$-form, which can be expressed as $\Theta_h(L) = -i\partial\overline{\partial}\log h$ in coordinates. The first Chern class of L is represented by $\{\frac{1}{2\pi}\Theta_h(L)\}$ in $H^{1,1}(X) \subset H^2(X,\mathbb{R})$.

1.1. Definition. *Let X be a compact Kähler manifold. A line bundle L on X is said to be*

a) *pseudo-effective if $c_1(L)$ is in the closed cone of $H^{1,1}_{\mathbb{R}}(X)$ generated by classes of d-closed positive $(1,1)$-currents,*

b) *nef (numerically effective) if $c_1(L)$ is in the closure of the Kähler cone, i.e. the closed cone generated by smooth non-negative d-closed $(1,1)$-forms.*

It is clear from the above definition that every nef line bundle is pseudo-effective (but the converse is in general *not true*). The names of these concepts stem from the following "more concrete" characterization in case X is projective.

1.2. Proposition. *Let X be a projective manifold and L a line bundle on X. Then*

a) *L is pseudo-effective if and only if $c_1(L)$ is in the closure $\bar{K}_{\mathrm{eff}}(X)$ of the cone generated by the effective divisors (modulo numerical equivalence) on X.*

b) *L is nef if and only if the degree $L \cdot C$ is non-negative for every effective curve $C \subset X$, or equivalently, if $c_1(L)$ is in the closure $K_{\mathrm{nef}}(X) = \bar{K}_{\mathrm{ample}}$ of the cone of ample divisors.*

Assume that L is pseudo-effective, and let T be a closed positive $(1,1)$-current such that $c_1(L) = [T]$. Choose a smooth Hermitian metric h_∞ on L. Let $\alpha = \frac{1}{2\pi}\Theta_{h_\infty}(L)$ denote its curvature. Since $\{T\} = \{\alpha\}$, we can write $T = \alpha + \frac{i}{\pi}\partial\overline{\partial}\psi$ for some locally integrable function ψ. Then $h = h_\infty \exp(-2\psi)$ is a singular metric on L, and

$$\frac{1}{2\pi}\Theta_h(L) = \alpha + \frac{i}{\pi}\partial\overline{\partial}\psi = T.$$

Hence, L is pseudo-effective if and only if there exists a singular Hermitian metric h on L such that its curvature current $\Theta_h(L) = -i\partial\overline{\partial}\log h$ is positive.

When X is projective, Hörmander's L^2 estimates show that L is pseudo-effective if and only if there exists an ample divisor A such that

$$H^0(X, \mathcal{O}_X(mL + A)) \neq 0$$

for $m \gg 0$ (see [De90]). Whence 1.2 a). We now come to the very important concept of multiplier ideal sheaf.

1.3. Multiplier ideal sheaves. *Let φ be a psh (plurisubharmonic) function on an open subset $\Omega \subset \mathbb{C}^n$. To φ is associated the ideal subsheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$ of germs of holomorphic functions $f \in \mathcal{O}_{\Omega,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x.*

A basic result of Nadel [Nad89] shows that the sheaf $\mathcal{I}(\varphi)$ is a coherent sheaf of ideals over Ω, generated by its global L^2 sections over Ω, provided Ω is e.g. bounded and pseudoconvex (this comes from standard L^2 estimates combined with the Krull lemma). If (L,h) is a pseudo-effective line bundle with $\Theta_h(L) \geqslant 0$, then the local weight functions φ of h are plurisubharmonic and we simply denote $\mathcal{I}(h) := \mathcal{I}(\varphi)$.

1.4. Definition. *Let L be a pseudo-effective line bundle on a compact complex manifold X. Consider two Hermitian metrics h_1, h_2 on L with curvature $\Theta_{h_j}(L) \geqslant 0$ in the sense of currents.*

a) *We write $h_1 \preccurlyeq h_2$, and say that h_1 is less singular than h_2, if there exists a constant $C > 0$ such that $h_1 \leqslant Ch_2$.*

b) *We write $h_1 \sim h_2$, and say that h_1, h_2 are singularity equivalent, if there exists a constant $C > 0$ such that $C^{-1}h_2 \leqslant h_1 \leqslant Ch_2$.*

Of course $h_1 \preccurlyeq h_2$ if and only if the local associated weights in suitable trivializations satisfy $\varphi_2 \leqslant \varphi_1 + C$. This implies in particular that the lelong numbers satisfy $\nu(\varphi_1, x) \leqslant \nu(\varphi_2, x)$ at every point. The above definition is motivated by the following observation.

1.5. Theorem. *For every pseudo-effective line bundle L over a compact complex manifold X, there exists up to equivalence of singularities a unique class of Hermitian metrics h with minimal singularities such that $\Theta_h(L) \geqslant 0$.*

Proof. The proof is almost trivial. We fix once for all a smooth metric h_∞ (whose curvature is of random sign and signature), and we write singular metrics of L under the form $h = h_\infty e^{-2\psi}$. The condition $\Theta_h(L) \geqslant 0$ is equivalent to $\frac{i}{\pi}\partial\overline{\partial}\psi \geqslant -u$ where $u = \Theta_{h_\infty}(L)$. This condition implies that ψ is plurisubharmonic up to the addition of the weight φ_∞ of h_∞, and therefore locally bounded from above. Since we are concerned with metrics only up to equivalence of singularities, it is always possible to adjust ψ by a constant in such a way that $\sup_X \psi = 0$. We now set

$$h_{\min} = h_\infty e^{2\psi_{\min}}, \qquad \psi_{\min}(x) = \sup_\psi \psi(x)$$

where the supremum is extended to all functions ψ such that $\sup_X \psi = 0$ and $\frac{i}{\pi}\partial\overline{\partial}\psi \geqslant -u$. By standard results on plurisubharmonic functions (see Lelong [Lel69]), $\psi_{\min}$ still satisfies $\frac{i}{\pi}\partial\overline{\partial}\psi_{\min} \geqslant -u$ (i.e. the weight $\varphi_\infty + \psi_{\min}$ of $h_{\min}$ is plurisubharmonic), and $h_{\min}$ is obviously the metric with minimal singularities that we were looking for. □

Now, given a section $\sigma \in H^0(X, mL)$, the expression $h(\xi) = |\xi^m/\sigma(x)|^{2/m}$ defines a singular metric on L, which therefore necessarily has at least as much singularity as $h_{\min}$ as, i.e. $\frac{1}{m}\log|\sigma|^2 \leqslant \varphi_{\min} + C$ locally. In particular, $|\sigma|^2 e^{-2m\varphi_{\min}}$ is locally bounded, hence $\sigma \in H^0(X, mL \otimes \mathcal{I}(h_{\min}^{\otimes m}))$. For all $m > 0$, we therefore get an isomorphism

$$H^0(X, mL \otimes \mathcal{I}(h_{\min}^{\otimes m})) \xrightarrow{\ \simeq\ } H^0(X, mL).$$

By the well-known properties of Lelong numbers (see Skoda [Sk72]), the union of all zero varieties of the ideals $\mathcal{I}(h_{\min}^{\otimes m})$ is equal to the Lelong sublevel set

$$E_+(h_{\min}) = \big\{x \in X\,;\ \nu(\varphi_{\min}, x) > 0\big\}. \tag{1.6}$$

We will call this set the *virtual base locus* of L. It is always contained in the "algebraic" base locus

$$B_{\|L\|} = \bigcap_{m>0} B_{|mL|}, \qquad B_{|mL|} = \bigcap_{\sigma\in H^0(X,mL)} \sigma^{-1}(0),$$

but there may be a strict inclusion. This is the case for instance if $L \in \mathrm{Pic}^0(X)$ is such that all positive multiples mL have no nonzero sections; in that case $E_+(h_{\min}) = \emptyset$ but $\bigcap_{m>0} B_{|mL|} = X$. Another general situation where $E_+(h_{\min})$ and $B_{\|L\|}$ can differ is given by the following result.

1.7. Proposition. *Let L be a big nef line bundle. Then $h_{\min}$ has zero Lelong numbers everywhere, i.e. $E_+(h_{\min}) = \emptyset$.*

Proof. Recall that L is big if its Kodaira-Iitaka dimension $\kappa(L)$ is equal to $n = \dim X$. In that case, it is well known that one can write $m_0L = A + E$ with A ample and E effective, for m_0 sufficiently large. Then $mL = ((m - m_0)L + A) + E$ is the sum of an ample divisor $A_m = (m - m_0)L + A$ plus a (fixed) effective divisor, so that there is a Hermitian metric h_m on L for which $\Theta_{h_m}(L) = \frac{1}{m}\Theta(A_m) + \frac{1}{m}[E]$, with a suitable smooth positive form $\Theta(A_m)$. This shows that the Lelong numbers of the weight of h_m are $O(1/m)$, hence in the limit those of $h_{\min}$ are zero. □

If h is a singular Hermitian metric such that $\Theta_h(L) \geqslant 0$ and

$$H^0(X, mL \otimes \mathcal{I}(h^{\otimes m})) \simeq H^0(X, mL) \qquad \text{for all } m \geqslant 0, \tag{1.8}$$

we say that h is an *analytic Zariski decomposition* of L. We have just seen that such a decomposition always exists and that $h = h_{\min}$ is a solution. The concept of analytic Zariski decomposition is motivated by its algebraic counterpart (the existence of which generally fails): one says that L admits an *algebraic Zariski decomposition* if there exists a modification $\mu : \tilde{X} \to X$ and an integer m_0 with $m_0\tilde{L} \simeq \mathcal{O}(E + D)$, where $\tilde{L} = \mu^\star L$, E is an effective divisor and D a nef divisor on $\tilde{X}$ such that

$$H^0(\tilde{X}, kD) = H^0(\tilde{X}, k(D + E)) \simeq H^0(X, km_0L) \qquad \text{for all } k \geqslant 0. \tag{1.9}$$

If $\mathcal{O}(*D)$ is generated by sections, there is a smooth metric with semi-positive curvature on $\mathcal{O}(D)$, and this metric induces a singular Hermitian metric $\tilde{h}$ on $\tilde{L} = \mu^\star(L)$ of curvature current $\frac{1}{m_0}(\Theta(\mathcal{O}(D)) + [E])$. Its poles are defined by the effective $\mathbb{Q}$-divisor $\frac{1}{m_0}E$. For this metric, we of course have $\mathcal{I}(\tilde{h}^{\otimes km_0}) = \mathcal{O}(-kE)$, hence assumption (1.9) can be rewritten

$$H^0\big(\tilde{X}, km_0\tilde{L} \otimes \mathcal{I}(\tilde{h}^{\otimes km_0})\big) = H^0(\tilde{X}, km_0\tilde{L}) \qquad \text{for all } k \geqslant 0. \tag{1.10}$$

When we take the direct image, we find a Hermitian metric h on L with curvature current $\Theta_h(L) = \mu_\star\Theta_{\tilde{h}}(\tilde{L}) \geqslant 0$ and

$$\mathcal{O}_X \supset \mathcal{I}(h^{\otimes km_0}) = \mu_\star\big(K_{\tilde{X}/X} \otimes \mathcal{I}(\tilde{h}^{\otimes km_0})\big) \supset \mu_\star\big(\mathcal{I}(\tilde{h}^{\otimes km_0})\big), \tag{1.11}$$

thus (1.8) holds true at least when m is a multiple of m_0. □

§2. Hard Lefschetz theorem with multiplier ideal sheaves

§2.1. Main statements

The goal of this section is to prove the following surjectivity theorem, which can be seen as an extension of the hard Lefschetz theorem.

2.1.1. Theorem. *Let (L, h) be a pseudo-effective line bundle on a compact Kähler manifold (X, ω) of dimension n, let $\Theta_h(L) \geqslant 0$ be its curvature current and $\mathcal{I}(h)$ the associated multiplier ideal sheaf. Then, for every nonnegative integer q, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism*

$$\Phi^q_{\omega,h} : H^0(X, \Omega^{n-q}_X \otimes L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h)).$$

The special case when L is nef is due to Takegoshi [Ta97]. An even more special case is when L is semi-positive, i.e. possesses a smooth metric with semi-positive curvature. In that case the multiplier ideal sheaf $\mathcal{I}(h)$ coincides with $\mathcal{O}_X$ and we get the following consequence already observed by Mourougane [Mou99].

2.1.2. Corollary. *Let (L, h) be a semi-positive line bundle on a compact Kähler manifold (X, ω) of dimension n. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism*

$$\Phi^q_{\omega} : H^0(X, \Omega^{n-q}_X \otimes L) \longrightarrow H^q(X, \Omega^n_X \otimes L).$$

The proof of Theorem 2.1.1 is based on the Bochner formula, combined with a use of harmonic forms with values in the Hermitian line bundle (L, h). The method can be applied only after h has been made smooth at least in the complement of an analytic set. However, we have to accept singularities even in the regularized metrics because only a very small incompressible loss of positivity is acceptable in the Bochner estimate (by the results of [De92], singularities can be removed, but only at the expense of a fixed, non zero, loss of positivity). Also, we need the multiplier ideal sheaves to be preserved by the smoothing process. This is possible thanks to a suitable "equisingular" regularization process.

§2.2. Equisingular approximations of quasi plurisubharmonic functions

A quasi-plurisubharmonic (quasi-psh) function is by definition a function φ which is locally equal to the sum of a psh function and of a smooth function, or equivalently, a locally integrable function φ such that $i\partial\overline{\partial}\varphi$ is locally bounded below by $-C\omega$ where ω is a Hermitian metric and C a constant. We say that φ has logarithmic poles if φ is locally bounded outside an analytic set A and has singularities of the form

$$\varphi(z) = c \log \sum_k |g_k|^2 + O(1)$$

with $c > 0$ and g_k holomorphic, on a neighborhood of every point of A. Our goal is to show the following

2.2.1. Theorem. *Let $T = \alpha + i\partial\overline{\partial}\varphi$ be a closed $(1,1)$-current on a compact Hermitian manifold (X,ω), where α is a smooth closed $(1,1)$-form and φ a quasi-psh function. Let γ be a continuous real $(1,1)$-form such that $T \geqslant \gamma$. Then one can write $\varphi = \lim_{\nu\to+\infty} \varphi_\nu$ where*

a) *φ_ν is smooth in the complement $X \smallsetminus Z_\nu$ of an analytic set $Z_\nu \subset X$;*

b) *(φ_ν) is a decreasing sequence, and $Z_\nu \subset Z_{\nu+1}$ for all ν;*

c) *$\int_X (e^{-2\varphi} - e^{-2\varphi_\nu})dV_\omega$ is finite for every ν and converges to 0 as $\nu \to +\infty$;*

d) *$\mathcal{I}(\varphi_\nu) = \mathcal{I}(\varphi)$ for all ν ("equisingularity");*

e) *$T_\nu = \alpha + i\partial\overline{\partial}\varphi_\nu$ satisfies $T_\nu \geqslant \gamma - \varepsilon_\nu\omega$, where $\lim_{\nu\to+\infty} \varepsilon_\nu = 0$.*

2.2.2. Remark. It would be interesting to know whether the φ_ν can be taken to have logarithmic poles along Z_ν. Unfortunately, the proof given below destroys this property in the last step. Getting it to hold true seems to be more or less equivalent to proving the semi-continuity property

$$\lim_{\varepsilon\to 0_+} \mathcal{I}((1+\varepsilon)\varphi) = \mathcal{I}(\varphi).$$

Actually, this can be checked in dimensions 1 and 2, but is unknown in higher dimensions (and probably quite hard to establish).

Proof of Theorem 2.2.1. Clearly, by replacing T with $T - \alpha$ and γ with $\gamma - \alpha$, we may assume that $\alpha = 0$ and $T = i\partial\overline{\partial}\varphi \geqslant \gamma$. We divide the proof in four steps.

Step 1. Approximation by quasi-psh functions with logarithmic poles.

By [De92], there is a decreasing sequence (ψ_ν) of quasi-psh functions with logarithmic poles such that $\varphi = \lim \psi_\nu$ and $i\partial\overline{\partial}\psi_\nu \geqslant \gamma - \varepsilon_\nu\omega$. We need a little bit more information on those functions, hence we first recall the main techniques used for the construction of (ψ_ν). For $\varepsilon > 0$ given, fix a covering of X by open balls $B_j = \{|z^{(j)}| < r_j\}$ with coordinates $z^{(j)} = (z_1^{(j)}, \ldots, z_n^{(j)})$, such that

$$0 \leqslant \gamma + c_j\, i\partial\overline{\partial}|z^{(j)}|^2 \leqslant \varepsilon\omega \qquad \text{on} \quad B_j, \tag{2.2.3}$$

for some real number c_j. This is possible by selecting coordinates in which γ is diagonalized at the center of the ball, and by taking the radii $r_j > 0$ small enough (thanks to the fact that γ is continuous). We may assume that these coordinates come from a finite sample of coordinates patches covering X, on which we perform suitable linear coordinate changes (by invertible matrices lying in some compact subset of the complex linear group). By taking additional balls, we may also assume that $X = \bigcup B''_j$ where

$$B''_j \Subset B'_j \Subset B_j$$

are concentric balls $B'_j = \{|z^{(j)}| < r'_j = r_j/2\}$, $B''_j = \{|z^{(j)}| < r''_j = r_j/4\}$. We define

$$\psi_{\varepsilon,\nu,j} = \frac{1}{2\nu} \log \sum_{k\in\mathbb{N}} |f_{\nu,j,k}|^2 - c_j|z^{(j)}|^2 \qquad \text{on} \quad B_j, \tag{2.2.4}$$

where $(f_{\nu,j,k})_{k\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_{\nu,j}$ of holomorphic functions on B_j with finite L^2 norm

$$\|u\|^2 = \int_{B_j} |u|^2 e^{-2\nu(\varphi + c_j|z^{(j)}|^2)} d\lambda(z^{(j)}).$$

(The dependence of $\psi_{\varepsilon,\nu,j}$ on ε is through the choice of the open covering (B_j)). Observe that the choice of c_j in (2.2.3) guarantees that $\varphi + c_j|z^{(j)}|^2$ is plurisubharmonic on B_j, and notice also that

$$\text{(2.2.5)} \qquad \sum_{k\in\mathbb{N}} |f_{\nu,j,k}(z)|^2 = \sup_{f\in\mathcal{H}_{\nu,j},\,\|f\|\leqslant 1} |f(z)|^2$$

is the square of the norm of the continuous linear form $\mathcal{H}_{\nu,j} \to \mathbb{C}$, $f \mapsto f(z)$. We claim that there exist constants C_i, $i = 1, 2, \ldots$ depending only on X and γ (thus independent of ε and ν), such that the following uniform estimates hold:

$$\text{(2.2.6)} \qquad i\partial\overline{\partial}\psi_{\varepsilon,\nu,j} \geqslant -c_j\, i\partial\overline{\partial}|z^{(j)}|^2 \geqslant \gamma - \varepsilon\omega \qquad \text{on } B'_j \ \ (B'_j \Subset B_j),$$

$$\text{(2.2.7)} \qquad \varphi(z) \leqslant \psi_{\varepsilon,\nu,j}(z) \leqslant \sup_{|\zeta - z|\leqslant r} \varphi(\zeta) + \frac{n}{\nu}\log\frac{C_1}{r} + C_2 r^2 \qquad \forall z \in B'_j,\ \ r < r_j - r'_j,$$

$$\text{(2.2.8)} \qquad |\psi_{\varepsilon,\nu,j} - \psi_{\varepsilon,\nu,k}| \leqslant \frac{C_3}{\nu} + C_4\varepsilon\big(\min(r_j, r_k)\big)^2 \qquad \text{on } B'_j \cap B'_k.$$

Actually, the Hessian estimate (2.2.6) is obvious from (2.2.3) and (2.2.4). As in the proof of ([De92], Prop. 3.1), (2.2.7) results from the Ohsawa-Takegoshi L^2 extension theorem (left hand inequality) and from the mean value inequality (right hand inequality). Finally, as in ([De92], Lemma 3.6 and Lemma 4.6), (2.2.8) is a consequence of Hörmander's L^2 estimates. We briefly sketch the idea. Assume that the balls B_j are small enough, so that the coordinates $z^{(j)}$ are still defined on a neighborhood of all balls $\overline{B}_k$ which intersect B_j (these coordinates can be taken to be linear transforms of coordinates belonging to a fixed finite set of coordinate patches covering X, selected once for all). Fix a point $z_0 \in B'_j \cap B'_k$. By (2.2.4) and (2.2.5), we have

$$\psi_{\varepsilon,\nu,j}(z_0) = \frac{1}{\nu}\log|f(z_0)| - c_j|z^{(j)}|^2$$

for some holomorphic function f on B_j with $\|f\| = 1$. We consider the weight function

$$\Phi(z) = 2\nu(\varphi(z) + c_k|z^{(k)}|^2) + 2n\log|z^{(k)} - z_0^{(k)}|,$$

on both B_j and B_k. The trouble is that we a priori have to deal with different weights, hence a comparison of weights is needed. By the Taylor formula applied at z_0, we get

$$\Big|c_k|z^{(k)} - z_0^{(k)}|^2 - c_j|z^{(j)} - z_0^{(j)}|^2\Big| \leqslant C\varepsilon\big(\min(r_j, r_k)\big)^2 \qquad \text{on } B_j \cap B_k$$

[the only nonzero term of degree 2 has type $(1,1)$ and its Hessian satisfies

$$-\varepsilon\omega \leqslant i\partial\overline{\partial}(c_k|z^{(k)}|^2 - c_j|z^{(j)}|^2) \leqslant \varepsilon\omega$$

by (2.2.3); we may suppose $r_j \ll \varepsilon$ so that the terms of order 3 and more are negligible]. By writing $|z^{(j)}|^2 = |z^{(j)} - z_0^{(j)}|^2 + |z_0^{(j)}|^2 + 2\,\mathrm{Re}\langle z^{(j)} - z_0^{(j)}, z_0^{(j)}\rangle$, we obtain

$$\begin{aligned} c_k|z^{(k)}|^2 - c_j|z^{(j)}|^2 &= 2c_k\,\mathrm{Re}\langle z^{(k)} - z_0^{(k)}, z_0^{(k)}\rangle - 2c_j\,\mathrm{Re}\langle z^{(j)} - z_0^{(j)}, z_0^{(j)}\rangle \\ &\qquad + c_k|z_0^{(k)}|^2 - c_j|z_0^{(j)}|^2 \pm C\varepsilon(\min(r_j,r_k))^2. \end{aligned}$$

We use a cut-off function θ equal to 1 in a neighborhood of z_0 and with support in $B_j \cap B_k$; as $z_0 \in B'_j \cap B'_k$, the function θ can be taken to have its derivatives uniformly bounded when z_0 varies. We solve the equation $\overline{\partial} u = \overline{\partial}(\theta f e^{\nu g})$ on B_k, where g is the holomorphic function

$$g(z) = c_k\langle z^{(k)} - z_0^{(k)}, z_0^{(k)}\rangle - c_j\langle z^{(j)} - z_0^{(j)}, z_0^{(j)}\rangle.$$

Thanks to Hörmander's L^2 estimates [Hör66], the L^2 solution for the weight Φ yields a holomorphic function $f' = \theta f e^{\nu g} - u$ on B_k such that $f'(z_0) = f(z_0)$ and

$$\begin{aligned} &\int_{B_k} |f'|^2 e^{-2\nu(\varphi + c_k|z^{(k)}|^2)} d\lambda(z^{(k)}) \leqslant C' \int_{B_j\cap B_k} |f|^2|e^{\nu g}|^2 e^{-2\nu(\varphi + c_k|z^{(k)}|^2)} d\lambda(z^{(k)}) \\ &\leqslant C' \exp\Big(2\nu\big(c_k|z_0^{(k)}|^2 - c_j|z_0^{(j)}|^2 + C\varepsilon(\min(r_j,r_k))^2\big)\Big) \int_{B_j} |f|^2 e^{-2\nu(\varphi + c_j|z^{(j)}|^2)} d\lambda(z^{(j)}). \end{aligned}$$

Let us take the supremum of $\frac{1}{\nu}\log|f(z_0)| = \frac{1}{\nu}\log|f'(z_0)|$ over all f with $\|f\| \leqslant 1$. By the definition of $\psi_{\varepsilon,\nu,k}$ ((2.2.4) and (2.2.5)) and the bound on $\|f'\|$, we find

$$\psi_{\varepsilon,\nu,k}(z_0) \leqslant \psi_{\nu,j}(z_0) + \frac{\log C'}{2\nu} + C\varepsilon(\min(r_j,r_k))^2,$$

whence (2.2.8) by symmetry. Assume that ν is so large that $C_3/\nu < C_4\varepsilon(\inf_j r_j)^2$. We "glue" all functions $\psi_{\varepsilon,\nu,j}$ into a function $\psi_{\varepsilon,\nu}$ globally defined on X, and for this we set

$$\psi_{\varepsilon,\nu}(z) = \sup_{j,\ B'_j \ni z} \Big(\psi_{\varepsilon,\nu,j}(z) + 12\,C_4\varepsilon(r'^2_j - |z^{(j)}|^2)\Big) \qquad \text{on } X.$$

Every point of X belongs to some ball B''_k, and for such a point we get

$$12\,C_4\varepsilon(r'^2_k - |z^{(k)}|^2) \geqslant 12\,C_4\varepsilon(r'^2_k - r''^2_k) > 2C_4 r_k^2 > \frac{C_3}{\nu} + C_4\varepsilon(\min(r_j,r_k))^2.$$

This, together with (2.2.8), implies that in $\psi_{\varepsilon,\nu}(z)$ the supremum is never reached for indices j such that $z \in \partial B'_j$, hence $\psi_{\varepsilon,\nu}$ is well defined and continuous, and by standard properties of upper envelopes of (quasi)-plurisubharmonic functions we get

$$i\partial\overline{\partial}\psi_{\varepsilon,\nu} \geqslant \gamma - C_5\varepsilon\omega \tag{2.2.9}$$

for $\nu \geqslant \nu_0(\varepsilon)$ large enough. By inequality (2.2.7) applied with $r = e^{-\sqrt{\nu}}$, we see that $\lim_{\nu\to+\infty} \psi_{\varepsilon,\nu}(z) = \varphi(z)$. At this point, the difficulty is to show that $\psi_{\varepsilon,\nu}$ is decreasing with ν – this may not be true formally, but we will see at Step 3 that this is essentially true. Another difficulty is that we must simultaneously let ε go to 0, forcing us to change the covering as we want the error to get smaller and smaller in (2.2.9).

Step 2. A comparison of integrals.
We claim that

$$I := \int_X \big(e^{-2\varphi} - e^{-2\max(\varphi,\frac{\ell}{\ell-1}\psi_{\nu,\varepsilon})+a}\big)dV_\omega < +\infty \tag{2.2.10}$$

for every $\ell \in \,]1,\nu]$ and $a \in \mathbb{R}$. In fact

$$\begin{aligned} I &\leqslant \int_{\{\varphi<\frac{\ell}{\ell-1}\psi_{\varepsilon,\nu}+a\}} e^{-2\varphi}dV_\omega = \int_{\{\varphi<\frac{\ell}{\ell-1}\psi_{\varepsilon,\nu}\}+a} e^{2(\ell-1)\varphi-2\ell\varphi}dV_\omega \\ &\leqslant e^{2(\ell-1)a}\int_X e^{2\ell(\psi_{\varepsilon,\nu}-\varphi)}dV_\omega \leqslant C\Big(\int_X e^{2\nu(\psi_{\varepsilon,\nu}-\varphi)}dV_\omega\Big)^{\frac{\ell}{\nu}} \end{aligned}$$

by Hölder's inequality. In order to show that these integrals are finite, it is enough, by the definition and properties of the functions $\psi_{\varepsilon,\nu}$ and $\psi_{\varepsilon,\nu,j}$, to prove that

$$\int_{B'_j} e^{2\nu\psi_{\varepsilon,\nu,j}-2\nu\varphi}d\lambda = \int_{B'_j}\Big(\sum_{k=0}^{+\infty}|f_{\nu,j,k}|^2\Big)e^{-2\nu\varphi}d\lambda < +\infty.$$

By the strong Noetherian property of coherent ideal sheaves ([Nar66] or [GR84]), we know that the sequence of ideal sheaves generated by the holomorphic functions $(f_{\nu,j,k}(z)\overline{f_{\nu,j,k}(\overline{w})})_{k\leqslant k_0}$ on $B_j\times B_j$ is locally stationary as k_0 increases, hence independent of k_0 on $B'_j\times B'_j \Subset B_j\times B_j$ for k_0 large enough. As the sum of the series $\sum_k f_{\nu,j,k}(z)\overline{f_{\nu,j,k}(\overline{w})}$ is bounded by

$$\Big(\sum_k |f_{\nu,j,k}(z)|^2\sum_k|f_{\nu,j,k}(\overline{w})|^2\Big)^{1/2}$$

and thus uniformly convergent on every compact subset of $B_j\times B_j$, and as the space of sections of a coherent ideal sheaf is closed under the topology of uniform convergence on compact subsets, we infer from the Noetherian property that the holomorphic function $\sum_{k=0}^{+\infty} f_{\nu,j,k}(z)\overline{f_{\nu,j,k}(\overline{w})}$ is a section of the coherent ideal sheaf generated by $(f_{\nu,j,k}(z)\overline{f_{\nu,j,k}(\overline{w})})_{k\leqslant k_0}$ over $B'_j\times B'_j$, for k_0 large enough. Hence, by restricting to the conjugate diagonal $w=\overline{z}$, we get

$$\sum_{k=0}^{+\infty}|f_{\nu,j,k}(z)|^2 \leqslant C\sum_{k=0}^{k_0}|f_{\nu,j,k}(z)|^2 \qquad \text{on } B'_j.$$

This implies

$$\int_{B'_j}\Big(\sum_{k=0}^{+\infty}|f_{\nu,j,k}|^2\Big)e^{-2\varphi}d\lambda \leqslant C\int_{B'_j}\Big(\sum_{k=0}^{k_0}|f_{\nu,j,k}|^2\Big)e^{-2\varphi}d\lambda = C(k_0+1).$$

Property (2.2.10) is proved.

Step 3. Subadditivity of the approximating sequence $\psi_{\varepsilon,\nu}$.
We want to compare $\psi_{\varepsilon,\nu_1+\nu_2}$ and ψ_{ε,ν_1}, ψ_{ε,ν_2} for every pair of indices ν_1, ν_2, first when the functions are associated with the same covering $X = \bigcup B_j$. Consider a function $f \in \mathcal{H}_{\nu_1+\nu_2,j}$ with

$$\int_{B_j} |f(z)|^2 e^{-2(\nu_1+\nu_2)\varphi_j(z)} d\lambda(z) \leqslant 1, \qquad \varphi_j(z) = \varphi(z) + c_j|z^{(j)}|^2.$$

We may view f as a function $\hat{f}(z,z)$ defined on the diagonal Δ of $B_j \times B_j$. Consider the Hilbert space of holomorphic functions u on $B_j \times B_j$ such that

$$\int_{B_j\times B_j} |u(z,w)|^2 e^{-2\nu_1\varphi_j(z)-2\nu_2\varphi_j(w)} d\lambda(z)d\lambda(w) < +\infty.$$

By the Ohsawa-Takegoshi L^2 extension theorem [OT87], there exists a function $\widetilde{f}(z,w)$ on $B_j \times B_j$ such that $\widetilde{f}(z,z) = f(z)$ and

$$\int_{B_j\times B_j} |\widetilde{f}(z,w)|^2 e^{-2\nu_1\varphi_j(z)-2\nu_2\varphi_j(w)} d\lambda(z)d\lambda(w)$$
$$\leqslant C_7 \int_{B_j} |f(z)|^2 e^{-2(\nu_1+\nu_2)\varphi_j(z)} d\lambda(z) = C_7,$$

where the constant C_7 only depends on the dimension n (it is actually independent of the radius r_j if say $0 < r_j \leqslant 1$). As the Hilbert space under consideration on $B_j \times B_j$ is the completed tensor product $\mathcal{H}_{\nu_1,j} \widehat{\otimes} \mathcal{H}_{\nu_2,j}$, we infer that

$$\widetilde{f}(z,w) = \sum_{k_1,k_2} c_{k_1,k_2} f_{\nu_1,j,k_1}(z) f_{\nu_2,j,k_2}(w)$$

with $\sum_{k_1,k_2} |c_{k_1,k_2}|^2 \leqslant C_7$. By restricting to the diagonal, we obtain

$$|f(z)|^2 = |\widetilde{f}(z,z)|^2 \leqslant \sum_{k_1,k_2} |c_{k_1,k_2}|^2 \sum_{k_1} |f_{\nu_1,j,k_1}(z)|^2 \sum_{k_2} |f_{\nu_2,j,k_2}(z)|^2.$$

From (2.2.3) and (2.2.4), we get

$$\psi_{\varepsilon,\nu_1+\nu_2,j} \leqslant \frac{\log C_7}{\nu_1+\nu_2} + \frac{\nu_1}{\nu_1+\nu_2}\psi_{\varepsilon,\nu_1,j} + \frac{\nu_2}{\nu_1+\nu_2}\psi_{\varepsilon,\nu_2,j},$$

in particular

$$\psi_{\varepsilon,2^\nu,j} \leqslant \psi_{\varepsilon,2^{\nu-1},j} + \frac{C_8}{2^\nu},$$

and we see that $\psi_{\varepsilon,2^\nu} + C_8 2^{-\nu}$ is a decreasing sequence. By Step 2 and Lebesgue's monotone convergence theorem, we infer that for every $\varepsilon, \delta > 0$ and $a \leqslant a_0 \ll 0$ fixed, the integral

$$I_{\varepsilon,\delta,\nu} = \int_X \Big(e^{-2\varphi} - e^{-2\max(\varphi,(1+\delta)(\psi_{2^\nu,\varepsilon}+a))}\Big) dV_\omega$$

converges to 0 as ν tends to $+\infty$ (take $\ell = \frac{1}{\delta} + 1$ and $2^\nu > \ell$ and a_0 such that $\delta \sup_X \varphi + a_0 \leqslant 0$; we do not have monotonicity strictly speaking but need only replace a by $a + C_8 2^{-\nu}$ to get it, thereby slightly enlarging the integral).

Step 4. Selection of a suitable upper envelope.

For the simplicity of notation, we assume here that $\sup_X \varphi = 0$ (possibly after subtracting a constant), hence we can take $a_0 = 0$ in the above. We may even further assume that all our functions $\psi_{\varepsilon,\nu}$ are nonpositive. By Step 3, for each $\delta = \varepsilon = 2^{-k}$, we can select an index $\nu = p(k)$ such that

$$(2.2.11) \qquad I_{2^{-k},2^{-k},p(k)} = \int_X \Big(e^{-2\varphi} - e^{-2\max(\varphi,(1+2^{-k})\psi_{2^{-k},2^{p(k)}})}\Big) dV_\omega \leqslant 2^{-k}$$

By construction, we have an estimate $i\partial\overline{\partial}\psi_{2^{-k},2^{p(k)}} \geqslant \gamma - C_5 2^{-k}\omega$, and the functions $\psi_{2^{-k},2^{p(k)}}$ are quasi-psh with logarithmic poles. Our estimates (especially (2.2.7)) imply that $\lim_{k\to+\infty} \psi_{2^{-k},2^{p(k)}}(z) = \varphi(z)$ as soon as $2^{-p(k)} \log\big(1/\inf_j r_j(k)\big) \to 0$ (notice that the r_j's now depend on $\varepsilon = 2^{-k}$). We set

$$(2.2.12) \qquad \varphi_\nu(z) = \sup_{k \geqslant \nu} (1 + 2^{-k})\psi_{2^{-k},2^{p(k)}}(z).$$

By construction (φ_ν) is a decreasing sequence and satisfies the estimates

$$\varphi_\nu \geqslant \max\big(\varphi, (1 + 2^{-\nu})\psi_{2^{-\nu},2^{p(\nu)}}\big), \qquad i\partial\overline{\partial}\varphi_\nu \geqslant \gamma - C_5 2^{-\nu}\omega.$$

Inequality (2.2.11) implies that

$$\int_X (e^{-2\varphi} - e^{-2\varphi_\nu}) dV_\omega \leqslant \sum_{k=\nu}^{+\infty} 2^{-k} = 2^{1-\nu}.$$

Finally, if Z_ν is the set of poles of $\psi_{2^{-\nu},2^{p(\nu)}}$, then $Z_\nu \subset Z_{\nu+1}$ and φ_ν is continuous on $X \smallsetminus Z_\nu$. The reason is that in a neighborhood of every point $z_0 \in X \smallsetminus Z_\nu$, the term $(1 + 2^{-k})\psi_{2^{-k},2^{p(k)}}$ contributes to φ_ν only when it is larger than $(1 + 2^{-\nu})\psi_{2^{-\nu},2^{p(\nu)}}$. Hence, by the almost-monotonicity, the relevant terms of the sup in (2.2.12) are squeezed between $(1 + 2^{-\nu})\psi_{2^{-\nu},2^{p(\nu)}}$ and $(1 + 2^{-k})(\psi_{2^{-\nu},2^{p(\nu)}} + C_8 2^{-\nu})$, and therefore there is uniform convergence in a neighborhood of z_0. Finally, condition c) implies that

$$\int_U |f|^2 (e^{-2\varphi} - e^{-2\varphi_\nu}) dV_\omega < +\infty$$

for every germ of holomorphic function $f \in \mathcal{O}(U)$ at a point $x \in X$. Therefore both integrals $\int_U |f|^2 e^{-2\varphi} dV_\omega$ and $\int_U |f|^2 e^{-2\varphi_\nu} dV_\omega$ are simultaneously convergent or divergent, i.e. $\mathcal{I}(\varphi) = \mathcal{I}(\varphi_\nu)$. Theorem 2.2.1 is proved, except that φ_ν is possibly just continuous instead of being smooth. This can be arranged by Richberg's regularization theorem [Ri68], at the expense of an arbitrary small loss in the Hessian form. □

2.2.13. Remark. By a very slight variation of the proof, we can strengthen condition c) and obtain that for every $t > 0$

$$\int_X (e^{-2t\varphi} - e^{-2t\varphi_\nu}) dV_\omega$$

is finite for ν large enough and converges to 0 as $\nu \to +\infty$. This implies that the sequence of multiplier ideals $\mathcal{I}(t\varphi_\nu)$ is a stationary decreasing sequence, with $\mathcal{I}(t\varphi_\nu) = \mathcal{I}(t\varphi)$ for ν large.

§2.3. A Bochner type inequality

Let (L, h) be a smooth Hermitian line bundle on a (non necessarily compact) Kähler manifold (Y, ω). We denote by $|\ \ | = |\ \ |_{\omega,h}$ the pointwise Hermitian norm on $\Lambda^{p,q}T_Y^\star \otimes L$ associated with ω and h, and by $\|\ \ \| = \|\ \ \|_{\omega,h}$ the global L^2 norm

$$\|u\|^2 = \int_Y |u|^2 dV_\omega \qquad \text{where} \quad dV_\omega = \frac{\omega^n}{n!}$$

We consider the $\overline{\partial}$ operator acting on (p, q)-forms with values in L, its adjoint $\overline{\partial}_h^\star$ with respect to h and the complex Laplace-Beltrami operator $\overline{\Box}_h = \overline{\partial}\overline{\partial}_h^\star + \overline{\partial}_h^\star\overline{\partial}$. Let v be a smooth $(n-q, 0)$-form with compact support in Y. Then $u = \omega^q \wedge v$ satisfies

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}_h^\star u\|^2 = \|\overline{\partial}v\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j\in J} \lambda_j\Big)|u_{IJ}|^2 \tag{2.3.1}$$

where $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ are the curvature eigenvalues of $\Theta_h(L)$ expressed in an orthonormal frame $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ (at some fixed point $x_0 \in Y$), in such a way that

$$\omega_{x_0} = i \sum_{1\leqslant j\leqslant n} dz_j \wedge d\overline{z}_j, \qquad \Theta_h(L)_{x_0} = i\partial\overline{\partial}\varphi_{x_0} = i \sum_{1\leqslant j\leqslant n} \lambda_j dz_j \wedge d\overline{z}_j.$$

The proof of (2.3.1) proceeds by checking that

$$(\overline{\partial}_\varphi^\star\,\overline{\partial} + \overline{\partial}\,\overline{\partial}_\varphi^\star)(v \wedge \omega^q) - (\overline{\partial}_\varphi^\star\,\overline{\partial}v) \wedge \omega^q = q\, i\partial\overline{\partial}\varphi \wedge \omega^{q-1} \wedge v, \tag{2.3.2}$$

taking the inner product with $u = \omega^q \wedge v$ and integrating by parts in the left hand side. In order to check (2.3.2), we use the identity $\overline{\partial}_\varphi^\star = e^\varphi\overline{\partial}^\star(e^{-\varphi}\bullet) = \overline{\partial}^\star + \nabla^{0,1}\varphi \,\lrcorner\, \bullet$. Let us work in a local trivialization of L such that $\varphi(x_0) = 0$ and $\nabla\varphi(x_0) = 0$. At x_0 we then find

$$(\overline{\partial}_\varphi^\star\,\overline{\partial} + \overline{\partial}\,\overline{\partial}_\varphi^\star)(\omega^q \wedge v) - \omega^q \wedge (\overline{\partial}_\varphi^\star\,\overline{\partial}v) =$$
$$\big[(\overline{\partial}^\star\,\overline{\partial} + \overline{\partial}\,\overline{\partial}^\star)(\omega^q \wedge v) - \omega^q \wedge (\overline{\partial}^\star\,\overline{\partial}v)\big] + \overline{\partial}(\nabla^{0,1}\varphi \,\lrcorner\, (\omega^q \wedge v)).$$

However, the term $[\ldots]$ corresponds to the case of a trivial vector bundle and it is well known in that case that $[\overline{\Box}, \omega^q \wedge \bullet] = 0$, hence $[\ldots] = 0$. On the other hand

$$\nabla^{0,1}\varphi \,\lrcorner\, (\omega^q \wedge v) = q(\nabla^{0,1}\varphi \,\lrcorner\, \omega) \wedge \omega^{q-1} \wedge v = -q\, i\partial\varphi \wedge \omega^{q-1} \wedge v,$$

and so

$$(\overline{\partial}_\varphi^\star\,\overline{\partial} + \overline{\partial}\,\overline{\partial}_\varphi^\star)(\omega^q \wedge v) - \omega^q \wedge (\overline{\partial}_\varphi^\star\,\overline{\partial}v) = q\, i\partial\overline{\partial}\varphi \wedge \omega^{q-1} \wedge v.$$

Our formula is thus proved when v is smooth and compactly supported. In general, we have:

2.3.3. Proposition. *Let (Y,ω) be a* complete *Kähler manifold and (L,h) a smooth Hermitian line bundle such that the curvature possesses a uniform lower bound $\Theta_h(L)\geqslant -C\omega$. For every measurable $(n-q,0)$-form v with L^2 coefficients and values in L such that $u=\omega^q\wedge v$ has differentials $\overline\partial u$, $\overline\partial^\star u$ also in L^2, we have*

$$\|\overline\partial u\|^2+\|\overline\partial_h^\star u\|^2=\|\overline\partial v\|^2+\int_Y\sum_{I,J}\Big(\sum_{j\in J}\lambda_j\Big)|u_{IJ}|^2$$

(*here, all differentials are computed in the sense of distributions*).

Proof. Since (Y,ω) is assumed to be complete, there exists a sequence of smooth forms v_ν with compact support in Y (obtained by truncating v and taking the convolution with a regularizing kernel) such that $v_\nu\to v$ in L^2 and such that $u_\nu=\omega^q\wedge v_\nu$ satisfies $u_\nu\to u$, $\overline\partial u_\nu\to\overline\partial u$, $\overline\partial^\star u_\nu\to\overline\partial^\star u$ in L^2. By the curvature assumption, the final integral in the right hand side of (2.3.1) must be under control (i.e. the integrand becomes nonnegative if we add a term $C\|u\|^2$ on both sides, $C\gg 0$). We thus get the equality by passing to the limit and using Lebesgue's monotone convergence theorem. □

§2.4. Proof of Theorem 2.1.1

To fix the ideas, we first indicate the proof in the much simpler case when (L,h) is Hermitian semipositive, and then treat the general case.

(2.4.1) Special case. (L,h) is (smooth) Hermitian semipositive

Let $\{\beta\}\in H^q(X,\Omega_X^n\otimes L)$ be an arbitrary cohomology class. By standard L^2 Hodge theory, $\{\beta\}$ can be represented by a smooth harmonic $(0,q)$-form β with values in $\Omega_X^n\otimes L$. We can also view β as a (n,q)-form with values in L. The pointwise Lefschetz isomorphism produces a unique $(n-q,0)$-form α such that $\beta=\omega^q\wedge\alpha$. Proposition 2.3.3 then yields

$$\|\overline\partial\alpha\|^2+\int_Y\sum_{I,J}\Big(\sum_{j\in J}\lambda_j\Big)|\alpha_{IJ}|^2=\|\overline\partial\beta\|^2+\|\overline\partial_h^\star\beta\|^2=0,$$

and the curvature eigenvalues λ_j are nonnegative by our assumption. Hence $\overline\partial\alpha=0$ and $\{\alpha\}\in H^0(X,\Omega_X^{n-q}\otimes L)$ is mapped to $\{\beta\}$ by $\Phi^q_{\omega,h}=\omega^q\wedge\bullet$.

(2.4.2) General case.

There are several difficulties. The first difficulty is that the metric h is no longer smooth and we cannot directly represent cohomology classes by harmonic forms. We circumvent this problem by smoothing the metric on an (analytic) Zariski open subset and by avoiding the remaining poles on the complement. However, some careful estimates have to be made in order to take the error terms into account.

Fix $\varepsilon=\varepsilon_\nu$ and let $h_\varepsilon=h_{\varepsilon_\nu}$ be an approximation of h, such that h_ε is smooth on $X\smallsetminus Z_\varepsilon$ (Z_ε being an analytic subset of X), $\Theta_{h_\varepsilon}(L)\geqslant-\varepsilon\omega$, $h_\varepsilon\leqslant h$ and $\mathcal{I}(h_\varepsilon)=\mathcal{I}(h)$. This is possible by Theorem 2.2.1. Now, we can find a family

$$\omega_{\varepsilon,\delta}=\omega+\delta(i\partial\overline\partial\psi_\varepsilon+\omega),\qquad \delta>0$$

of *complete Kähler* metrics on $X \smallsetminus Z_\varepsilon$, where ψ_ε is a quasi-psh function on X with $\psi_\varepsilon = -\infty$ on Z_ε, ψ_ε on $X \smallsetminus Z_\varepsilon$ and $i\partial\overline{\partial}\psi_\varepsilon + \omega \geqslant 0$ (see e.g. [De82], Théorème 1.5). By construction, $\omega_{\varepsilon,\delta} \geqslant \omega$ and $\lim_{\delta\to 0}\omega_{\varepsilon,\delta} = \omega$. We look at the L^2 Dolbeault complex $K^\bullet_{\varepsilon,\delta}$ of $(n,\bullet)$-forms on $X \smallsetminus Z_\varepsilon$, where the L^2 norms are induced by $\omega_{\varepsilon,\delta}$ on differential forms and by h_ε on elements in L. Specifically

$$K^q_{\varepsilon,\delta} = \Big\{u : X\smallsetminus Z_\varepsilon \to \Lambda^{n,q}T^\star_X\otimes L;\int_{X\smallsetminus Z_\varepsilon}(|u|^2_{\Lambda^{n,q}\omega_{\varepsilon,\delta}\otimes h_\varepsilon}+|\overline{\partial}u|^2_{\Lambda^{n,q+1}\omega_{\varepsilon,\delta}\otimes h_\varepsilon})dV_{\omega_{\varepsilon,\delta}}<\infty\Big\}.$$

Let $\mathcal{K}^q_{\varepsilon,\delta}$ be the corresponding sheaf of germs of locally L^2 sections on X (the local L^2 condition should hold on X, not only on $X \smallsetminus Z_\varepsilon$!). Then, for all $\varepsilon > 0$ and $\delta \geqslant 0$, $(\mathcal{K}^q_{\varepsilon,\delta}, \overline{\partial})$ is a resolution of the sheaf $\Omega^n_X \otimes L \otimes \mathcal{I}(h_\varepsilon) = \Omega^n_X \otimes L \otimes \mathcal{I}(h)$. This is because L^2 estimates hold locally on small Stein open sets, and the L^2 condition on $X \smallsetminus Z_\varepsilon$ forces holomorphic sections to extend across Z_ε ([De82], Lemme 6.9).

Let $\{\beta\} \in H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h))$ be a cohomology class represented by a smooth form with values in $\Omega^n_X \otimes L \otimes \mathcal{I}(h)$ (one can use a Čech cocycle and convert it to an element in the C^∞ Dolbeault complex by means of a partition of unity, thanks to the usual De Rham-Weil isomorphism). Then

$$\|\beta\|^2_{\varepsilon,\delta} \leqslant \|\beta\|^2 = \int_X |\beta|^2_{\Lambda^{n,q}\omega\otimes h}dV_\omega < +\infty.$$

The reason is that $|\beta|^2_{\Lambda^{n,q}\omega\otimes h}dV_\omega$ decreases as ω increases. This is just an easy calculation, shown by comparing two metrics ω, ω' which are expressed in diagonal form in suitable coordinates; the norm $|\beta|^2_{\Lambda^{n,q}\omega\otimes h}$ turns out to decrease faster than the volume dV_ω increases; see e.g. [De82], Lemme 3.2; a special case is $q = 0$, then $|\beta|^2_{\Lambda^{n,q}\omega\otimes h}dV_\omega = i^{n^2}\beta\wedge\overline{\beta}$ with the identification $L\otimes\overline{L} \simeq \mathbb{C}$ given by the metric h, hence the integrand is even independent of ω in that case.

By the proof of the De Rham-Weil isomorphism, the map $\alpha \mapsto \{\alpha\}$ from the cocycle space $Z^q(\mathcal{K}^\bullet_{\varepsilon,\delta})$ equipped with its L^2 topology, into $H^q(X, \Omega^n_X\otimes L\otimes\mathcal{I}(h))$ equipped with its finite vector space topology, is continuous. Also, Banach's open mapping theorem implies that the coboundary space $B^q(\mathcal{K}^\bullet_{\varepsilon,\delta})$ is closed in $Z^q(\mathcal{K}^\bullet_{\varepsilon,\delta})$. This is true for all $\delta \geqslant 0$ (the limit case $\delta = 0$ yields the strongest L^2 topology in bidegree (n,q)). Now, β is a $\overline{\partial}$-closed form in the Hilbert space defined by $\omega_{\varepsilon,\delta}$ on $X \smallsetminus Z_\varepsilon$, so there is a $\omega_{\varepsilon,\delta}$-harmonic form $u_{\varepsilon,\delta}$ in the same cohomology class as β, such that

$$\|u_{\varepsilon,\delta}\|_{\varepsilon,\delta} \leqslant \|\beta\|_{\varepsilon,\delta}.$$

2.4.3. Remark. The existence of a harmonic representative holds true only for $\delta > 0$, because we need to have a complete Kähler metric on $X \smallsetminus Z_\varepsilon$. The trick of employing $\omega_{\varepsilon,\delta}$ instead of a fixed metric ω, however, is not needed when Z_ε is (or can be taken to be) empty. This is the case if (L, h) is such that $\mathcal{I}(h) = \mathcal{O}_X$ and L is nef. Indeed, in that case, from the very definition of nefness, it is easy to prove that we can take the φ_ν's to be everywhere smooth in Theorem 2.2.1. However, we will see in § 2.5 that multiplier ideal sheaves are needed even in case L is nef, when $\mathcal{I}(h) \neq \mathcal{O}_X$.

Let $v_{\varepsilon,\delta}$ be the unique $(n-q,0)$-form such that $u_{\varepsilon,\delta} = v_{\varepsilon,\delta} \wedge \omega^q_{\varepsilon,\delta}$ ($v_{\varepsilon,\delta}$ exists by the pointwise Lefschetz isomorphism). Then

$$\|v_{\varepsilon,\delta}\|_{\varepsilon,\delta} = \|u_{\varepsilon,\delta}\|_{\varepsilon,\delta} \leqslant \|\beta\|_{\varepsilon,\delta} \leqslant \|\beta\|.$$

As $\sum_{j\in J} \lambda_j \geqslant -q\varepsilon$ by the assumption on $\Theta_{h_\varepsilon}(L)$, the Bochner formula yields

$$\|\overline{\partial} v_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} \leqslant q\varepsilon \|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} \leqslant q\varepsilon\|\beta\|^2.$$

These uniform bounds imply that there are subsequences $u_{\varepsilon,\delta_\nu}$ and $v_{\varepsilon,\delta_\nu}$ with $\delta_\nu \to 0$, possessing weak-L^2 limits $u_\varepsilon = \lim_{\nu\to+\infty} u_{\varepsilon,\delta_\nu}$ and $v_\varepsilon = \lim_{\nu\to+\infty} v_{\varepsilon,\delta_\nu}$. The limit $u_\varepsilon = \lim_{\nu\to+\infty} u_{\varepsilon,\delta_\nu}$ is with respect to $L^2(\omega) = L^2(\omega_{\varepsilon,0})$. To check this, notice that in bidegree $(n-q,0)$, the space $L^2(\omega)$ has the weakest topology of all spaces $L^2(\omega_{\varepsilon,\delta})$; indeed, an easy calculation as in ([De82], Lemme 3.2) yields

$$|f|^2_{\Lambda^{n-q,0}\omega\otimes h} dV_\omega \leqslant |f|^2_{\Lambda^{n-q,0}\omega_{\varepsilon,\delta}\otimes h} dV_{\omega_{\varepsilon,\delta}} \qquad \text{if } f \text{ is of type } (n-q,0).$$

On the other hand, the limit $v_\varepsilon = \lim_{\nu\to+\infty} v_{\varepsilon,\delta_\nu}$ takes place in all spaces $L^2(\omega_{\varepsilon,\delta})$, $\delta > 0$, since the topology gets stronger and stronger as $\delta \downarrow 0$ [possibly not in $L^2(\omega)$, though, because in bidegree (n,q) the topology of $L^2(\omega)$ might be strictly stronger than that of all spaces $L^2(\omega_{\varepsilon,\delta})$]. The above estimates yield

$$\begin{aligned}
&\|v_\varepsilon\|^2_{\varepsilon,0} = \int_X |v_\varepsilon|^2_{\Lambda^{n-q,0}\omega\otimes h_\varepsilon} dV_\omega \leqslant \|\beta\|^2,\\
&\|\overline{\partial} v_\varepsilon\|^2_{\varepsilon,0} \leqslant q\varepsilon\|\beta\|^2_{\varepsilon,0},\\
&u_\varepsilon = \omega^q \wedge v_\varepsilon \equiv \beta \qquad \text{in } \ H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h_\varepsilon)).
\end{aligned}$$

Again, by arguing in a given Hilbert space $L^2(h_{\varepsilon_0})$, we find L^2 convergent subsequences $u_\varepsilon \to u$, $v_\varepsilon \to v$ as $\varepsilon \to 0$, and in this way get $\overline{\partial} v = 0$ and

$$\begin{aligned}
&\|v\|^2 \leqslant \|\beta\|^2,\\
&u = \omega^q \wedge v \equiv \beta \qquad \text{in } \ H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h)).
\end{aligned}$$

Theorem 2.1.1 is proved. Notice that the equisingularity property $\mathcal{I}(h_\varepsilon) = \mathcal{I}(h)$ is crucial in the above proof, otherwise we could not infer that $u \equiv \beta$ from the fact that $u_\varepsilon \equiv \beta$. This is true only because all cohomology classes $\{u_\varepsilon\}$ lie in the same fixed cohomology group $H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h))$, whose topology is induced by the topology of $L^2(\omega)$ on $\overline{\partial}$-closed forms (e.g. through the De Rham-Weil isomorphism). □

§2.5. A counterexample

In view of Corollary 2.1.2, one might wonder whether the morphism Φ^q_ω would not still be surjective when L is a nef vector bundle. We will show that this is unfortunately not so, even in the case of algebraic surfaces.

Let B be an elliptic curve and V the rank let 2 vector bundle over B which is defined as the (unique) non split extension

$$0 \to \mathcal{O}_B \to V \to \mathcal{O}_B \to 0.$$

In particular, the bundle V is numerically flat, i.e. $c_1(V) = 0$, $c_2(V) = 0$. We consider the ruled surface $X = \mathbb{P}(V)$. On that surface there is a unique section $C = \mathbb{P}(\mathcal{O}_B) \subset X$ with $C^2 = 0$ and

$$\mathcal{O}_X(C) = \mathcal{O}_{\mathbb{P}(V)}(1)$$

is a nef line bundle. It is easy to see that

$$h^0(X, \mathcal{O}_{\mathbb{P}(V)}(m)) = h^0(B, S^m V) = 1$$

for all $m \in \mathbb{N}$ (otherwise we would have $mC = aC + M$ where aC is the fixed part of the linear system $|mC|$ and $M \neq 0$ the moving part, thus $M^2 \geqslant 0$ and $C \cdot M > 0$, contradiction). We claim that

$$h^0(X, \Omega^1_X(kC)) = 2$$

for all $k \geqslant 2$. This follows by tensoring the exact sequence

$$0 \to \Omega^1_{X|C} \to \Omega^1_X \to \pi^*\Omega^1_C \simeq \mathcal{O}_C \to 0$$

by $\mathcal{O}_X(kC)$ and observing that

$$\Omega^1_{X|C} = K_X = \mathcal{O}_X(-2C).$$

From this, we get

$$0 \to H^0(X, \mathcal{O}_X((k-2)C)) \to H^0(X, \Omega^1_X \mathcal{O}(kC)) \to H^0(X, \mathcal{O}_X(kC))$$

where $h^0(X, \mathcal{O}_X((k-2)C)) = h^0(X, \mathcal{O}_X(kC)) = 1$ for all $k \geqslant 2$. Moreover, the last arrow is surjective because we can multiply a section of $H^0(X, \mathcal{O}_X(kC))$ by a nonzero section in $H^0(X, \pi^*\Omega^1_B)$ to get a preimage. Our claim follows. We now consider the diagram

$$\begin{array}{ccc} H^0(X, \Omega^1_X(2C)) & \xrightarrow{\wedge\omega} & H^1(X, K_X(2C)) \\ \simeq \big\downarrow & & \big\downarrow \varphi \\ H^0(X, \Omega^1_X(3C)) & \xrightarrow[\psi]{\wedge\omega} & H^1(X, K_X(3C)). \end{array}$$

Since $K_X(2C) \simeq \mathcal{O}_X$ and $K_X(3C) \simeq \mathcal{O}_X(C)$, the cohomology sequence of

$$0 \to K_X(2C) \to K_X(3C) \to K_X(3C)|C \simeq \mathcal{O}_C \to 0$$

immediately implies $\varphi = 0$ (notice that $h^1(X, K_X(2C)) = h^1(X, K_X(3C)) = 1$, since $h^1(B, \mathcal{O}_B) = h^1(B, V) = 1$, and $h^2(X, K_X(2C)) = h^2(B, \mathcal{O}_B) = 0$). Therefore the diagram implies $\psi = 0$, and we get:

2.5.1. Proposition. *$L = \mathcal{O}_{\mathbb{P}(V)}(3)$ is a counterample to 2.1.2 in the nef case.*

By Corollary 2.1.2, we infer that $\mathcal{O}_X(3)$ cannot be Hermitian semi-positive and we thus again obtain – by a quite different method – the result of [DPS94], example 1.7.

2.5.2. Corollary. *Let B be an elliptic curve, V the vector bundle given by the unique non-split extension*

$$0 \to \mathcal{O}_B \to V \to \mathcal{O}_B \to 0.$$

Let $X = \mathbb{P}(V)$. Then $L = \mathcal{O}_X(1)$ is nef but not Hermitian semi-positive (nor does any multiple, e.g. the anticanonical line bundle $-K_X = \mathcal{O}_X(-2)$ is nef but not semi-positive).

We now show that the above counterample is the only one that can occur on a surface, at least when $L = \mathcal{O}_X(\lambda C)$ and C is an elliptic curve with $C^2 = 0$ (if C is a curve with $C^2 > 0$, L is big and therefore the conclusion is positive as well).

2.5.3. Proposition. *Let X be a smooth minimal compact Kähler surface with Kähler form ω. Let $C \subset X$ be a smooth elliptic curve with $C^2 = 0$. Then the natural map*

$$\Phi^1_\omega : H^0(X, \Omega^1_X \otimes \mathcal{O}_X(\lambda C)) \to H^1(X, K_X \otimes \mathcal{O}_X(\lambda C))$$

is surjective for all $\lambda \in \mathbb{N}$ with the following single exception: $X = \mathbb{P}(V)$ where V is the unique non-split extension $0 \to \mathcal{O}_B \to V \to \mathcal{O}_B \to 0$ and $L = \mathcal{O}_X(1) = \mathcal{O}_X(C_0)$ where $C_0 = \mathbb{P}(\mathcal{O}_B) \subset \mathbb{P}(V)$ is the section with $C_0^2 = 0$.

Proof. (0) First notice that by 2.1.2 the result is positive if $\mathcal{O}_X(\mu C)$ is generated by global sections for some $\mu \in \mathbb{N}$, or if $\mathcal{O}_X(\mu C) \otimes G$ is generated by global sections for some $G \in \mathrm{Pic}^0(X)$. Since $K_X \cdot C = K_C \cdot C = 0$, K_X cannot be ample, hence $\kappa(X) \leqslant 1$. If $\kappa(X) = 1$, then $|mK_X|$ defines an elliptic fibration $f : X \to B$ and the equality $K_X \cdot C = 0$ implies $\dim f(C) = 0$. Therefore we conclude by (0).

If X is a torus or an hyperelliptic surface, we can directly apply (0). If X is a K3 surface, Riemann-Roch gives $\chi(\mathcal{O}_X(C)) = 2$, hence $h^0(\mathcal{O}_X(C)) \geqslant 2$, so that $\mathcal{O}_X(C)$ is generated by global sections. If X is Enriques, choose a 2:1 unramified cover $h : \tilde{X} \to X$ with $\tilde{X}$ a K3 surface. Then $h^*(\mathcal{O}_X(C))$ is generated by global sections and therefore $\mathcal{O}_X(C)$ is semi-positive, so that 2.1.2 applies again.

It remains to treat the case $\kappa(X) = -\infty$. Since $X \neq \mathbb{P}_2$, the surface X carries a $\mathbb{P}_1$-bundle structure $f : X \to B$, and $f(C) = B$. In particular the genus $g(B) \leqslant 1$. We cannot have $B = \mathbb{P}_1$, since there is no rational ruled surface X carrying an elliptic curve C with $C^2 = 0$, as we check immediately by [Ha77, V.2]. Hence B is elliptic. In that case [Ha77, V.2] gives immediately that $X = \mathbb{P}(V)$ with a semi-stable rank 2 vector bundle V on B. We normalize V in such a way that $c_1(V) \in \{0, 1\}$.

(a) $c_1(V) = 0$.

Then either $V = \mathcal{O} \oplus L$ with $L \in \mathrm{Pic}^0(B)$ or there is a non-split extension

$$0 \to \mathcal{O}_B \to V \to \mathcal{O}_B \to 0. \tag{2.5.4}$$

If V splits, $\mathcal{O}_{\mathbb{P}(V)}(1)$ is semi-positive. Since

$$\mathcal{O}_X(C) \equiv \mathcal{O}_{\mathbb{P}(V)}(\alpha)$$

for some $\alpha \in \mathbb{N}$, we conclude by 2.1.2. In the non split case, we claim that the curve C must be equal to $C_0 = \mathbb{P}(\mathcal{O}_B) \subset \mathbb{P}(V)$. In fact, sequence (2.5.4) implies that

$$H^0(\mathcal{O}_{\mathbb{P}(V)}(\alpha) \otimes \pi^*(L)) = 0$$

for all $L \in \mathrm{Pic}^0(B), L \neq \mathcal{O}_B$. Hence

$$\mathcal{O}_X(C) \simeq \mathcal{O}_X(\alpha\, C_0)$$

for some $\alpha \in \mathbb{N}$. If $\alpha \neq 1$ or $C \neq C_0$, then V would split – possibly after taking a finite étale cover $C \to B$, which is not the case.

The latter case was already discussed and leads to the exception mentioned in the theorem.

(b) $c_1(V) = 1$.
We perform a base change $h : C \to B, \tilde{V} = h^*(V)$. Let $\tilde{X} = \mathbb{P}(\tilde{V})$. Then $\tilde{V}$ is semi-stable with $c_1(\tilde{V})$ even so that we are in case (a). By [At57], C is 2:1 (étale) over B. Therefore $h^{-1}(C)$ consists of two sections, hence $\tilde{V}$ splits. Consequently we can easily reduce ourselves to the splitting case of (a) and obtain surjectivity. □

§2.6. A direct image theorem

We state here, for later use, the following useful direct image theorem.

2.6.1. Theorem. *Let $f : X \to Y$ be a holomorphic (smooth) submersion between compact Kähler manifolds, and let L be a pseudo-effective line bundle. We assume that the direct image $\mathcal{E} = f_*(K_{X/Y} \otimes L)$ is locally free and that the zero variety of $\mathcal{I}(h_{\min})$ does not project onto Y. Set $\mathcal{J} = f_\star(\mathcal{I}(h_{\min})) \subset \mathcal{O}_Y$, $\mathcal{J} \neq 0$.*

a) *If Y is projective, there exists a very ample line bundle G on Y such that the global sections of $\mathcal{E}^{\otimes m} \otimes G$ generate a subsheaf containing $\mathcal{E}^{\otimes m} \otimes G \otimes \mathcal{J}^m$ for every integer $m > 0$.*

b) *If Y is projective or Kähler, then $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is pseudo-effective on $\mathbb{P}(\mathcal{E})$.*

c) *If $\mathcal{E}$ is of rank 1, then $\mathcal{E}$ is a pseudo-effective line bundle on Y.*

Proof. The proof closely follows ideas already described by Viehweg, [Kol86], [DPS94] and [Mou97], so we will be rather quick on details.

a) One can take for instance $G = K_Y + (n+1)A$ where A is very ample on Y and $n = \dim Y$. Then

$$K_{X/Y} \otimes L \otimes f^\star G = K_X \otimes L \otimes f^\star A$$

and $L \otimes f^\star A$ can be equipped with the tensor product of the metric $h_{\min}$ of L by a singular metric on A which is smooth of positive curvature outside a point $y \in Y$, with a single isolated pole of Lelong number $> 1-\varepsilon$ at y ([De90], § 6). Hörmander's standard L^2 estimates show that sections of $K_{X/Y} \otimes L \otimes f^\star G \otimes \mathcal{I}(h_{\min})$ on $X_y = f^{-1}(y)$ can be extended to global sections on X; actually, given such a section h defined on $f^{-1}(V)$, where V is a neighborhood of y, we solve the $\overline{\partial}$ equation $\overline{\partial} u = \overline{\partial}(\theta(f)h) = h\, f^\star \overline{\partial}\theta$ where θ is a cut-off function with support in V, equal to 1 near y. By construction, the curvature current of $L \otimes f^\star G$ satisfies $\Theta \geqslant f^\star \omega_Y$ for some Kähler form ω_Y on Y. The curvature need not be positive on X, but this is nevertheless sufficient to solve the $\overline{\partial}$-equation in virtue of ([De82], Théorème 4.1), since the norm $|f^\star \overline{\partial}\theta|_\Theta$ is bounded (in the notation of [De82]). Moreover, the Lelong number of the induced metric on

$L \otimes f^\star G$ along the fiber X_y will be $(n+1)(1-\varepsilon)$, thus in the range $]n, n+1[$, so that the resulting Nadel multiplier ideal sheaf $\mathcal{I}'$ of that metric satisfies $\mathcal{I}' \subset \mathcal{I}(h_{\min}) \cap \mathcal{I}_{X_y}$ by [Sk72]. This implies that the solution u vanishes along X_y and that

$$\widetilde{h} = \theta h - u \in H^0(X, K_{X/Y} \otimes L \otimes f^\star G \otimes \mathcal{I}(h_{\min}))$$

coincides with h in restriction to X_y. In other words, the direct image sheaf

$$f_*\big(K_{X/Y} \otimes L \otimes f^\star G \otimes \mathcal{I}(h_{\min})\big)$$

is generated by global sections. However, this sheaf is obviously contained in $\mathcal{E} \otimes G$ and its global sections contain those of $\mathcal{E} \otimes G \otimes \mathcal{J}$. The assertion for $\mathcal{E}^m$ follows by the usual fiber product trick, where $X \to Y$ is replaced by $X_m = X \times_Y \ldots \times_Y X \to Y$ (recall that $f : X \to Y$ is supposed to be smooth). Then $K_{X_m/Y} \otimes (L \boxtimes_Y \ldots \boxtimes_Y L)$ has direct image $\mathcal{E}^{\otimes m}$ on Y.

b) is an straightforward consequence of a), at least in the projective situation, since G gets multiplied by $1/m$ as m goes to $+\infty$. The Kähler case (which we will not need anyway) can be dealt with as in Mourougane [Mou97], by using metrics and local sections over a fixed Stein covering of Y.

c) special case of b). $\square$

§2.7. Applications

Our applications mostly concern compact Kähler manifolds such that either the canonical or anticanonical line bundle is pseudo-effective. The first one has been observed independently by M. Paun [Pau98].

2.7.1. Proposition. *Let X be a compact Kähler manifold with $-K_X$ pseudo-effective. Assume that $-K_X$ has a (singular) Hermitian metric h with semi-positive curvature such that $\mathcal{I}(h) = \mathcal{O}_X$ (i.e. the singularities of the weights φ are mild enough to warrant that $e^{-\varphi}$ is locally integrable). Then*

a) *The natural pairing*
$$H^0(X, T_X) \times H^0(X, \Omega^1_X) \to \mathbb{C}$$
is non degenerate on the $H^0(X, \Omega^1_X)$ side, and the non zero holomorphic 1-forms do not vanish at all.

b) *The Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is a submersion, and there is a group of automorphisms of X which lies above the translations of the Albanese torus.*

Proof. a) The hard Lefschetz theorem applied with $L = K_X^{-1}$ and $q = 1$ implies that there is a surjective map

$$H^0(X, \Omega^{n-1}_X \otimes K_X^{-1}) \xrightarrow{\omega \wedge \bullet} H^1(X, K_X \otimes K_X^{-1}) = H^1(X, \mathcal{O}_X).$$

However, $H^0(X, \Omega^{n-1}_X \otimes K_X^{-1}) \simeq H^0(X, T_X)$ and the arrow $\omega \wedge \bullet$ can then be seen as the contraction mapping $\xi \mapsto \xi \lrcorner\, \omega$ of the Kähler form by a holomorphic vector field ξ. Since the group $H^1(X, \mathcal{O}_X)$ is conjugate to $H^0(X, \Omega^1_X)$ by Hodge symmetry, the (non degenerate) L^2 pairing between the $(0,1)$-class $\{\xi \lrcorner\, \omega\} \in H^1(X, \mathcal{O}_X)$ and a form $\eta \in H^0(X, \Omega_X^1)$ is given by

$$\int_X \langle \xi \lrcorner\, \omega, \overline{\eta} \rangle_\omega \omega^n = \int_X \langle \xi, \eta \rangle_{T_X \times \Omega_X^1} \omega^n = C \langle \xi, \eta \rangle_{T_X \times \Omega_X^1}, \qquad C > 0.$$

($\langle \xi, \eta \rangle_{T_X \times \Omega_X^1}$ is a holomorphic function, hence constant). Because of surjectivity onto $H^1(X, \mathcal{O}_X)$, there exists for every nonzero holomorphic 1-form η a vector field ξ such that $\langle \xi, \eta \rangle_{T_X \times \Omega_X^1} \neq 0$. This implies that η does not vanish and a) is proved.

b) Let $u_1, \ldots, u_q$ be a basis of $H^0(X, \Omega_X^1)$. The Albanese map is given by

$$\alpha : X \to \mathrm{Alb}(X), \qquad x \mapsto \alpha(x) = \Big(\int_{x_0}^x u_j \Big)_{1 \leqslant j \leqslant q} \ \text{modulo periods.}$$

Hence $d\alpha \simeq (du_1, \ldots, du_q)$, and we know by a) that $du_1(x), \ldots, du_q(x) \in T^\star_{X,x}$ are linearly independent at every point. This means that α has maximal rang q at every point, i.e. is a submersion. The existence of vector fields shown in a) easily imply the assertion on automorphisms. □

We next consider the case when K_X is pseudo-effective.

2.7.2. Abundance conjecture. *If X is a compact Kähler manifold with K_X pseudo-effective, then the Kodaira dimension $\kappa(X)$ is non-negative, i.e. there exist non trivial sections of $H^0(X, mK_X)$ for some $m > 0$.*

The abundance conjecture is presently known only in the projective case, and even then, only for $\dim X \leqslant 3$. What we can prove from our hard Lefschetz theorem is the following partial result in the Kähler case.

2.7.3. Theorem. *Let X be a compact Kähler manifold with K_X pseudo-effective. Assume that K_X has a singular Hermitian metric of non-negative curvature, possessing "algebraic singularities" in the following sense: there exists a modification $\mu : \tilde{X} \to X$ such that the pullbacks of the local plurisubharmonic weights φ take the form*

$$\varphi \circ \mu = \sum_j \lambda_j \log |g_j| + O(1)$$

where $O(1)$ is a bounded term, $D_j = \{g_j = 0\}$ is a family of normal crossing divisors in $\tilde{X}$ and λ_j are positive rational numbers. Then X satisfies at least one of the following two properties:

a) $\chi(X, \mathcal{O}_X) = \chi(X, K_X) = 0$ *and there exists a nonzero holomorphic p-form in $H^0(X, \Omega_X^p)$ for some odd integer p.*

b) *There exists $q = 0, 1, \ldots, n$ and infinitely many positive integers m such that*

$$H^0(X, \Omega_X^q \otimes \mathcal{O}(mK_X)) \neq 0.$$

Proof. Observe that sections of $H^0(X, \Omega_X^q \otimes \mathcal{O}(mK_X))$ are bimeromorphic invariants, hence we can assume that $\tilde{X} = X$. Suppose that b) fails, i.e. that there is $m_0 > 0$ such that $H^0(X, \Omega_X^q \otimes \mathcal{O}(mK_X)) = 0$ for all q and $m \geqslant m_0$. Then a fortiori

$$H^0(X, \Omega_X^q \otimes \mathcal{O}(mK_X) \otimes \mathcal{I}(h^{\otimes m})) = 0$$

and the hard Lefschetz theorem implies that

$$H^q(X, K_X \otimes \mathcal{O}(mK_X) \otimes \mathcal{I}(h^{\otimes m})) = 0 \qquad \text{for all } q \text{ and } m \geqslant m_0.$$

Therefore $\chi(X, \mathcal{O}((m+1)K_X) \otimes \mathcal{I}(h^{\otimes m})) = 0$ for $m \geqslant m_0$. However, the assumption on the singularities of h shows that $\mathcal{I}(h^{\otimes m}) = \mathcal{O}_X(-[m\lambda_j]D_j)$ where $[\ \ ]$ denotes the integral part. By Riemann-Roch, we have

$$\begin{aligned} f(m) &= \chi(X, \mathcal{O}((m+1)K_X) \otimes \mathcal{I}(h^{\otimes m})) \\ &= \int_X \exp\Big((m+1)c_1(K_X) - \sum [m\lambda_j]c_1(D_j)\Big)\,\mathrm{Todd}(X). \end{aligned}$$

By expanding the integral, we find a (constant) integer N such that

$$N\,f(m) = P(m, [m\lambda_1], \ldots, [m\lambda_r])$$

for some polynomial $P \in \mathbb{Z}[t_0, t_1, \ldots, t_r]$ of degree $\leqslant n$. Take m to be a large multiple kd of a common denominator of the λ_j's. Then $f(kd)$ is a polynomial in k and vanishes for k large, thus $f(0) = \chi(K_X) = 0$. Therefore $\chi(\mathcal{O}_X) = (-1)^n\chi(K_X) = 0$, and as $h^0(X, \mathcal{O}_X) = 1$, we conclude that one of the odd degree groups $H^p(X, \mathcal{O}_X)$ must be nonzero. By Hodge symmetry, we get $H^0(X, \Omega^p_X) \neq 0$; property a) is proved.

It seems likely that in Theorem 2.7.2 (at least in case b), the Kodaira dimension $\kappa(X)$ should be non-negative. For our purposes it suffices to have

2.7.4. Proposition. *Let X be a compact manifold, E a vector bundle and L a line bundle on X. Suppose that $H^0(X, E \otimes L^m) \neq 0$ for infinitely many m. Then $a(X) \geq 1$ or $\kappa(L) \geq 0$.*

Proof. By our assumption we have inclusions $L^{-m} \to E$. Considering the smallest subsheaf in E containing all the images and taking determinants, we obtain a line bundle F and infinitely many inclusions $L^{-m} \to F$. So $H^0(X, F \otimes L^m) \neq 0$ for infinitely many m. If X carries infinitely many irreducible hypersurfaces, then $a(X) > 0$ ([Kra75], see also [FiFo79]). So suppose that X has only finitely many irreducible hypersufaces Y_i. Then consider the cone K generated by the Y_i, say in $\mathrm{Pic}(X) \otimes \mathbb{R}$. Then $F \otimes L^m \in K$, hence $L \in K$, which implies $\kappa(L) \geq 0$. □

2.7.5. Theorem. *Let X be a smooth compact Kähler threefold with K_X pseudo-effective. Assume that K_X has a singular metric as in Theorem 2.7.3, e.g. that K_X is Hermitian semi-positive. Then $\kappa(X) \geq 0$.*

Proof. Since K_X is pseudo-effective, X cannot be uniruled. Therefore the main result in [CP00] implies that $\kappa(X) \geq 0$ unless possibly if X is simple (and $\kappa(X) = -\infty$), which means that there is no positive-dimensional subvariety through the very general point of X. From 2.7.3 we obtain that

$$\chi(X, \mathcal{O}_X) = 0$$

or that

$$H^0(X, \Omega^q_X \otimes \mathcal{O}_X(mK_X)) \neq 0$$

for infinitely many m. In the first case we have a 3-form (so that $\kappa(X) \geq 0$ or a $1-$form, so that we have a non-trivial Albanese. X being simple, the Albanese map must be generically finite over the Albanese torus. But then $\kappa(X) \geq 0$. In the second case we conclude by Proposition 2.7.4. □

§3. Pseudo-effective versus almost nef line bundles

In this section we study pseudo-effective line bundles and their "numerical" counterparts, which we call almost nef line bundles.

3.1. Definition. *Let X be a projective manifold, L a line bundle on X. The bundle L is almost nef if and only if there is a family $A_i \subset X$, $i \in \mathbb{N}$, of proper algebraic subvarieties such that $L \cdot C \geqslant 0$ for all irreducible curves $C \not\subset \bigcup_i A_i$. The Zariski closure of the union of all curves C with $L \cdot C < 0$ will be called the non-nef locus of L.*

3.2. Remark. We say that $(C_t)_{t \in T}$ is a covering family of curves on X if T is compact, $X = \bigcup_{t \in T} C_t$, and if C_t is irreducible for general $t \in T$.

With this notation, L is almost nef if and only if $L \cdot C_t \geqslant 0$ for all covering families (C_t) of curves. Indeed, one direction is clear, the other is an obvious Hilbert scheme argument.

3.3. Proposition. *Let X be a projective manifold. Assume that L is pseudo-effective on X. Then L is almost nef.*

Proof. This follows from [DPS96a, 4.3] but for the convenience of the reader we give here a proof using directly the definition. Choose A ample such that $mL + A$ is effective for $m \geqslant m_0$ sufficiently divisible. Denote this set of m's by M. Then we can write

$$mL + A = E_m$$

with an effective divisor $E_m \subset X$. Hence $E_m \cdot C \geqslant 0$ for all $C \not\subset E_m$ and therefore $(mL + A) \cdot C \geqslant 0$ for $C \not\subset E_m$. Consequently we have

$$(mL + A) \cdot C \geqslant 0$$

for all $m \in M$ and all $C \not\subset \bigcup_{m \in M} E_m$. Hence $L \cdot C \geqslant 0$ for all those C. □

3.4. Problem. *Let X be a projective manifold and L an almost nef divisor on X. Is L pseudo-effective?*

3.5. Comments. This is in general a very hard problem (maybe even the answer is negative). We here point out some circumstances when (3.4) has a positive answer.

a) $\dim X = 2$ and L is arbitrary.

This is already observed in [DPS96a, 4.5]. The reason is simply that the cone of effective divisors is the cone effective curves, hence, by dualizing, the ample cone is the dual cone to the cone of effective divisors. So $L \in \bar{K}_{\text{eff}}(X)$ if and only if $L \cdot C \geqslant 0$ for all $C \subset X$ with $C^2 \geqslant 0$.

b) Consider now the case $L = K_X$.

Notice first that K_X almost nef just says that X is not uniruled. In fact, if X is uniruled, we have a covering family (C_t) of rational curves (C_t) with $K_X \cdot C_t < 0$, and conversely, if there is a covering family (C_t) with $K_X \cdot C_t < 0$, then X is uniruled by [MM86].

Now suppose that K_X is almost nef and $\dim X = 3$. Then X has a minimal model X', i.e. $K_{X'}$ is nef. By abundance, $\kappa(X) = \kappa(X') \geqslant 0$, in particular K_X is pseudo-effective. Notice that this is more than what we asked for, because a priori K_X could be pseudo-effective and $\kappa(X) = -\infty$.

In order to prove (3.4) in the case $L = K_X$ and $\dim X = n$, we will "only" need the existence and finiteness of flips but we can avoid the use of the abundance conjecture. Since X is not uniruled, X has a birational model X' with $K_{X'}$ nef, therefore $K_{X'}$ is pseudo-effective. In order to see that K_X itself is pseudo-effective, take a divisor A' on X' such that $H^0(mK_{X'} + A') \neq 0$ for $m \gg 0$ and sufficiently divisible. Then we only need to consider the two following situations

(3.5.1) $\lambda : X \to X'$ is a divisorial contraction,

(3.5.2) $\lambda : X \rightharpoonup X'$ is of flipping type.

In case (3.5.1) $K_X = \lambda^*(K_{X'}) + \mu E$, where E is the exceptional divisor and $\mu > 0$, therefore clearly K_X is pseudo-effective. In case (3.5.2) let A be the strict transform of A' in X and, λ being an isomorphism in codimension 1, we have by the Riemann extension theorem

$$H^0(X, mK_X + A) = H^0(X', mK_{X'} + A'),$$

thus $H^0(X, mK_X + A) \neq 0$ for $m \gg 0$ sufficiently divisible. There is a slight difficulty: A is a priori only a Weil divisor, but since X is $\mathbb{Q}$-factorial, we find λ such that λA is Cartier and moreover

$$H^0(X, \lambda m K_X + \lambda A) \neq 0.$$

In total the flip conjectures imply (3.4) for $L = K_X$ (in any dimension). □

§4. Varieties with pseudo-effective anticanonical bundles

In this section we study compact Kähler manifolds and projective varieties with pseudo-effective and nef anticanonical bundles. We shall begin with the nef case, in which already a substantial number of results have been obtained. In fact, concerning the structure of compact Kähler manifolds X with $-K_X$ nef, we have the following:

4.1. Conjecture ([DPS93, 96b]). *Let X be a compact Kähler manifold with $-K_X$ nef. Then*

a) *If $\varphi : X \to Y$ is a surjective map to the normal complex space Y, then $\kappa(Y) \leqslant 0$ where $\kappa(Y) = \kappa(\hat{Y})$, $\hat{Y}$ a desingularization).*

b) *The Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is a surjective submersion.*

c) $\pi_1(X)$ *is almost Abelian, i.e. Abelian up to a subgroup of finite index.*

4.2. Remark. The status of the conjecture is as follows

(4.2.1) a), b) *and* c) *hold if* $-K_X$ *is Hermitian semi-positive* ([DPS93, DPS96b]).

(4.2.2) a) *holds and therefore surjectivity of* α, *if* X *is projective* [Zh96].

(4.2.3) a), b) *hold if* X *is a projective 3-fold* ([PS97]) *and* c) holds if $q(X) > 0$.

(4.2.4) *a compact Kähler* n*-fold,* $n \leq 4$ *with* $-K_X$ *nef, does not admit a surjective map to a normal projective variety of general type and therefore* a) *holds for* n*-folds,* $n \leq 4$. [CPZ98].

(4.2.5) *If* $-K_X$ *is nef, then* X *does not admit a map to a curve* C *of genus* $g(C) \geqslant 2$ [DPS93].

Our main aim here is to show that the Albanese map of a compact Kähler threefold is a surjective submersion also in the non-algebraic case.

4.3. Theorem. *Let* X *be a compact Kähler 3-fold with* $-K_X$ *nef. Then the Albanese map* $\alpha : X \to \mathrm{Alb}(X)$ *is surjective and has connected fibers.*

Proof. We may assume X non-algebraic by (4.2) and $\kappa(X) = -\infty$ by [Bea83]. Let $Y = \alpha(X)$ and assume $Y \neq \mathrm{Alb}(X)$. By (4.2.4) we have $\dim Y = 2$. Let $\hat{Y} \to Y$ be a desingularization. Then $\kappa(\hat{Y}) > 0$ and $q(\hat{Y}) \geqslant 3$ by [Ue75]. If necessary, substitute α by its Stein factorization. The general fiber F of $\alpha : X \to Y$ must be a smooth rational curve; otherwise $\kappa(X) \geqslant 0$ by $C_{3,1}$ ([Ue87]). Next observe that the algebraic dimension $a(Y) \leqslant 1$. In fact otherwise Y would be Moishezon and induces 2 independent meromorphic functions on X. But then X is clearly algebraically connected, i.e. any two points can be joined by a finite union of irreducible compact curves. Hence X is Moishezon (and therefore projective) by [Cam81]. So $a(Y) \leqslant 1$; in particular $\kappa(\hat{Y}) = 1 = a(Y)$. Let $\hat{f} : \hat{Y} \to C$ be the algebraic reduction (= Iitaka fibration). Then clearly $\kappa(C) = \kappa(\hat{Y}) = 1$, and C is of general type. Therefore the induced meromorphic map $X \to Y$ is actually holomorphic, contradicting (4.2.5). It remains to prove that α has connected fibers. If $\dim A = 1$, this is a general fact [Ue75]. If $\dim A = 2$, we argue as follows. If $a(A) = 2$, then, using again $C_{3,1}$, X is projective and we refer to (4.2.3). If $a(A) = 1$, consider the algebraic reduction $f : A \to B$ to the elliptic curve. If α is not connected, then so does $\beta : X \to B$. Let $\gamma : X \to \tilde{B}$ be the Stein factorization of β. By (4.2.4), $\tilde{B}$ must be an elliptic curve. This contradicts clearly the universal property of the Albanese torus. If finally $a(A) = 0$, then consider the Stein factorization $g : X \to S$ of α. Since A contains no curves, the map $h : S \to A$ is unramified, hence S is a torus itself. Therefore $h = \mathrm{Id}$. □

In order to investigate further the structure of compact Kähler 3-folds with $-K_X$ nef, we quote the

4.4. Proposition. *Let* Z *be a compact Kähler threefold and* $f : Z \longrightarrow C$ *be a surjective map with connected fibers to a smooth curve of genus* $g \geq 1$. *Assume that the general fiber* F *has* $\kappa(F) = -\infty$ *and* $q(F) = 1$. *If* $-K_{Z|C}$ *is nef, then the only singular fibers of* f *are multiples of smooth surfaces. In particular,* f *is smooth after a finite base change* $\tilde{C} \longrightarrow C$. *If* C *is elliptic, so that* $-K_X$ *is nef, the original* f *is already smooth.*

Furthermore there is a smooth minimal surface Y, a $\mathbb{P}_1$-bundle structure $g : Z \longrightarrow Y$ and an elliptic fibration $h : Y \longrightarrow C$ with at most multiple fibers as singular fibers such that $f = h \circ g$.

The proof can be found in [CPZ98]. For the convenience of the reader, we sketch the idea of the proof; we argue locally over the base. Since Z is Kähler and f is locally Moishezon, f is projective [CP00]. Since K_Z is not f-nef, a theorem of Kawamata allows us to construct locally over the base C a relative contraction φ, and it turns out that the dimension of the image is always 2 unless φ is birational. Then one proves that all these local relative contractions glue to a global relative contraction $g : Z \longrightarrow Y$. In the birational case one has to repeat this construction and one finally ends up with some fibration analogous to g which has to be studied with the methods of [PS97].

4.5. Corollary. *Let X be a compact Kähler threefold with $-K_X$ nef. If $q(X) = 1$, then X is projective unless $K_X \equiv 0$.*

Proof. Let $\alpha : X \to A$ be the Albanese to the elliptic curve A and suppose X not projective and $K_X \not\equiv 0$. By $C_{3,1}$, we have $\kappa(F) = -\infty$ for the general fiber F of α. Since $-K_F$ is nef, the irregularity $q(F) \leq 1$. If $q(F) = 0$, then X is algebraic; see [CP00] for that and further references. Actually we can also conclude as follows; the following arguments also settle the case $q(F) = 1$. Let ω be a non-zero 2-form. Consider the exact sequence

$$0 \longrightarrow N_F^* \otimes \Omega_F^1 \longrightarrow \Omega_X^2 | F \longrightarrow \Omega_F^2 \longrightarrow 0;$$

then $\omega | F$ induces a 1-form on F. So by (4.4), α is a submersion and there is a smooth surface Y and submersions $g : X \to Y$ and $h : Y \to A$ such that $f = h \circ g$. Moreover g is a $\mathbb{P}_1$-bundle and h is a smooth elliptic fibration. Since A is an elliptic curve, h is locally trivial and therefore $\kappa(Y) = 0$; actually Y is a torus or hyperelliptic. However Y cannot be projective, otherwise X is projective, so Y is a torus. This contradicts $q(X) = 1$. □

Continuing the study of non-algebraic compact Kähler threefolds X with $-K_X$ nef and $K_X \not\equiv 0$, we therefore either have $q(X) = 0$ or $q(X) = 2$.

4.6. Theorem. *Let X be a compact Kähler 3-fold with $-K_X$ nef, $K_X \not\equiv 0$ and $q(X) = 2$. Let $\alpha : X \to A$ be the Albanese map and assume $a(A) > 0$. Then X is a $\mathbb{P}_1$-bundle over A.*

Proof. By (4.3), it is clear that dim $A = 2$. Note that by $(C_{3,2})$ the general fiber of α must be $\mathbb{P}_1$. Therefore $a(A) = 2$ implies $a(X) = 3$ and X is projective. We conclude by (4.2.3). So we may suppose $a(A) = 1$; let $\pi : A \longrightarrow B$ be the algebraic reduction, an elliptic bundle. Now we can apply (4.4) to the composite map $X \longrightarrow B$ to obtain our claim. □

We now investigate the case $a(A) = 0$.

4.7. Theorem. *Let X be a compact Kähler 3-fold with $-K_X$ nef, $K_X \not\equiv 0$ and $q(X) = 2$. Let $\alpha : X \to A$ be the Albanese map and assume $a(A) = 0$. Then α is a projectively flat $\mathbb{P}_1$-bundle.*

We already know in the situation of 4.7 that $\alpha : X \longrightarrow A$ is surjective with connected fibers and the general fiber is $\mathbb{P}_1$. Let us first see that α is projectively flat once we know that α is smooth. In fact, the exact sequence

$$0 \longrightarrow T_{X/A} \longrightarrow T_X \longrightarrow T_A$$

and the observation $T_{X/A} = -K_X$ show that T_X is nef. Then the claim follows from [CP91,8.2] (the non-algebraic case is just the same).

Now we show that α is a $\mathbb{P}_1-$bundle. Since A does not contain subvarieties of positive dimension, α can have at most a finite number of singular fibers; therefore there is a finite set $E \subset A$ such that α is smooth over $A_0 = A \setminus E$. We let $X_0 = \alpha^{-1}(A_0)$ and we must prove $E = \emptyset$.

We prepare the proof of (4.7) by the following

4.8. Lemma. *In the situation of (4.6) we have*

$$H^0(X, -K_X \otimes \alpha^*(L)) = 0$$

for all $L \in \mathrm{Pic}(A)$, *unless* X *is a* $\mathbb{P}_1$*-bundle.*

Proof. Suppose $H^0(X, -K_X \otimes \alpha^*(L)) \neq 0$ and take $D \in |-K_X \otimes \alpha^*(L)|$. Let D_0 be a component of D with $\alpha(D_0) = A$. Then $\kappa(D_0) \geqslant 0$. Let λ_0 be the multiplicity of D_0 in D. Then

$$K_{\lambda_0 D_0} = K_D | \lambda_0 D_0 - E$$

with E effective. By adjunction

$$\lambda_0 K_{D_0} = K_{\lambda_0 D_0} | D_0 + (\lambda_0 - 1) K_X | D_0.$$

Taking into account $K_D | D_0 = \alpha^*(L) | D_0$ (again by adjunction), we obtain

$$K_{D_0} \equiv -B_1 - B_2 + \alpha^*(L')$$

with B_1 effective and B_2 nef, where L' is a rational multiple of L. Let $\tau : \hat{D}_0 \to D_0$ be a desingularization of D_0; namely a minimal desingularization after normalization. Then still

$$K_{\hat{D}_0} \equiv -\hat{B}_1 - \hat{B}_2 + \hat{\alpha}^*(L')$$

with $\hat{B}_1$ effective, $\hat{B}_2$ nef and $\hat{\alpha}$ is the induced map. But $\kappa(\hat{D}_0) \geqslant 0$. So $\hat{\alpha}^*(L')$ is pseudo-effective; on the other hand L' is of signature $(1,1)$ if not numerically trivial ([LB92,p.318,Ex.8). This implies $K_{\hat{D}_0} \equiv L \equiv 0$, and $\hat{B}_1 = \hat{B}_2 = 0$, hence $\hat{D}_0$ is a torus and $\hat{D}_0 \to A$ is étale. From $\hat{B}_1 = \hat{B}_2 = 0$ we conclude that $B_1 = B_2 = 0$ and D_0 is normal with at most rational double points. Since $\hat{D}_0$ is a torus, it has no rational curves, hence D_0 is smooth. Write

$$-K_X \equiv \sum_{i \geqslant 0} \lambda_i D_i + \sum \mu_j R_j$$

with $\alpha(D_i) = A$, and $R_j \cdot F = 0$, F the general fiber of α. Since $a(A) = 0$, we have

$$\dim \alpha(R_j) = 0$$

for all j. Moreover we know already $D_i \cap R_j = \emptyset$ for all i, j. Hence $\sum \mu_j R_j$ must be nef. In fact, first observe that

$$-K_X | \sum \mu_j R_j = \sum \mu_j R_j | \sum \mu_j R_j$$

is nef (i.e. nef on the reduction); then apply [Pet98, 4.9]). But $\sum \mu_j R_j$ cannot be nef, unless all $\mu_j = 0$. Hence $-K_X \equiv \sum \lambda_j D_j$.

Since $K_X \cdot F = -2$, we observe by the way $-K_X \equiv D_0 + D_1$ or $-K_X \equiv 2D_0$. We claim that α is a submersion. Suppose that $G \subset \alpha^{-1}(a)$ is a 2-dimensional fiber component. Then we deduce, $\alpha | D_i$ being étale, that $D_i \cap G$ is at most finite, hence empty. From

$$-K_X \equiv \sum \lambda_i D_i$$

we deduce $K_X \cdot G = 0$. Hence there must be a 1-dimensional irreducible component $C_0 \subset \alpha^{-1}(a)$ with $D_0 \cdot C_0 \neq 0$. In particular $K_X \cdot C_0 < 0$. In case that $\dim \alpha^{-1}(a) = 1$, this is clear anyway, so that C_0 always exists. Consequently [Kol96] C_0 moves in an at least 1-dimensional family, say $(C_t)_{t \in T}$. Obviously the general C_t is a fiber of α and $\dim \alpha(C_t) = 0$ for all t. So C_0 actually moves in a 2-dimensional family and $K_X \cdot C_0 = -2$. Now consider the graph of (C_t) and it follows immediately that α is a $\mathbb{P}_1$-bundle. □

Proof of 4.7. The proof of 4.7 will now be completed by proving

4.8.0. Claim. *There exists a line L on A such that*

$$H^0(A, \alpha_*(-K_X) \otimes L) \neq 0.$$

There is a slight technical difficulty: a priori $\alpha_*(-K_X)$ need not be locally free. Therefore we will consider its dual W, compare the cohomology of W and $\alpha_*(-K_X)$ via several direct image calculations and then prove equality of both sheaves. Then it will be easy to conclude. To begin with, consider the exact sequence

$$0 \to \alpha_*(-K_X) \to W \to Q \to 0$$

where $W = \alpha_*(-K_X)^{**}$ (a rank 3 vector bundle) and Q is just the cokernel. Hence

$$\chi(W) = \chi(\alpha_*(-K_X)) + \chi(Q). \tag{4.8.1}$$

We claim that

$$c_1(W) = 0. \tag{4.8.2}$$

This is seen as follows. By Malgrange's theorem [Mal55], we have $H^q(A_0, \mathcal{O}) = 0$ for $q \geqslant 2$, since $\dim A = q(X) = 2$ and A_0 is not compact by the hypothesis $E \neq \emptyset$. The exponential sequence plus the vanishing $H^q(A_0, \mathcal{O}) = 0$ for $q \geq 2$ yields

$$H^2(A_0, \mathcal{O}^*_{A_0}) \simeq H^3(A_0, \mathbb{Z}),$$

From that we see easily - using e.g. Mayer-Vietoris that $H^2(\mathcal{O}^*_{A_0})$ is torsion free. Now the obstruction for a projective bundle to come from a vector bundle is a torsion element in $H^2(A_0, \mathcal{O}^*)$. Therefore there is a vector bundle V_0 on A_0 such that

$$X_0 = \mathbb{P}(V_0).$$

Now

$$-K_{X_0} = \mathcal{O}_{\mathbb{P}(V_0)}(2) \otimes \alpha^*(\det V_0^*),$$

hence

$$W_0 = W|A_0 = S^2 V_0 \otimes \det V_0^*.$$

Therefore we have $c_1(W_0) = 0$, hence $c_1(W) = 0$, E being finite, and this proves (4.8.2).

By the Leray spectral sequence we get

$$\chi(-K_X) = \chi(\alpha_*(-K_X)) - \chi(R^1\alpha_*(-K_X)) + \chi(R^2\alpha_*(-K_X)). \tag{4.8.3}$$

We next claim that

$$R^2\alpha_*(-K_X) = 0. \tag{4.8.4}$$

Of course, $R^2\alpha_*(-K_X) = 0$ generically. Now let $a \in A$ and assume that $\dim \alpha^{-1}(a) = 2$. Let $F = \alpha^{-1}(a)$, equipped with its reduced structure. Then

$$R^2\alpha_*(-K_X)_a = 0$$

if

$$H^2(F, -K_X|F \otimes N_F^{*\mu}) = 0 \tag{4.8.5}$$

for all $\mu \geqslant 0$. Let $S \subset F$ be an irreducible 2-dimensional component. Then (4.8.5) comes down to show

$$H^2(S, -K_X|S \otimes N_F^{*\mu}|S) = 0. \tag{4.8.6}$$

By Serre duality, this means that

$$H^0(S, 2K_X|S \otimes N_F^{\mu}|S \otimes N_S) = 0. \tag{4.8.7}$$

If $X_a = \varphi^{-1}(a)$ denotes the full complex-analytic fiber (with natural structure), then $N^*_{X_a}$ is generated by global sections. It follows that $N_F^{*\mu_1}|S$ and $N_S^{*\mu_2}$ have non-zero sections for suitable $\mu_1, \mu_2 > 0$. If therefore (4.8.7) does not hold, we conclude – having in mind that $-K_X|S$ is nef – that $K_X|S \equiv 0$, $N_F|S \equiv 0$ and $N_S \equiv 0$. However any section of $N_F^{*\mu_1}|S$ (resp. $N_S^{*\mu_2}$) is free of zeroes, and this implies $\alpha^{-1}(a) = S$ set-theoretically. Then N_S^* clearly cannot be numerically trivial, since $N^*_{X_a}$ is generated by at least two sections and $N^*_{X_a}|S = N_S^{*\lambda}|S$ generically suitable for λ. Hence (4.8.7) holds and (4.8.4) is proved.

In completely the same way we prove that

$$R^2\alpha_*(\mathcal{O}_X) = 0. \tag{4.8.8}$$

Since – as already seen – α is generically a $\mathbb{P}_1$-bundle with at most finitely many singular fibers, $R^1\alpha_*(\mathcal{O}_X)$ is a torsion sheaf. Together with (4.8.8) and the Leray spectral sequence we deduce

$$\dim H^2(X, \mathcal{O}_X) = 1. \tag{4.8.9}$$

Since $\dim H^1(X, \mathcal{O}_X) = q(X) = 2$, we obtain from (4.8.9):

$$\chi(X, \mathcal{O}_X) = 0. \tag{4.8.10}$$

Since $-K_X$ is nef, we have $(-K_X)^3 \geqslant 0$. If however $(-K_X)^3 > 0$, then the holomorphic Morse inequalities imply the projectivity of X so that $(-K_X)^3 = 0$. Riemann-Roch and (4.8.10) therefore yield

$$\chi(-K_X) = 0. \tag{4.8.11}$$

Then (4.8.3), (4.8.4) and (4.8.11) imply

$$\chi(\alpha_*(-K_X)) \geqslant 0 \tag{4.8.12}$$

and $\chi(\alpha_*(-K_X)) = 0$ if and only if $R^1\alpha_*(-K_X) = 0$.

In order to bring W into the game via (4.8.1), we first show:

$$\chi(W) \leqslant 0. \tag{4.8.13}$$

In fact, Riemann-Roch and (4.8.2) say

$$\chi(W) = -c_2(W)$$

Suppose $\chi(W) > 0$. Then $c_2(W) < 0$. Therefore the Bogomolov inequality

$$c_1^2(W) \leqslant 4\, c_2(W)$$

is violated and W is not semi-stable with respect to a fixed Kähler metric ω. Let $\mathcal{F} \subset W$ be a maximal destabilizing subsheaf with respect to ω; we may assume $\mathcal{F}$ locally free of rank 1 or 2. So

$$c_1(\mathcal{F}) \cdot \omega > 0.$$

First suppose that $\mathcal{F}$ has rank 2. Then we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow W \longrightarrow \mathcal{I}_Z \otimes L \longrightarrow 0$$

with a finite set Z of length $l(Z)$ and a line bundle L. Thus

$$c_2(W) = 2c_2(\mathcal{F}) + l(Z) - c_1(\mathcal{F})^2$$

and the condition $c_2(W) < 0$ yields

$$c_1(\mathcal{F})^2 > 2c_2(\mathcal{F}).$$

Now $c_1(\mathcal{F})^2 \leq 0$ since A is non-algebraic and moreover $c_1(\mathcal{F})^2 \leq 4c_2(\mathcal{F})$ since $\mathcal{F}$ is $\omega-$semi-stable. These last three inequalities give a contradiction. The calculations in case that $\mathcal{F}$ has rank 1 are the same, working with W^* instead of W, if $W/\mathcal{F} =: \mathcal{A}$ is $\omega-$semistable. If however $\mathcal{A}$ is unstable, we argue as follows. Write

$$0 \longrightarrow L_1 \longrightarrow \mathcal{A} \longrightarrow \mathcal{I}_B \otimes L_2 \longrightarrow 0$$

with L_1 a maximally $\omega-$destabilizing subsheaf and B a finite set. Then $c_2(W) < 0$ yields (as before)

$$c_1(\mathcal{A})^2 > l(Z) + c_2(\mathcal{A}). \tag{4.8.14}$$

Now

$$c_1(\mathcal{A}) = c_1(L_1) + c_1(L_2), \qquad c_2(\mathcal{A}) = c_1(L_1) \cdot c_1(L_2) + l(B), \tag{4.8.15}$$

hence, by combining (4.8.14) and (4.8.15) we obtain that

$$c_1(L_1)^2 + 2c_1(L_1) \cdot c_1(L_2) + c_1(L_2)^2 > l(Z) + l(B) + c_1(L_1) \cdot c_1(L_2),$$

hence

$$c_1(L_1)^2 + c_1(L_2)^2 + c_1(L_1) \cdot c_1(L_2) > l(Z) + l(B) \geq 0,$$

so $c_1(L_1) \cdot c_1(L_2) \geq 0$. On the other hand $c_1(\mathcal{A})^2 \leq 0$, hence $c_2(\mathcal{A}) < 0$ by (4.8.14), so (4.8.15) gives a contradiction and (4.8.3) is verified.

Observe now that (4.8.12), (4.8.13) and (4.8.1) imply

$$\chi(\alpha_*(-K_X)) = \chi(W) = 0$$

and $Q = 0$, i.e. $\alpha_*(-K_X) = W$. Thus $c_2(W) = 0$, W is semi-stable but not stable and one can find $\mathcal{F} \subset W$ as above with $c_1(\mathcal{F}) = 0$. If $\mathrm{rk}\,\mathcal{F} = 1$, we have

$$H^0(\alpha_*(-K_X) \otimes \mathcal{F}^*) \neq 0,$$

proving Claim 4.8.0 in that case.

In case $\mathrm{rk}\,\mathcal{F} = 2$, we consider the sequence

$$0 \to \mathcal{F} \to W \to \mathcal{R} \to 0. \tag{4.8.16}$$

$\mathcal{R}$ is a torsion free coherent sheaf whose singular locus $Z = \mathrm{Sing}\,\mathcal{R}$ is at most finite (since A has no compact positive dimensional subvarieties). Consequently $\mathcal{R} = \mathcal{J} \otimes L$ with an ideal sheaf $\mathcal{J}$ such that Supp $(\mathcal{O}_A/\mathcal{J}) = Z$. With the same argument as for W, we have

$$c_2(\mathcal{F}) \geqslant 0.$$

The exact sequence (4.8.16) yields

$$c_2(W) = c_2(\mathcal{R}) + c_2(\mathcal{F}).$$

Now $c_2(\mathcal{R}) = \sharp Z$, counted with multiplicities. Therefore $Z = \emptyset, \mathcal{R} = L$ and $c_2(\mathcal{F}) = 0$. Let $\zeta \in H^1(\mathcal{F} \otimes L^*)$ be the extension class of

$$0 \longrightarrow \mathcal{F} \longrightarrow W \longrightarrow L \longrightarrow 0.$$

If $\zeta \neq 0$, then $H^1(\mathcal{F} \otimes L^*) \neq 0$, hence $H^0(\mathcal{F} \otimes L^*) \neq 0$ or $H^0(\mathcal{F} \otimes \det \mathcal{F}^* \otimes L) \neq 0$ by Riemann-Roch and duality. Hence $H^0(W \otimes G) \neq 0$ for some G. If $\zeta = 0$, then $W = \mathcal{F} \oplus L$, hence again $H^0(W \otimes L^*) \neq 0$. □

Combining all our results – as explained in detail in the proof – we can state:

4.9. Theorem. *Let X be a compact Kähler 3-fold with $-K_X$ nef. Then the Albanese map is a surjective submersion. Moreover $\pi_1(X)$ is almost Abelian.*

Proof. The Albanese map is a surjective submersion by 4.3, 4.6 and 4.7. Thus it remains to prove the statement concerning $\pi_1(X)$. Suppose first $q(X) > 0$. By (4.2.3) we may assume X non-projective. We also may assume that $K_X \not\equiv 0$, i.e. $\kappa(X) = -\infty$, otherwise [Bea83] gives the claim. Hence $q(X) \geq 2$ by 4.5, so $q(X) = 2$ by $K_X \equiv 0$ (the Albanese map is surjective). Then 4.6 implies the claim.
The rest of the proof has been communicated to us by the referee. It only remains to show that if

$$\tilde{q}(X) = \sup\{q(\tilde{X}) \mid \tilde{X} \to X \ \text{ finite étale } \} = 0,$$

then $\pi_1(X)$ is finite. By Paun [Pau96], see (4.20(b)), G has polynomial growth. Hence G is almost nilpotent by Gromov's theorem, hence we can assume G nilpotent. Therefore the claim follows from the fact that a nilpotent group with finite Abelianization is actually finite. Indeed, consider the lower central series $(C^k(G))$ of G and let $G(j) = C^j(G)/C^{j+1}(G)$. Then we have a natural surjective map $G(0) \otimes G(j) \to G(j+1)$, induced by taking commutators. Using the nilpotency of G and the finiteness of $G(1)$, we easily deduce the finiteness of G.

4.10. Remark. It remains to investigate the structure of compact Kähler 3-folds X with $-K_X$ nef and $q(X) = 0$. Since our theory is only up to finite étale covers we should assume

$$\tilde{q}(X) = \sup\{q(\tilde{X}) \mid \tilde{X} \to X \ \text{ finite étale } \} = 0.$$

In case $-K_X$ is Hermitian semi-positive, the structure of X is as follows ([DPS96b]):

Up to a finite étale cover, X is of one of the following types:

a) *a Calabi-Yau manifold,*

b) $\mathbb{P}_1 \times S$, *where S is a K3 surface,*

c) *a rationally connected manifold.*

In class c), we have all Fano 3-folds, 3-folds with $-K_X$ big and nef but also e.g. $X = \mathbb{P}_1 \times Y$, where Y is $\mathbb{P}_2$ blown up in 9 general points in such a way that $-K_X$ is Hermitian semi-positive, in particular $K_X^3 = 0$. In case $-K_X$ nef one should expect the same result. Especially, if X is not projective then we should have $X = \mathbb{P}_1 \times S$ with S a non-projective $K3$ surface. Observe that rationally connected compact Kähler manifolds are automatically projective ([Cam81]). □

We finish this section by proving a weak "rational" version of our expectation in case $-K_X$ is nef and X is projective.

4.11. Proposition. *Let X be a compact Kähler 3-fold with $-K_X$ nef and $K_X \not\equiv 0$. Assume $\tilde{q}(X) = 0$. Then either X is rationally connected (in particular $\pi_1(X) = 0$) or there is a dominant meromorphic map*

$$f : X \rightharpoonup S$$

to surface birational to a K3 or Enriques surface S with general fiber a rational curve (hence $\pi_1(X) = 0$ or $\pi_1(X) = \mathbb{Z}_2$).

Proof. Assume X not to be rationally connected. Since $-K_X$ is nef and $K_X \not\equiv 0$, we have $\kappa(X) = -\infty$. Therefore X is uniruled. By [Cam92], [KoMM92] there is a meromorphic fibration

$$f : X \rightharpoonup S$$

contracting the general rational curves of a given (and fixed) covering family $(C_t)_{t\in T}$ of rational curves. Properly speaking, we have $f(x) = f(y)$ for general points $x, y \in X$, if x and y can be joined by a chain of rational curves of type C_t. Since X is not rationally connected, we have dim $S > 0$. Next notice that dim $S = 2$. In fact, if dim $S = 1$, then $S \simeq \mathbb{P}_1$ by $\tilde{q}(X) = 0$. Since the fibers of f are rationally connected, [KoMM92] gives rational connectedness. Hence dim $S = 2$. Of course we may assume S smooth. We first verify that $\kappa(S) \leqslant 0$.

Assume on the contrary that $\kappa(S) \geqslant 1$.

We can obtain f – after possibly changing the family (C_t) and changing S birationally and admitting rational double points on S – by a composite of birational Mori contractions and flips, say $X \rightharpoonup X'$, and a Mori fibration

$$f' : X' \to S$$

(just perform the Mori program on X). Now $-K_{X'}$ has the following property (cf. [PS97]):

$$-K_{X'} \cdot C' \geqslant 0$$

for all curves C' but a finite number of rational curves. In particular $-K_{X'}$ is almost nef with non-nef locus not projecting onto S, hence (4.7) applies and gives $\kappa(S) \leqslant 0$.

In total we know $\kappa(S) \leqslant 0$. Since $\tilde{q}(S) = 0$, S is birationally a K3, Enriques or rational surface. In the last case X is clearly rationally connected. □

4.12. Remark. (1) It seems rather plausible that methods similar to those of [PS97] will prove that, in case X is a projective 3-fold with $-K_X$ nef, $\kappa(S) = 0$, $\tilde{q}(X) = 0$ and X not rationally connected, the meromorphic map $f : X \rightharpoonup S$ is actually a holomorphic map with S being a K3 or Enriques surface and actually f is a submersion, i.e. a $\mathbb{P}_1$-bundle. Then we see immediately that $X \simeq \mathbb{P}_1 \times S$. Of course one difficulty arises from the fact that S contains some rational curves. In the non-algebraic case one would further need a more complete "analytic Mori theory".

(2) We discuss the Kähler analogue of (4.21). So let X be a compact Kähler threefold with $-K_X$ nef and $K_X \not\equiv 0$. This first difficulty is that X might be simple, i.e. there is no positive dimensional proper subvariety through the general point of X. Ruling out this potential case (which is expected not to exist), we conclude by [CP00] that X is uniruled. As in (4.21), we can form the rational quotient $f : X \rightharpoonup S$. Again $\dim S \neq 1$, because otherwise $S = \mathbb{P}_1$ and clearly X is projective (X cannot carry a 2-form). So S is a non-projective surface and automatically $\kappa(S) \leq 1$. In fact, otherwise X is algebraically connected (any two points can be joined by a chain of compact curves) and therefore projective by [Cam81]. One would like to exclude the case $\kappa(S) = 1$. However we do not know how to do this at the moment. The method of 4.7 does not work because we do not have enough curves in S. What we can say is the following. S admits an elliptic fibration $h : S \to C \simeq \mathbb{P}_1$. Since S is not algebraic, h has no multi-section. Now f is almost holomorphic ([Cam92]), i.e. there are $U \subset X$ and $V \subset S$ Zariski open such that $f : U \to V$ is holomorphic and proper. Let F be a general fiber of h. Let $A = S \smallsetminus V$. Then $A \cap F = \emptyset$ because h has no multi-sections, therefore f is holomorphic over F and the composition yields a holomorphic map $X \longrightarrow C$. Let $X_F = f^{-1}(F)$. Then X_F is a ruled surface over an elliptic curve of the form $\mathbb{P}(\mathcal{O} \oplus L)$ with L of degree 0 but not torsion.

In the second part of this section we investigate more generally the structure of normal projective varieties X such that $-K_X$ is pseudo-effective. Our leitfaden is the following

4.13. Problem. *Let X and Y be normal projective $\mathbb{Q}$-Gorenstein varieties. Let $\varphi : X \to Y$ be a surjective morphism. If $-K_X$ is pseudo-effective* (*almost nef*)*, is $-K_Y$ pseudo-effective* (*almost nef*) *?*

This problem in general has a negative answer:

4.14. Example. *Let C be any curve of genus $g \geqslant 2$. Let L be a line bundle on C and put $X = \mathbb{P}(\mathcal{O} \oplus L)$. Then we have*

$$\begin{aligned} H^0(-K_X) &= H^0(X, \mathcal{O}_{\mathbb{P}(\mathcal{O}\oplus L)}(2) \otimes \pi^*(L^* \otimes -K_C)) \\ &= H^0(C, S^2(\mathcal{O} \oplus L) \otimes L^* \otimes -K_C) \neq 0, \end{aligned}$$

if deg $L^* > 3g - 2$. So $-K_X$ is effective, hence pseudo-effective (hence also almost nef by (1.5)), however $-K_C$ is not pseudo-effective.

It is easily checked in this example that $-K_X \cdot B \geqslant 0$ for all curves $B \subset X$ with the only exception $B = C_0$, C_0 the unique exceptional section of X. In particular the non-nef locus of $-K_X$ projects onto C. This leads us to reformulate (4.3) in the following way:

4.13.a. Problem. *Assume moreover in* (4.13) *that the non-nef locus of $-K_X$ does not project onto Y. Is $-K_Y$ pseudo-effective?*

In case $-K_X$ is pseudo-effective the answer to (4.13.a) is positive at least in the case of submersions but with a slightly stronger assumption than in (4.13.a), replacing the non-nef locus by the zero locus of the multiplier ideal sheaf associated with a metric of minimal singularities (4.15 below) while in the almost nef case we have only a weak answer (4.17 below), which is however valid for general ϕ and deals with the non-nef locus.

4.15. Theorem. *Let X and Y be compact Kähler manifolds. Let $\varphi : X \to Y$ be a surjective submersion. Suppose that $-K_X$ is pseudo-effective and that the zero locus of the multiplier ideal of a minimal metric of $-K_X$ does not project onto Y. Then $-K_Y$ is pseudo-effective.*

Proof. This is a consequence of (2.6.1). In fact, apply (2.6.1) with $L = -K_X$. □

4.16. Definition. *A ($\mathbb{Q}$-)line bundle L on a normal n-dimensional projective variety X is generically nef if*

$$L \cdot H_1 \cdots H_{n-1} \geqslant 0$$

for all ample divisors H_i on X.

4.17. Theorem. *Let X and Y be normal projective $\mathbb{Q}-$Gorenstein varieties and let $\varphi : X \to Y$ be surjective. Let $-K_X$ be almost nef with non-nef locus B. Assume $\varphi(B) \neq Y$. Then $-K_Y$ is generically nef.*

Proof. We will use the method of [Zh96] in which Zhang proves the surjectivity of the Albanese map for projective manifolds with $-K_X$ nef. As a generalization of Prop. 1 in [Zh96] we claim

(4.17.1) *Let $\pi : X \to Z$ be a surjective morphism of smooth projective varieties. Then there is no ample divisor A on Z such that $-K_{X|Z} - \delta\varphi^*(A)$ is pseudo-effective (almost nef) with non-nef locus B not projecting onto Z for some $\delta > 0$, unless* $\dim Z = 0$.

Proof of (4.17.1). The proof is essentially the same as the one of Theorem 2 in [Miy93], where Miyaoka proves that $-K_{X|Z}$ cannot be ample. Just replace $\mathrm{Sing}(\pi)$ in Lemma 10 by $\mathrm{Sing}(\pi) \cup B$. Then, assuming the existence of A and δ, the old arguments work also in our case. This proves (4.17.1).

Now, coming to our previous situation, we argue as in [Zh96]. Take $C \subset Y$ be a general complete intersection curve cut out by $m_1H_1, \ldots, m_{n-1}H_{n-1}$, with H_i ample, $m_i > 0$. Let $X_C = \varphi^{-1}(C)$. By Bertini, C and X_C are smooth. Applying (4.17.1) to $X_C \to C$ it follows that

$$-K_{X_C|C} - \delta\varphi^*(A)$$

is never pseudo-effective (almost nef) with non-nef locus not projecting onto C for any choice of A and δ. On the other hand

$$-K_{X_C/C} = -K_{X/Z}|X_C = -K_X|X_C + \varphi^*(K_Z|C).$$

Since $-K_X$ is pseudo-effective with non-nef locus not projecting onto Z, so does $-K_X|X_C$ and we conclude that $K_Z|C$ cannot be ample, i.e. $K_Z \cdot C \leqslant 0$, which was to be proved. □

4.18. Corollary. *Let X be a normal projective $\mathbb{Q}$-Gorenstein variety. Assume $-K_X$ almost nef with non-nef locus B.*

a) *If $\varphi : X \to Y$ is a surjective morphism to a normal projective $\mathbb{Q}$-Gorenstein variety Y with $\varphi(B) \neq Y$, then $\kappa(Y) \leqslant 0$.*

b) *The Albanese map* $\alpha : X \to \mathrm{Alb}(X)$ *is surjective, if* $\alpha(B) \neq \alpha(X)$.

The next question asks for the fundamental group of varieties X with $-K_X$ pseudo-effective and small non-nef locus. Here of course we need more assumptions on the singularities. A *terminal* (*canonical*) n-fold is a normal projective $\mathbb{Q}$-factorial variety with at most *terminal* (*canonical*) singularities.

4.19. Question. Let X be a terminal (or canonical) variety with $-K_X$ pseudo-effective (almost nef). Assume that $\alpha(B) \neq \alpha(X)$, with $\alpha : X \to \mathrm{Alb}(X)$ the Albanese and B the non-nef locus of $-K_X$. Is $\pi_1(X)$ almost Abelian (i.e. Abelian up to finite index)?

If $\alpha : X \to Y$ is a morphism, we will say that a line bundle L on X is *properly pseudo-effective (properly almost nef)* with respect to α, if L is pseudo-effective (resp. nef) and the non-nef locus satisfies $\alpha(B) \neq \alpha(X)$.

4.20. Remarks.

a) If $-K_X$ is Hermitian semi-positive, then $\pi_1(X)$ is almost Abelian by [DPS96b].

b) If $-K_X$ nef, then $\pi_1(X)$ has at most polynomial growth by Paun [Pau97].

c) If X is a projective surface such that $-K_X$ is properly pseudo-effective with respect to the Albanese map, then $\pi_1(X)$ is almost Abelian. This is a consequence of the Kodaira-Enriques classification and is proved as follows. First it is clear that $\kappa(X) \leqslant 0$. If $\kappa(X) = 0$, there is nothing to be proved, X being birationally a torus, hyperelliptic, K3 or Enriques surface. So let $\kappa(X) = -\infty$. By (4.4),(4.5), X does not admit a map to a curve of genus $\geqslant 2$, so that X is either rational or its minimal model is a ruled surface over an elliptic curve. Thus $\pi_1(X) = 0$ or $\pi_1(X) = \mathbb{Z}^2$. $\square$

In dimension 3, (4.20) has still a positive answer, at least if X is projective.

4.21. Theorem. *Let X be a terminal 3-fold such that $-K_X$ is properly pseudo-effective with respect to the Albanese map. Then $\pi_1(X)$ is almost Abelian.*

Proof. By [Mor88] there exists a finite sequence $\varphi : X \rightharpoonup X'$ of extremal birational "divisorial" contractions and flips such that either $K_{X'}$ is nef or X' carries a Fano fibration $\varphi : X' \to Y$, i.e. an extremal contraction with $\dim Y < \dim X'$. Since extremal contractions and flips leave the fundamental group unchanged, we have

$$\pi_1(X) = \pi_1(X').$$

a) We claim that $-K_{X'}$ is properly pseudo-effective with respect to Albanese. For that we need to prove the following. If $\lambda : X \to Z$ is an extremal divisorial contraction (i.e. λ contracts a divisor) or if $\lambda : X \rightharpoonup X^+$ is a flip, then $-K_Z$ (resp $-K_{X^+}$) is properly pseudo-effective if $-K_X$ is properly pseudo-effective. Indeed, if $\lambda : X \to Z$ is divisorial, then $\lambda_*(-K_X) = -K_Z$ as $\mathbb{Q}$-Cartier divisors and moreover $\lambda_* : N^1(X) \to N^1(Z)$ maps effective divisors to effective divisors, hence pseudo-effective divisors to pseudo-effective divisors. If $\lambda : X \rightharpoonup X^+$ is small, we still have a natural map

$$\lambda_* : N^1(X) \to N^1(Z)$$

with the same properties as above. In fact, if $\mathcal{L} \in \mathrm{Pic}(X)$, then let $C \subset X$ and $C^+ \subset X^+$ be the 1-dimensional indeterminacy sets so that $X \smallsetminus C \simeq X^+ \smallsetminus C^+$, and consider $\mathcal{L}^+ = \lambda_*(\mathcal{L}|X \setminus C)$. $\mathcal{L}^+$ can be extended to a reflexive sheaf on X^+, however since X is $\mathbb{Q}$-factorial, so does X^+ ([KMM87]), and some $(\mathcal{L}^+)^m$ extends to a line bundle $\tilde{\mathcal{L}}^{[m]}$ on X^+. Now

$$H^0(X^+, \tilde{\mathcal{L}}^{[m]}) = H^0(X^+ \setminus C^+, (\mathcal{L}^+)^m) = H^0(X \setminus C, \mathcal{L}^m) = H^0(X, \mathcal{L}^m).$$

In total $-K_{X^+}$ is again pseudo-effective.

It is now clear that if $-K_X$ is *properly* pseudo-effective with respect to Albanese, then so is $-K_Z$ (resp. $-K_{X^+}$).

b) Now let $\varphi : X' \to Y$ be an extremal contraction with $\dim Y \leqslant 2$.

By a) we may assume $X = X'$. If $\dim Y = 0$, then X is $\mathbb{Q}$-Fano, hence rationally connected by [KoMM92], and in particular $\pi_1(X) = 0$. If $\dim Y = 1$, then Y is a smooth curve with genus $g \leqslant 1$ by (4.17). Therefore $\pi_1(X) = \pi_1(Y) = 0$ or $\mathbb{Z}^2$. If $\dim Y = 2$, then $\kappa(\hat{Y}) \leqslant 0$, where $\hat{Y}$ is a desingularization, again by (4.17). If $\kappa(\hat{Y}) = 0$, then $\hat{Y}$ is birational to a torus, a hyperelliptic surface, a $K3$ surface or an Enriques surface. Thus $\pi_1(\hat{Y})$ is (almost) Abelian. Since Y has at most quotient singularities, we have $\pi_1(\hat{Y}) = \pi_1(Y)$, hence $\pi_1(X) = \pi_1(Y)$ is (almost) Abelian.

c) Finally assume K_X nef. Since $-K_X$ is pseudo-effective, we must have $K_X \equiv 0$. Therefore $mK_X = \mathcal{O}_X$ for suitable $m > 0$, and $\pi_1(X)$ is almost Abelian by [Kol95]. □

4.22. Remarks.

a) If $-K_X$ is properly almost nef in (4.18) instead of properly pseudo-effective, then our conclusion still holds. The proof is essentially the same. The only change concerns the invariance under flips. This follows from [KMM87, 5-1-11].

b) To prove (4.21) in any dimension with the methods presented here, would require several deep things. First of all we would need the minimal model program working in any dimension.

Second we need to know that, given an extremal contraction $\varphi : X \to Y$ with $< \dim X$, then $-K_Y$ is properly pseudo-effective, if $-K_X$ is properly pseudo-effective; i.e. we would need a positive answer to (4.13a)) at least in the case of an extremal contraction.

And last we would need to know that $\pi_1(X)$ is almost Abelian if $K_X \equiv 0$. This is well known if X is smooth [Bea83] but hard if X is singular, $\dim X \geqslant 4$. Compare [Pet93]. □

§5. Threefolds with pseudo-effective canonical classes

If X is a smooth projective threefold or more generally a normal projective threefold with at most terminal singularities such that K_X is pseudo-effective, then actually some multiple mK_X is effective, i.e. $\kappa(X) \geq 0$. This is one of the main results of Mori theory and is in fact a combination of Mori's theorem that a projective threefold X has a model X' with either X' uniruled or with $K_{X'}$ nef and of Miyaoka's theorem that threefolds with K_X nef have $\kappa(X) \geq 0$. In particular reduction to char p is used and it is very much open whether the analogous result holds in the Kähler category. In this section we give a very partial result in this direction.

5.1. Lemma. *Let X be a normal compact Kähler space with at most isolated singularities and L a pseudo-effective line bundle on X. Let h be a singular metric on L with curvature $\Theta_h(L) \geq 0$ (in the sense of currents). Let φ be the weight function of h. Assume that the Lelong numbers satisfy $\nu(\varphi, x) = 0$ for all $x \in X$ but a countable set. Then L is nef.*

Proof. See [Dem92], Corollary 6.4. □

Now consider a pseudo-effective line bundle L which is not nef. Choose a metric h_0 with minimal singularities in the sense of Theorem 1.5 and denote by T_0 its curvature current. Let

$$E_c = \{x \in X | \nu(x, \varphi_0(x)) \geq c\}.$$

By [Siu74] (see also [Dem87]), E_c is a closed analytic set. Then Lemma 5.1 tells us that $\dim E_c \geq 1$ for sufficiently small $c > 0$. Furthermore we notice

5.2. Lemma. *There exists $c > 0$ such that $L|E_c$ is not nef. Moreover, given an irreducible codimension 1 (in X) component $D \subset E_c$, the line bundle $L \otimes \mathcal{O}_X(-aD)|D$ is pseudo-effective for a suitable $a > 0$.*

Proof. In fact, by [Pau98b, Théorème 2], a closed positive current T has a nef cohomology class $\{T\}$ if and only if the restriction of $\{T\}$ to all components Z of all sets $E_c(T)$ is nef. Thus, as $\{T_0\}$ is not nef, there must be some E_c such that $\{T_0\}|E_c$ is not nef. The second assertion follows from Siu's decomposition

$$T_0 = \sum a_j D_j + R,$$

where the D_j are irreducible divisors, say $D_1 = D$, and R is a closed positive current such that $\operatorname{codim} E_c(R) \geqslant 2$ for every c. Then $R|D$ is pseudo-effective (as one sees by applying the main regularization theorem for $(1,1)$-currents in [Dem92]), and $D_j|D$ is pseudo-effective for $j \geqslant 2$, thus $\{T_0 - a_1 D_1\}|D$ is pseudo-effective. □

5.3. Corollary. *Let S be a compact Kähler surface and L a line bundle on S. Assume that L is pseudo-effective and that $L \cdot C \geq 0$ for all curves $C \subset S$. Then L is nef.*

Proof. Introduce a singular metric h on L whose curvature current is positive (see 5.1). Let $c > 0$ and let E_c be the associated Lelong set of the weight function of h. Then by 5.1 we can find c such that $\dim E_c = 1$. Suppose that L is not nef. Then by Lemma 5.2, there exists a curve $C \subset E_c$ such that $L \cdot C < 0$. This is a contradiction. □

Of course this proof does not extend to dimension 3, because now we might find a surface A such that $L|A$ is not nef. However $L|A$ might not be pseudo-effective, so there is no conclusion. On the other hand, the situation for $L = K_X$ is much better:

5.4. Theorem. *Let X be a $\mathbb{Q}$-factorial normal 3-dimensional compact Kähler space with at most isolated singularities. Suppose that K_X is pseudo-effective but not nef. Then there exists an irreducible curve $C \subset X$ with $K_X \cdot C < 0$.*

If X is smooth, then C can be choosen to be rational, and there exists a surjective holomorphic map $f : X \longrightarrow Y$ with connected fibers contracting C to a point, such that $-K_X$ is f-ample.

Proof. We only need to prove the existence of an irreducible curve C with $K_X \cdot C < 0$; the rest in case X is smooth follows from [Pet98, Pet99].

By Lemma 5.2, a suitable set E_c contains a positive-dimensional component S with $K_X|S$ not nef. If $\dim S = 1$, we are done, so suppose that S is an irreducible surface. Let $\mu : \tilde{S} \longrightarrow S$ denotes normalization followed by the minimal desingularization. Let $L = \mu^*(K_X|S)$. Then we need to show that there is a curve $C \subset \tilde{S}$ such that

$$L \cdot C < 0.$$

Since L is not nef, this is clear if $\tilde{S}$ is projective. So we may assume $\tilde{S}$ non-algebraic.

By Lemma 5.2, we find a positive number a such that $L + \mu^*(N_S^{*a})$ is pseudo-effective, hence by adjunction

$$\mu^*(K_X^{(1+a)}|S - K_S^a)$$

is pseudo-effective. On the other hand by subadjunction, $\mu^*(K_S) = K_{\tilde{S}} + B$ with B effective. Since $K_{\tilde{S}}$ is effective, $\tilde{S}$ being non-algebraic, we conclude that L is pseudo-effective. Now we apply Corollary 5.3 and conclude. □

The reason why we can construct a contraction in Theorem 5.4 is as follows. If X is a smooth Kähler threefold with $K_X \cdot C < 0$, then C moves in a positive-dimensional family, and therefore one can pass to a non-splitting family. This family provides the contraction, see [CP97]. These arguments are likely to work also in the Gorenstein case but break down in the presence of non-Gorenstein singularities. Here it can happen that C does not deform and new arguments are needed.

§6. Pseudo-effective vector bundles

In this section we discuss pseudo-effective vector bundles with special emphasis on the tangent bundle of a projective manifold.

6.1. Definition. *Let X be a projective manifold and E a holomorphic vector bundle on X. Then E is pseudo-effective, if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective and the union of all curves C with $\mathcal{O}(1) \cdot C < 0$ (i.e. the non-nef locus of the almost nef line bundle $\mathcal{O}(1)$) is contained in a countable union of subvarieties which do not project onto X.*

6.2. Remark. *Notice that E is pseudo-effective if and only if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective and additionally there is a countable union S of proper subvarieties of X such that $E|C$ is nef for every curve not contained in S.*

We have the following cohomological criterion for pseudo-effectivity.

6.3. Proposition. *Let X be a projective manifold and E a vector bundle on X. Then E is pseudo-effective if and only if there exists an ample line bundle A on X and* *positive integers m_0 and $n_0(m)$ such that $H^0(X, S^n(S^m E \otimes A))$ generically generates $S^n(S^m E \otimes A)$ for all $m \geqslant m_0$ and $n \geq n_0(m)$.*

Proof. If such an A exists, then clearly E is pseudo-effective, using (1.2). Working in the other direction, we choose an ample line bundle H on $\mathbb{P}(E)$ by $H = \mathcal{O}(1) \otimes \pi^*(A)$ with A ample on X. Now let $x \in X$ be a very general point and let $F = \pi^{-1}(x)$ the fiber over $\pi : \mathbb{P}(E) \to X$. We must prove that

$$H^0(\mathbb{P}(E), (\mathcal{O}(m) \otimes H)^{\otimes k}) \to H^0(F, (\mathcal{O}(m) \otimes H)^{\otimes k}|F)$$

is surjective for $k \gg 0$. Suppose first E nef. Then reduce inductively to $\dim X = 1$; the necessary H^1-vanishing is provided by Kodaira's vanishing theorem for sufficiently large k. In case $\dim X = 1$, the ideal sheaf $\mathcal{I}_F$ is locally free of rank 1 and

$$H^1(\mathbb{P}(E), \mathcal{I}_F \otimes (\mathcal{O}(m) \otimes H)^k) = 0$$

again holds by Kodaira for large k. In the general case one introduces multiplier ideal sheaves associated with singular metrics on $\mathcal{O}(m)$ whose support do not meet F and substitutes Kodaira's vanishing theorem by Nadel's vanishing theorem. We leave the easy details to the reader. □

6.4. Definition. *Let X be a projective manifold and E a vector bundle on X. Then E is said to be almost nef, if and only if there is a countable family A_i of proper subvarieties of X such that $E|C$ is nef for all $C \not\subset \bigcup_i A_i$. The non-nef locus of E is the smallest countable union S of analytic subsets such that $E|C$ is nef for all $C \not\subset S$.*

It is immediately seen that E is almost nef if and only if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is almost nef with non-nef locus not projecting onto X. Hence we have in analogy to (3.3) the following

6.5. Proposition. *Let X be a projective manifold and E a vector bundle on X. If E is pseudo-effective, then E is almost nef.*

6.6. Problem. Is every almost nef vector bundle pseudo-effective ?

Here are some basic properties of pseudo-effective and almost nef vector bundles.

6.7. Theorem. *Let X be a projective manifold and E a vector bundle on X.*

a) *If E is pseudo-effective (almost nef) and Γ^a is any tensor representation, then $\Gamma^a E$ is again pseudo-effective (almost nef). In particular $S^m E$ and $\Lambda^q E$ are pseudo-effective (almost nef).*

b) *If E is almost nef and if $s \in H^0(E^*)$ is a non-zero section, then s has no zeroes at all.*

c) *If either E is pseudo-effective or if E is almost nef with non-nef locus S having codimension at least 2, and if $\det E^*$ is almost nef, then E is numerically flat i.e. both E and E^* are nef, and then E has a filtration by Hermitian flat bundles.*

Proof. a) By standard representation theory it is sufficient to prove the statement for $E^{\otimes m}$. But in that case the claim is immediate.

b) Let $S \subset X$ be the non-nef locus. Suppose $s(x) = 0$. Choose a curve $C \not\subset S$ such that $x \in C$. Then $E|C$ is nef, on the other hand $E^*|C$ has a section with zeroes. So $s|C = 0$. Varying C we conclude $s = 0$.

c) First notice that our claim is easy for line bundles L, even without assumption on the codimension: if both L and L^* are almost nef, then clearly $L \cdot H_1 \cdots H_{n-1} = 0$ for all ample line bundles H_i on X ($n = \dim X$). Thus $L \equiv 0$. In particular we conclude from (a) that $\det E \equiv 0$.

We now treat the case when E is almost nef with $\operatorname{codim} S \geq 2$.

c.1) First we claim that E is H-semistable for all ample line bundles H. In fact, otherwise we find some H and a torsion free subsheaf $\mathcal{S} \subset E$ such that $c_1(\mathcal{S}) \cdot H^{n-1} > 0$. Now, assuming H very ample, let C be a general complete intersection curve cut out by H. Then $\mathcal{S}|C$ is locally free and $c_1(\mathcal{S}|C) > 0$. On the other hand $E|C$ is numerically flat (since it is nef and $\det E^*|C$ is nef, see [DPS94]). This is impossible. So E is H-semistable for all H.

c.2) As a consequence we obtain the inequality

$$0 = (r-1)c_1^2(E) \cdot H^{n-2} \leq 2rc_2(E) \cdot H^{n-2}$$

for all H ample, r denoting the rank of E.

c.3) Next suppose that E is H-stable for some H (and still that $\operatorname{codim} S \geq 2$). Let Y be a general surface cut out by hyperplane sections in H (again assume H very ample). Then $Y \cap S = \emptyset$ by our assumption on the codimension of S, hence $E|Y$ is nef, hence numerically flat and we conclude

$$c_2(E) \cdot H^{n-2} = 0. \tag{6.7.1}$$

Since E is H-stable, E is Hermite-Einstein and from (6.7.1) we deduce (see e.g. [Kob87, p.115]) that E is numerically flat. Hence we may assume that E is H-semistable for all H but never H-stable.

c.4) Fix some ample line bundle H and let $\mathcal{S}$ the maximal H-destabilizing subsheaf of E, so that

$$c_1(\mathcal{S}) \cdot H^{n-1} = 0. \tag{6.7.2}$$

On the other hand the generically surjective map $E^* \to \mathcal{S}^*$ proves that

$$c_1(\mathcal{S}^*) \cdot H_1 \cdots H_{n-1} \geq 0$$

for all ample divisors H_i. Together with (6.7.2) this yields

$$c_1(\mathcal{S}) = 0.$$

Now we follows the arguments of [DPS94]. Let $p = \mathrm{rk}\mathcal{S}$. Then $\det \mathcal{S}$ is a numerically flat line bundle, moreover it is a subsheaf of $\Lambda^p E$, hence by (6.7 a), $\det \mathcal{S}$ is a subbundle of $\Lambda^p E$, and thus by [DPS94,1.20], $\mathcal{S}$ is a subbundle of E. Now $\mathcal{S}$ being almost nef with trivial determinant, an induction on the rank of E yields the numerical flatness of $\mathcal{S}$.

For the same reason the quotient bundle $E/\mathcal{S}$ is numerically flat, too, so that E is numerically flat.

c.5) Finally assume that E is pseudo-effective and suppose $S \neq \emptyset$. Choose a general smooth curve C meeting S in a finite set. Then E_C is nef with $\det E_C \equiv 0$. Therefore E_C is numerically flat. Let $x \in C \cap S$ with maximal ideal sheaf $\mathcal{I}_x \subset \mathcal{O}_C$ and let A be ample on X. Then, applying 6.3 and having in mind that $S^m E|C'$ is not nef (with C' a suitable curve in S passing through x), the image of the restriction map

$$H^0(X, S^k(S^m E \otimes A)) \longrightarrow H^0(C, S^k(S^m(E_C \otimes A))$$

has non-zero intersection with

$$H^0(C, S^k(S^m(E_C \otimes \mathcal{I}_x) \otimes A))$$

for m large and sufficiently divisible and for large k. This contradicts the numerical flatness of E_C. $\square$

6.8. Remark. Of course we expect that every almost nef bundle E with $\det E \equiv 0$ is numerically flat. The above considerations show that it is sufficient to prove this only on surfaces and for bundles which are stable (for all polarizations). We formulate the problem precisely as follows.

6.9. Problem. Let Y be a smooth projective surface and E a vector bundle on Y of rank at least 2. Let E be an almost nef vector bundle on X. *If* $\det E \equiv 0$, *show that* E *is nef, hence numerically flat.*

It would be sufficient to prove that $c_2(E) = 0$ and also the proof of (6.7) shows that one may assume E to be $H-$stable for any ample polarization H on Y. It is *a priori* clear that E is always $H-$semi-stable.

We now want to study projective manifolds with almost nef tangent bundles. A class of examples is provided by the almost homogeneous manifolds X, i.e. the automorphism group acts with an open orbit, or equivalently, the tangent bundle is generically generated. A question we have in mind is how far the converse is from being true.

6.10. Proposition. *Let* X *be a projective manifold with* T_X *almost nef. Then* $\kappa(X) \leqslant 0$. *If* $\kappa(X) = 0$, *then* $K_X \equiv 0$.

Proof. Since T_X is almost nef, $-K_X$ is almost nef, too. If $\kappa(X) \geqslant 0$, then mK_X is effective, therefore $K_X \cdot H_1 \cdots H_{n-1} = 0$ for all hyperplane sections H_i, hence $K_X \equiv 0$. $\square$

6.11. Proposition. *Let* X *be a projective manifold with* $K_X \equiv 0$ *and* T_X *pseudo-effective. Then* X *is Abelian after a suitable finite étale cover.*

Proof. By (6.7) T_X is numerically flat and the claim follows by Yau's theorem. $\square$

6.12. Corollary. *Let* X *be a Calabi-Yau or projective symplectic manifold. Then neither* T_X *nor* Ω^1_X *is pseudo-effective. Moreover the union of curves* $C \subset X$ *such that* $T_X|C$ *(resp.* $\Omega^1_X|C$*)) is not nef is not contained in a countable union of analytic sets of codimension at least* 2.

Of course we expect (see Problem 6.6) that the tangent bundle of a Calabi-Yau or symplectic manifold is not almost nef, but this seems rather delicate already for K3 surfaces. Since both bundles are generically nef by Miyaoka's theorem, we get examples of generically nef vector bundles (of rank at least 2) which are not pseudo-effective.

6.13. Example. Let $X \subset \mathbb{P}_3$ be a general quartic (with $\rho(X) = 1$). Then T_X is not pseudo-effective by (6.11). More precisely :

$$H^0(X, S^k(S^m T_X \otimes \mathcal{O}_X(1))) = 0 \qquad \text{for all } m \geqslant 1 \text{ and all } k \geqslant 1.$$

Here $\mathcal{O}_X(1)$ is the ample generator of $Pic(X) \cong \mathbb{Z}$. This is proved by direct calculation in [DPS96a] and actually shows that $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ is not pseudo-effective. The same argument works for smooth hypersurfaces $X \subset \mathbb{P}_{n+1}$ of degree $n+2$ with $\rho(X) = 1$.

Continuing this example we are now able to exhibit a generically nef line bundle L which is not pseudo-effective. Namely, let

$$L = \mathcal{O}_{\mathbb{P}(T_X)}(1).$$

As already noticed, L is not pseudo-effective. So it remains to verify that L is generically nef which is of course based on the generic nefness of T_X. Let H_1 and H_2 be ample divisors on $\mathbb{P}(T_X)$, then we must prove

$$L \cdot H_1 \cdot H_2 \geq 0.$$

Let $m_i \gg 0$ and choose $D_i \in |m_i H_i|$. Let

$$\tilde{C} = D_1 \cap D_2$$

and let $C = \pi(\tilde{C})$, where $\pi : \mathbb{P}(T_X) \to X$ is the projection. Then certainly $\deg(\tilde{C}/C) \geq 2$; moreover

$$C \in |\mathcal{O}_X(k)|,$$

where k is so large that $T_X|C_0$ is nef for the general $C_0 \in |\mathcal{O}_X(1)|$. Thus $\mathcal{O}_{\mathbb{P}(T_X|C_0)}(1)$ is nef. By semi-continuity (for $\mathcal{O}_{\mathbb{P}(T_X|C_0)}(m) \otimes A$ with A ample on $\mathbb{P}(T_X)$), we conclude that $\mathcal{O}_{\mathbb{P}(T_X|C)}(1)$ is at least pseudo-effective. Since by reasons of degree, $\tilde{C}$ cannot be the exceptional section of $\mathbb{P}(T_X|C)$, we conclude that

$$m_1 m_2 (L \cdot H_1 \cdot H_2) = c_1(\mathcal{O}_{\mathbb{P}(T_X|C)}(1)|\tilde{C}) \geq 0,$$

proving our claim. □

We now study projective manifolds X with T_X almost nef and $\kappa(X) = -\infty$.

6.14. Proposition. *Let X be a projective manifold with T_X almost nef and $\kappa(X) = -\infty$. Then X is uniruled.*

Proof. In fact, since T_X is almost nef we have

$$K_X \cdot H_1 \cdots H_{n-1} \leq 0$$

for all ample divisors H_i on X. We must have strict inequality for some choice of H_i because otherwise $K_X \equiv 0$ and $\kappa(X) = 0$. Now [MM86] gives the conclusion. □

6.15. Proposition. *Let X be a projective manifold with T_X almost nef. Then*

a) *the Albanese map is a surjective submersion,*

b) *there is no surjective map onto a variety Y with $\kappa(Y) > 0$.*

Proof. In fact, all holomorphic 1-forms on X have no zeroes, and this proves a) (compare with [DPS94]). Point b) follows from the analogous fact that sections in $S^m\Omega^p_X$ cannot have zeroes. □

For the following considerations we recall from (4.20) the following notation:

$$\tilde{q}(X) = \sup\{q(\tilde{X}) \mid \tilde{X} \to X \text{ finite étale}\}.$$

6.16. Proposition. *Let T_X be pseudo-effective and suppose that $\pi_1(X)$ does not contain a non-Abelian free subgroup. Then if $H^0(\Omega^p_X) \neq 0$ for some p, we have $q(\tilde{X}) > 0$.*

Proof. The proof is contained in [DPS94, 3.10]. □

Notice that $\pi_1(X)$ does not contain a free non-Abelian free subgroup if $-K_X$ is nef [DPS93].

In order to make further progress we need informations on manifolds with T_X almost nef which do not have p-forms for all p, the same also being true for every finite étale cover. □

6.17. Conjecture. *Let T_X be almost nef. Assume that $H^q(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for all finite étale covers and all $q \geq 1$. Then $\pi_1(X) = 0$, in fact X should even be rationally connected.*

The evidence for the conjecture is that it holds if T_X is nef by [DPS94]; it is furthermore true if X is almost homogeneous and it is true in low dimensions, as we shall see below. With the same arguments as in [DPS94], we obtain

6.18. Proposition. *Let T_X be almost nef and suppose that $\pi_1(X)$ does not contain a non Abelian free subgroup. If Conjecture 6.17 holds, then $\pi_1(X)$ is almost Abelian, i.e. $\pi_1(X)$ contains a subgroup $\mathbb{Z}^r$ of finite index.*

6.19. Proposition. *Let X be a smooth projective surface with T_X almost nef and $\kappa(X) = -\infty$.*

a) *If $q(X) > 0$, then X is ruled surface over an elliptic curve and every ruled surface over an elliptic curve has T_X almost nef.*

b) *The minimal model Y has again T_Y almost nef.*

c) *Every rational ruled surface has an almost nef tangent bundle.*

Proof. a) The first statement is clear from (6.15), the second part is done as follows. Let $p : X \to C$ be the projection to the elliptic curve. Then consider the tangent bundle sequence

$$0 \to T_{X/C} \to T_X \to \mathcal{O}_X \to 0.$$

Then $T_{X/C} = -K_X$, and the claim follows from the pseudo-effectivity of $-K_X$, which is immediately checked by [Ha77,V.2].

b) is obvious.

c) Such an X is actually almost homogeneous, see [Po69]. □

We do not investigate the rather tedious problem of determining which rational blow-ups have almost nef tangent bundles, and instead turn ourselves to the case of 3-folds.

6.20. Theorem. *Let X be a smooth projective 3-fold with T_X almost nef. Then $\pi_1(X)$ is almost Abelian and if $\tilde{q}(X) = 0$, then X is rationally connected unless (possibly) T_X is not pseudo-effective and X is a $\mathbb{P}_1-$bundle over a K3- or an Enriques surface (this case should not exist). Suppose $\kappa(X) = -\infty$. Then the finer structure of X is as follows:*

a) *If $q(X) = 2$, then X is a $\mathbb{P}_1$-bundle over an Abelian surface.*

b) *If $q(X) = 1$, then the Albanese map $\alpha : X \to A$ is a fiber bundle over the elliptic curve with general fiber F having T_F almost nef and $\kappa(F) = -\infty$.*

   α) *If F is rational, then there is a factorization $X \to Y \to A$ such that either $f : X \to Y$ is birational, in that case f is a succession of blow-ups of étale multisections over A. Or $\dim Y < \dim X$, in that case $f : X \to Y$ and $g : Y \to A$ are both $\mathbb{P}_1$-bundles or $\alpha = \phi$ is a $\mathbb{P}_2$- or a $\mathbb{P}_1 \times \mathbb{P}_1$-bundle.*

   β) *If F is a ruled surface over an elliptic curve, then f is a $\mathbb{P}_1$-bundle and $g : Y \to A$ is a hyperelliptic surface. Then there is an étale $2:1$-cover $\tilde{X} \to X$ such that $q(\tilde{X}) = 2$.*

Proof. If $\kappa(X) = 0$, then we have $K_X \equiv 0$ and $\pi_1(X)$ is Abelian by [Bea83]. So we will now suppose $\kappa(X) = -\infty$.

(I) First we treat the case $\tilde{q}(X) = 0$. By (6.14), X is uniruled, so we can form the rational quotient $f : X \rightharpoonup Y$ with respect to some covering family of rational curves ([Cam81, Cam92], [KoMM92]). Then Y is a projective manifold with $\dim Y \leq 2$. If $\dim Y = 0$, then X is rationally connected. If $\dim Y = 1$, then $\tilde{q}(X) = 0$ implies $Y = \mathbb{P}_1$, hence X is rationally connected by [KoMM92]. If finally $\dim Y = 2$, then from $\tilde{q}(Y) = 0$ and $\kappa(Y) \leq 0$ (Proposition 6.15), we deduce that either Y is rational or a K3 resp. an Enriques surface. In the rational case it is easy to see and well-known that X is rationally connected. So suppose Y is not rational. Since an Enriques surface has a finite étale cover which is K3, we may assume that Y is actually K3. Then X carries a holomorphic 2-form ω. Since T_X is almost nef, ω cannot have zeroes by (6.7) (b)). Therefore X is a $\mathbb{P}_1-$bundle over Y by [CP00]. Now suppose T_X pseudo-effective. Then $f^*(T_Y)$ is pseudo-effective as quotient of T_X. Since $c_1(f^*(T_Y)) = 0$, the bundle $f^*(T_Y)$ is numerically flat by (6.7), in particular $f^*(c_2(Y)) = 0$ which is absurd.

(II) We will now assume $q(X) > 0$ and shall examine the structure of the Albanese map $\alpha : X \to A$. Once we have proved the structure of α as described in 1) and 2), it is clear that $\pi_1(X)$ is almost Abelian. We already know that α is a surjective submersion.

a) If $\dim A = 2$, then α is a $\mathbb{P}_1$-bundle.

b) Suppose now $\dim A = 2$. Let F be a general fiber of α. Then $\kappa(F) = -\infty$, since X is uniruled (Proposition 6.14), moreover T_F is almost nef, hence either F is rational or a ruled surface over an elliptic curve (Proposition 6.19). We will examine both cases by studying a Mori contraction $\phi : X \to Z$ which induces a factorization $\beta : Z \to A$ such that

$$\alpha = \beta \circ \phi.$$

Note that all possible ϕ are classified in [Mor82].

b.1) Assume that F is rational. If $\dim Z = 2$, then ϕ is a $\mathbb{P}_1$-bundle and β is again a $\mathbb{P}_1$-bundle. If $\dim Z = 1$, then $A = Z$ and $\alpha = \phi$ is a $\mathbb{P}_2$- or a $\mathbb{P}_1 \times \mathbb{P}_1$-bundle. If finally $\dim Z = 3$, then ϕ must be the blow-up of an étale multisection of β and T_Z is again almost nef so that we can argue by induction on $b_2(X)$.

b.2) If F is irrational, then ϕ is necessarily a $\mathbb{P}_1$-bundle over Z and then $\beta : Z \to A$ is an elliptic bundle with $\kappa(Z) = 0$, hence Z is hyperelliptic. □

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