COMPACT COMPLEX MANIFOLDS WITH NUMERICALLY EFFECTIVE TANGENT BUNDLES

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Contents

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0. Introduction

The compact Riemann surfaces which are not of general type are those which are of semipositive curvature. In higher dimensions the situation is much more subtle and it has been a long standing conjecture due to Frankel to characterize $\mathbb{P}_n$ as the only compact Kähler manifold with positive holomorphic bisectional curvature. Hartshorne extended Frankel's conjecture and weakened the assumption to the case of ample tangent bundle. In his famous paper [Mo79] Mori solved the Hartshorne conjecture by using characteristic p methods. Around the same time Siu and Yau [SY80] gave an (analytic) proof of the Frankel conjecture. Combining algebraic and analytic tools Mok [Mk88] classified all compact Kähler manifolds with semipositive holomorphic bisectional curvature.

In algebraic geometry a powerful and flexible notion of semipositivity is *numerical effectivity* ("nefness"). A line bundle L on a projective manifold X is said to be *nef* if $L \cdot C \geq 0$ for all compact curves $C \subset X$. It is clear that a line bundle with semipositive curvature is nef. The converse had been conjectured by Fujita [Fu83]. Unfortunately this is not true as we shall see in §1, example 1.7.

A vector bundle E is called *numerically effective* (nef) if the line bundle $\mathcal{O}_E(1)$ on $\mathbb{P}(E)$, the projective bundle of hyperplanes in the fibres of E, is nef. Again it is clear that a vector bundle E which admits a metric with semipositive curvature (in the sense of Griffiths) is nef. A compact Kähler manifold X having semipositive holomorphic bisectional curvature has by definition a tangent bundle T_X with semipositive curvature. Again the converse does not hold. From an algebraic geometric point of view it is natural to consider the class of projective manifolds X whose tangent bundle is nef. This has been done by Campana and Peternell [CP91] and –in case of dimension 3– by Zheng [Zh90]. In particular, a complete classification is obtained for dimension at most three.

The purpose of this paper is to investigate compact Kähler manifolds with nef tangent bundles. One difficulty in carrying over the algebraic definition of nefness to the Kähler case is the possible lack of curves. This is overcome by the following

Definition. — *Let X be a compact complex manifold with a fixed hermitian metric ω. A line bundle L over X is <u>nef</u> if for every $\varepsilon > 0$ there exists a smooth hermitian metric h_ε on L such that the curvature satisfies*

$$\Theta_{h_\varepsilon} \geq -\varepsilon\,\omega\,.$$

This means that the curvature of L can have an arbitrarily small negative part. Clearly a nef line bundle L satisfies $L \cdot C \geq 0$ for all curves $C \subset X$, but the converse is not true. For projective algebraic X both notions coincide. Our main result is a structure theorem on the Albanese map.

Main Theorem. — *Let X be a compact Kähler manifold with nef tangent bundle T_X. Let $\widetilde{X}$ be a finite étale cover of X of maximum irregularity $q = q(\widetilde{X}) = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Then*

(i) $\pi_1(\widetilde{X}) \simeq \mathbb{Z}^{2q}$.

(ii) *The Albanese map $\alpha : \widetilde{X} \to A(\widetilde{X})$ is a smooth fibration over a q-dimensional torus with nef relative tangent bundle.*

(iii) *The fibres F of α are Fano manifolds with nef tangent bundles.*

As a consequence one gets the

Corollary. — *As an abstract group, the fundamental group $\pi_1(X)$ is an extension of a finite group by $\mathbb{Z}^{2q}$.*

It is conjectured in [CP91] that Fano manifolds (i.e. manifolds such that $-K$ is ample) with nef tangent bundle are rational homogeneous. Once this is proved our main theorem classifies the compact Kähler manifolds with nef tangent bundle up to étale cover. In §1 we prove basic properties of nef vector bundles. An important technical point – used over and over – is the following:

Lemma. — *Let E be a nef vector bundle on a compact complex connected manifold X and let $\sigma \in H^0(X, E^\star)$ be a non trivial section. Then σ has no zeros at all.*

The main result of §1 characterizes vector bundles E which are numerically flat, i.e. such that E and $E^\star$ are nef.

Theorem. — *Let E be numerically flat on the compact Kähler manifold X. Then E admits a filtration by vector bundles whose graded pieces are hermitian flat, i.e. given by unitary representations of $\pi_1(X)$.*

The proof uses two deep facts: the Uhlenbeck-Yau theorem on the existence of Hermite-Einstein metrics on stable vector bundles and Kobayashi's flatness criterion derived from Lübke's inequality on Chern classes. In §2 it is shown that the Fulton-Lazarsfeld inequalities [FL83] for Chern classes of ample vector bundles still hold for nef vector bundles on compact Kähler manifolds.

Theorem. — *Let X be a compact Kähler manifold and let E be a nef vector bundle on X. Then E is numerically semipositive, i.e. $P(c(E)) \geq 0$ for all positive polynomials P.*

As a consequence we get the following

Corollary. — *If E is nef and $c_1(E)^n = 0$, $n = \dim X$, then all Chern polynomials $P(c(E))$ of degree $2n$ vanish.*

This will be used in the proof of our main theorem. §3 contains the proof of our main theorem cited above. One of the key points is the following

PROPOSITION. — *Let X be a compact Kähler n-fold with T_X nef. Then*

(i) *If $c_1(X)^n > 0$, then X is a Fano manifold.*

(ii) *If $c_1(X)^n = 0$, then $\chi(\mathcal{O}_X) = 0$ and there exists a non zero holomorphic p-form, p suitable odd and a finite étale cover $\widetilde{X} \to X$ such that $q(\widetilde{X}) > 0$.*

The difficult part is (ii). Since $c_1(X)^n = 0$ we get by the above corollary that $\chi(\mathcal{O}_X) = 0$. From this we infer the existence of a non zero p-form u for some suitable odd number p. By contraction with u we get a map $S : \Lambda^{p-1}T_X \to \Omega^1_X$. The image E has constant rank and is therefore a subbundle of Ω^1_X. Hence E is numerically flat. The theorem in §1 implies the existence of a hermitian flat subbundle $E_1 \subset E$. If E_1 would be trivial after some finite étale cover the proposition would be proved. We use a result of Tits that every subgroup of a linear group contains either a non abelian free subgroup or a solvable subgroup of finite index. The first case cannot occur since we show in [DPS92] that $\pi_1(X)$ has sub-exponential growth. The second case leads to the desired conclusion.

In §4 we give algebraic proofs of the following two results, stated resp. proved in 3.6 and 3.10 (i).

THEOREM. — *Any Moishezon manifold with nef tangent bundle is projective.*

THEOREM. — *Let X be a compact Kähler n-fold with T_X nef and $c_1(X)^n > 0$. Then X is a Fano manifold.*

In §5 we show that a Mori contraction $\varphi : X \to Y$ of a projective manifold X with T_X nef is smooth; moreover Y is smooth too. As a consequence we get an algebraic proof of the crucial proposition 3.10. In §6 we classify all non algebraic surfaces with nef tangent bundles.

THEOREM. — *The smooth non algebraic compact complex surfaces with nef tangent bundles are precisely the following:*

(i) *Non algebraic tori*;

(ii) *Kodaira surfaces*;

(iii) *Hopf surfaces.*

In the final section we give a classification of all non algebraic Kähler 3-folds.

THEOREM. — *Let X be a non algebraic 3-dimensional compact Kähler manifold. Then T_X is nef if and only if X is up to finite étale cover either a torus or of the form $\mathbb{P}(E)$ where E is a numerically flat rank 2-bundle over a 2-dimensional torus.*

Let us mention that the classification of projective surfaces and 3-folds with nef tangent bundle is contained in [CP91] and [Zh90].

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1. Basic properties of nef vector bundles

1.A. Nef line bundles

Let X be a compact complex manifold and let L be a holomorphic line bundle over X. When X is projective or Moishezon, L is said to be numerically effective (nef for short) if $L \cdot C = \int_C c_1(L) \geq 0$ for every curve C in X. This definition can no longer be used for a general compact manifold X because X need not possess any curve. An alternative definition is suggested by the following lemma, which is an easy consequence of Seshadri's ampleness criterion (namely, a line bundle A is ample if and only if there exists $\varepsilon > 0$ such that $A \cdot C \geq \varepsilon \cdot m(C)$ for every curve C, where $m(C)$ is the maximum multiplicity of singular points in C).

Lemma 1.1. — *Suppose that X is projective algebraic and let A be an ample line bundle on X. Then L is nef if and only if $L^k \otimes A$ is ample for every integer $k \geq 0$.*

Now it is well known that a line bundle A is ample if and only if it carries a smooth hermitian metric h such that the associated Chern curvature form

$$\Theta_h(A) = \frac{i}{2\pi} D^2_{A,h}$$

is positive definite everywhere. Suppose that h is such a metric on A. If L is nef, lemma 1.1 shows that there is a smooth hermitian metric h_k on L such that

$$\Theta\left(L^k \otimes A\right) = k\Theta_{h_k}(L) + \Theta_h(A) > 0,$$

hence

$$\Theta_{h_k}(L) \geq -\frac{1}{k}\Theta_h(A)$$

can have an arbitrary small negative part. We take this property as the general definition.

Definition 1.2. — *Let X be an arbitrary compact complex manifold, equipped with a hermitian metric ω. A line bundle L over X is said to be nef if for every $\varepsilon > 0$ there exists a smooth hermitian metric h_ε on L such that*

$$\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega.$$

If C is a curve in X, we infer from this

$$L \cdot C = \int_C \Theta_{h_\varepsilon}(L) \geq -\varepsilon \int_C \omega$$

for every $\varepsilon > 0$, so $L \cdot C \geq 0$. However this property does not imply nefness in general, as shown by:

EXAMPLE 1.3. — Let $X = \left(\mathbb{C}^2 \setminus \{0\}\right)/\Gamma$ be the primary Hopf surface associated to the infinite cyclic group $\Gamma = \{\lambda^p; p \in \mathbb{Z}, \lambda > 1\}$. Then $X \simeq S^3 \times S^1$ has zero second Betti number, so every line bundle L is numerically trivial (i.e. $c_1(L) = 0$), in particular $L \cdot C = 0$ for every curve C. However, if $\pi : X \to \mathbb{P}^1$ is the natural projection, the pull-back $L = \pi^\star \mathcal{O}(-1)$ cannot be nef, otherwise $\mathcal{O}(-1)$ itself would be nef by property 1.8 (ii) below. □

One may wonder whether ε can be made equal to zero in Definition 1.2. This is true if the hermitian metric h is allowed to be singular.

PROPOSITION 1.4. — *If L is nef, there exists a singular hermitian metric h on L (trivialized locally by weight functions $\varphi \in L^1_{loc}$ with $h \simeq e^{-\varphi}$) such that*

$$\Theta_h(L) = \frac{i}{2\pi}\partial\overline{\partial}\varphi \geq 0$$

in the sense of currents, i.e. such that the weight functions φ are plurisubharmonic.

Proof. Let ω be a hermitian metric on X and let h_0 be a fixed smooth metric on L. For every $\varepsilon \in \,]0,1[$ there is a hermitian metric $h_\varepsilon = h_0 e^{-\psi_\varepsilon}$ such that $\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$. We can always multiply h_ε by a constant in such a way that $\max_X \psi_\varepsilon = 0$. Then

$$\frac{i}{2\pi}\partial\overline{\partial}\psi_\varepsilon = \Theta_{h_\varepsilon}(L) - \Theta_{h_0}(L) \geq -\omega + \Theta_{h_0}(L).$$

In particular, there is a covering (U_j) of X by coordinate patches and a constant $C > 0$ such that $\psi_\varepsilon(z) + C|z^{(j)}|^2$ is plurisubharmonic in each U_j. The well known properties of subharmonic functions (namely, the mean value inequality and the Green-Riesz formula) show that the family (ψ_ε) is bounded in L^1 norm on each patch U_j, and therefore relatively compact in $L^1(X)$. Hence there exists a subsequence (ψ_{ε_ν}) converging to a limit ψ in $L^1(X)$. The possibly singular hermitian metric $h = h_0 e^{-\psi}$ has curvature $\Theta_h(L) = \lim_{\nu\to+\infty}\Theta_{h_{\varepsilon_\nu}}(L)$ in the weak topology of currents. Therefore $\Theta_h(L) \geq 0$. □

COROLLARY 1.5. — *A line bundle L is nef as well as its dual $L^\star = L^{-1}$ if and only if L is hermitian flat, that is, L is given by a representation $\pi_1(X) \to S^1$ in the unit circle.*

Proof. If L is hermitian flat, then of course both L and $L^\star$ are nef. Conversely, suppose that L and $L^\star$ are nef. By Proposition 1.4, there are (possibly singular) hermitian metrics h on L and h' on $L^\star$ with semipositive curvature in the sense of currents. Then $h \otimes h'$ is a metric of semipositive curvature on the trivial line bundle $L \otimes L^\star$, given by a weight $h \times h' = e^{-\psi}$ where ψ is a global plurisubharmonic function on X. Since X is compact, we infer that ψ is a constant, hence $\Theta_h(L) + \Theta_{h'}(L^\star) = 0$ and therefore $\Theta_h(L) = \Theta_{h'}(L) = 0$, each term being semipositive. This implies that h, h' are smooth and flat. □

Remark 1.6. — If L admits a singular metric h with $\Theta_h(L) \geq 0$, it does not necessarily follow that L is nef. In fact, let X be the blow-up of some manifold at one point and let $L = \mathcal{O}(E)$ be the line bundle associated to the exceptional divisor E. Then L is not nef because $L \cdot C < 0$ for every curve $C \subset E$. However the inclusion morphism $\mathcal{O} \subset \mathcal{O}(E)$ gives a singular metric h on $\mathcal{O}(E)$ with poles along E, such that

$$\Theta_h(\mathcal{O}(E)) = [E]$$

is the current of integration on E. In particular $\Theta_h(L) \geq 0$. $\square$

It is also important to observe that a smooth hermitian metric h with $\Theta_h(L) \geq 0$ need not exist when L is nef, thus ε cannot be taken equal to 0 in Definition 1.2, disproving a conjecture of Fujita ([Fu83], 4.14):

Example 1.7. — Let $\Gamma = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, $\operatorname{Im}\tau > 0$, be an elliptic curve and let E be the rank 2 vector bundle over Γ defined by

$$E = \mathbb{C} \times \mathbb{C}^2/(\mathbb{Z} + \mathbb{Z}\tau)$$

where the action of $\mathbb{Z} + \mathbb{Z}\tau$ is given by the two automorphisms

$$\begin{aligned} g_1(x, z_1, z_2) &= (x + 1, z_1, z_2), \\ g_\tau(x, z_1, z_2) &= (x + \tau, z_1 + z_2, z_2), \end{aligned}$$

and where the projection $E \to \Gamma$ is induced by the first projection $(x, z_1, z_2) \mapsto x$. Then $\mathbb{C} \times \mathbb{C} \times \{0\}/(\mathbb{Z} + \mathbb{Z}\tau)$ is clearly a trivial line subbundle $\mathcal{O} \hookrightarrow E$, and the quotient $E/\mathcal{O} \simeq \Gamma \times \{0\} \times \mathbb{C}$ is also trivial. Let L be the line bundle $L = \mathcal{O}_E(1)$ over the ruled surface $X = \mathbb{P}(E)$. The exact sequence

$$0 \to \mathcal{O} \to E \to \mathcal{O} \to 0$$

shows by Prop. 1.14 (ii) below that L is nef over X. Now, we are going to determine all hermitian metrics (including singular ones) such that the curvature $\Theta(L)$ is semipositive in the sense of currents. As the total space of L^{-1} is equal to $E^\star$ blown up along the zero section, the function

$$\varphi(\zeta) = \log \parallel \zeta \parallel^2_{h^{-1}}, \quad \zeta \in L^{-1}$$

associated to any hermitian metric h on L can also be seen as a function on $E^\star$ satisfying the log-homogeneity condition

$$\varphi(\lambda\zeta) = \log \mid \lambda \mid + \varphi(\zeta)$$

for every $\lambda \in \mathbb{C}$. We have

$$\frac{i}{2\pi}\partial\overline{\partial}\varphi(\zeta) = \pi^\star_{L^{-1}}\Theta_h(L), \quad \pi_{L^{-1}} : L^{-1} \to X,$$

thus $\Theta_h(L) \geq 0$ if and only if φ is plurisubharmonic on $E^\star$. The total space of $E^\star$ is the quotient

$$E^\star = \mathbb{C} \times \mathbb{C}^2/(\mathbb{Z} + \mathbb{Z}\tau)$$

by the dual action

$$g_1^\star(x,w_1,w_2) = (x+1,w_1,w_2),$$
$$g_\tau^\star(x,w_1,w_2) = (x+\tau,w_1,w_1+w_2).$$

The function φ gives rise to a plurisubharmonic function $\widetilde{\varphi}$ on $\mathbb{C}\times\mathbb{C}^2$ which is invariant by $g_1^\star$ and $g_\tau^\star$ and log-homogeneous with respect to (w_1,w_2). We claim that the only such functions are

$$(+)\qquad \widetilde{\varphi}(x,w_1,w_2) = \log|w_1| + a, \quad a\in\mathbb{R}.$$

Hence the associated metric h^{-1} on L^{-1} must have zeros along the curve $C = \{w_1=0\}\simeq\{z_2=0\}$ in $\mathbb{P}(E^\star)\simeq\mathbb{P}(E)$. By the Lelong-Poincaré equation, we then find

$$\Theta_h(L) = [C]$$

where $[C]$ denotes the current of integration over C.

To check (+), we first observe that any periodic subharmonic function u on $\mathbb{C}$ with logarithmic growth at infinity must be constant: suppose $u(z)\le A\log_+|z|+B$ and $u(z+1)=u(z)$, then $v(z)=\sup_{t\in[0,1]}u(z+t)$ satisfies a similar upper bound and depends only on $\mathrm{Im}\,z$; since v is a convex function of $\mathrm{Im}\,z$ it must be a constant, so u is bounded above and constant, too. As

$$\widetilde{\varphi}(x+p+q\tau,w_1,w_2) = \widetilde{\varphi}(x,w_1,w_2+qw_1)$$

for all $p,q\in\mathbb{Z}$, the log-homogeneity of $\widetilde{\varphi}$ in (w_1,w_2) implies that $\widetilde{\varphi}$ has logarithmic growth also in x. But $x\mapsto\widetilde{\varphi}(x,w_1,w_2)$ is 1-periodic, so $\widetilde{\varphi}$ must be independent of x. Now, the invariance by $g_\tau^\star$ shows that $w_2\mapsto\widetilde{\varphi}(w_1,w_2)$ is periodic for $w_1\ne 0$. Therefore $\widetilde{\varphi}$ is also independent of w_2. The log-homogeneity implies

$$\widetilde{\varphi}(w_1) = \log|w_1| + \widetilde{\varphi}(1),$$

as desired.

We have thus shown that $\mathcal{O}_E(1)$ does not admit any smooth hermitian metric with semipositive curvature. In particular, E is not semipositive in the sense of Griffiths (see 1.10 (i) for the definition). This weaker statement could have been proved directly as follows: if $\Theta_h(E)\ge 0$, then $\mathrm{Tr}\,\Theta_h(E)=\Theta_h(\det E)\ge 0$ and must be zero, for $\det E\simeq\mathcal{O}$; therefore the full curvature tensor $\Theta_h(E)$ would itself vanish and so E would be hermitian flat; then the exact sequence $0\to\mathcal{O}\to E\to\mathcal{O}\to 0$ would split holomorphically (by the orthogonal splitting), which is easily seen not to be the case. □

Proposition 1.8. — *Let X, Y be compact complex manifolds, let L be a line bundle on X and let $f:Y\to X$ be an analytic map.*

(i) *If L is nef, then $f^\star L$ is nef.*

(ii) *Suppose that f is surjective and has equidimensional fibres. If $f^\star L$ is nef, then L is nef.*

These results are of course well known in the algebraic case (see e.g. [Fu83]).

Proof. Let ω, ω' be hermitian metrics on X, Y respectively. There exists a constant $a > 0$ such that

$$f^\star\omega \le a\omega'.$$

(i) If h_ε is as in Definition 1.2, then

$$\Theta_{f^\star h_\varepsilon}(f^\star L) = f^\star\Theta_{h_\varepsilon}(L) \ge -\varepsilon f^\star\omega \ge -\varepsilon a\omega'.$$

This shows that $f^\star L$ is nef.

(ii) Fix an arbitrary hermitian metric h on L and set $\alpha = \Theta_h(L)$. If $f^\star L$ is nef, there is for every $\varepsilon > 0$ a metric $h'_\varepsilon = e^{-\psi_\varepsilon} f^\star h$ on $f^\star L$, where ψ_ε is some smooth real function on Y, such that

$$\Theta_{h'_\varepsilon}(f^\star L) = f^\star\alpha + \frac{i}{2\pi}\partial\overline{\partial}\psi_\varepsilon \ge -\varepsilon\omega'.$$

The idea, in very rough terms, is to define a metric h_ε on L by

$$|\xi|_{h_\varepsilon(x)} = \inf_{y\in f^{-1}(x)} |\xi|_{h'_\varepsilon(y)}, \qquad \xi \in L_x = (f^\star L)_y, \quad y \in f^{-1}(x),$$

that is, $h_\varepsilon = e^{-\varphi_\varepsilon} h$ with $\varphi_\varepsilon(x) = \sup_{y\in f^{-1}(x)} \psi_\varepsilon(y)$. However, this simple-minded idea would not give the desired lower bound for $\Theta_{h_\varepsilon}(L)$ in a neighborhood of the critical values of f in X. Therefore, a slightly modified construction is needed.

Let p be the dimension of the fibres of f. For any point $y \in Y$, there are local coordinates $(w) = (w_1, \ldots, w_m)$ centered at y such that the map $z \mapsto (f(z), w_1(z), \ldots, w_p(z))$ is proper and finite from a neighborhood of y onto a neighborhood of $f(y)\times\{0\}$ in $X \times \mathbb{C}^p$. Thus we can find local coordinates centered at $f(y)$ on X and a neighborhood U of y such that

$$|F(z) - F(y)|^2 + \sum_{1\le j\le p} |w_j(z)|^2 > 0$$

on ∂U, where $F = (F_1, \ldots, F_n)$ denote the local coordinate components of f. As Y is compact, we can find finitely many points $y_k \in Y$ and local coordinates $(w^{(k)}) = (w_1^{(k)}, \ldots, w_m^{(k)})$ on neighborhoods $\overline{U}_k$ of y_k (with $w^{(k)}(y_k) = 0$), resp. local coordinates on X centered at $f(y_k)$, such that the corresponding components $F^{(k)} = (F_1^{(k)}, \ldots, F_n^{(k)})$ of f satisfy

$$2\delta_k := \left|F^{(k)}(z) - F^{(k)}(y_k)\right|^2 + \sum_{1\le j\le p} \left|w_j^{(k)}(z)\right|^2 > 0 \quad \text{on } \partial U_k,$$

and such that the collection of open sets (V_k) defined by

$$V_k := \left\{ z \in U_k\,;\, \left|F^{(k)}(z) - F^{(k)}(y_k)\right|^2 + \sum_{1\le j\le p} \left|w_j^{(k)}(z)\right|^2 < \delta_k \right\}$$

is a covering of Y. For $z \in \overline{U}_k$ and $x \in X$ we define

$$\lambda_\varepsilon^{(k)}(z) := \varepsilon^3\left|w^{(k)}(z)\right|^2 - \varepsilon^2 \left(\left|F^{(k)}(z) - F^{(k)}(y_k)\right|^2 + \sum_{1\le j\le p} \left|w_j^{(k)}(z)\right|^2 - \delta_k \right),$$

$$\varphi_\varepsilon(x) := \sup_{y\in f^{-1}(x),\, \overline{U}_k\ni y} \psi_{\varepsilon^4}(y) + \lambda_\varepsilon^{(k)}(y),$$

where ε is chosen $\leq \varepsilon_0$ so small that $\lambda_\varepsilon^{(k)} < 0$ on ∂U_k. Since $\lambda_\varepsilon^{(k)} \geq 0$ on V_k, it follows that the supremum in $\varphi_\varepsilon(x)$ is never attained for an index j such that $\partial U_j \ni y$: indeed we then have $\lambda_\varepsilon^{(j)}(y) < 0$, while $\lambda_\varepsilon^{(k)}(y) \geq 0$ for some k such that $V_k \ni y$. Therefore φ_ε is continous. Moreover, for $\varepsilon \leq \varepsilon_1$ small

$$\frac{i}{2\pi}\partial\overline{\partial}\left(\varepsilon^3 |w^{(k)}|^2\right) \geq \varepsilon^4 \omega',$$

$$\frac{i}{2\pi}\partial\overline{\partial}\left(\varepsilon^2 |F^{(k)} - F^{(k)}(y_k)|^2\right) \leq \frac{\varepsilon}{2} f^\star \omega$$

on each U_k. Hence by definition of ψ_ε we get

$$f^\star\alpha + \frac{i}{2\pi}\partial\overline{\partial}\left(\psi_{\varepsilon^4} + \lambda_\varepsilon^{(k)}\right) \geq -\frac{\varepsilon}{2} f^\star\omega - \varepsilon^2 \frac{i}{2\pi}\partial\overline{\partial} \sum_{1\leq j\leq p} |w_j^{(k)}|^2.$$

We will check below that this implies

$$(\star) \qquad \alpha + \frac{i}{2\pi}\partial\overline{\partial}\varphi_\varepsilon \geq -\frac{\varepsilon}{2}\omega.$$

By Richberg's regularization theorem [Ri68], we can find a smooth function $\widetilde{\varphi}_\varepsilon$ such that $\alpha + \frac{i}{2\pi}\partial\overline{\partial}\widetilde{\varphi}_\varepsilon \geq -\varepsilon\omega$. Then the metric $h_\varepsilon = e^{-\widetilde{\varphi}_\varepsilon} h$ on L satisfies

$$\Theta_{h_\varepsilon}(L) = \alpha + \frac{i}{2\pi}\partial\overline{\partial}\widetilde{\varphi}_\varepsilon \geq -\varepsilon\omega, \quad \text{QED.}$$

To check $(\star)$, fix a point $x_0 \in X$ and take $y_0 \in Y$ and $U_k \ni y_0$ such that $\varphi_\varepsilon(x_0) = \psi_{\varepsilon^4}(y_0) + \lambda_\varepsilon^{(k)}(y_0)$. Take an arbitrary quadratic function $q(x)$ near x_0 such that $\frac{i}{2\pi}\partial\overline{\partial}q > \alpha + \frac{\varepsilon}{2}\omega$ at x_0. Then

$$\frac{i}{2\pi}\partial\overline{\partial}\Psi \geq 0 \quad \text{where} \quad \Psi = \psi_{\varepsilon^4} + \lambda_\varepsilon^{(k)} + q\circ f + \varepsilon^2 \sum_{1\leq j\leq p} |w_j^{(k)}|^2$$

i.e. the function Ψ is plurisubharmonic near y_0. However, f defines a finite map from the submanifold

$$S = \{z \in U_k\,;\, w_{1,k}(z) = w_{1,k}(y_0), \ldots, w_{p,k}(z) = w_{p,k}(y_0)\}$$

onto a neighborhood of x_0 (by construction of $(w^{(k)})$). Hence the function

$$\Phi(x) := \sup_{y\in f^{-1}(x)\cap S} \Psi(y)$$

is well defined near x_0 and satisfies

$$\Phi(x) \leq \varphi_\varepsilon(x) + q(x) + \varepsilon^2 \sum_{1\leq j\leq p} |w_j^{(k)}(y_0)|^2$$

with equality at $x = x_0$. Since Φ is plurisubharmonic, we conclude that $\varphi_\varepsilon + q$ satisfies the mean value inequality at x_0. This is also true at all nearby points where $\frac{i}{2\pi}\partial\overline{\partial}q > \alpha + \frac{\varepsilon}{2}\omega$, so $\varphi_\varepsilon + q$ is plurisubharmonic near x_0. Taking $\frac{i}{2\pi}\partial\overline{\partial}q$ as close as possible to $\alpha + \frac{\varepsilon}{2}\omega$ at x_0, we conclude that $(\star)$ holds. $\square$

Of course we expect that property 1.8 (ii) holds for arbitrary surjective maps $f : Y \to X$, but there are serious technical difficulties to overcome.

1.B. Nef vector bundles

Let E be a holomorphic vector bundle of rank r over a compact complex manifold. We denote by $\mathbb{P}(E)$ the projectivized bundle of hyperplanes of E and by $\mathcal{O}_E(1)$ the associated canonical line bundle.

Definition 1.9. — *We say that E is nef over X if $\mathcal{O}_E(1)$ is nef over $\mathbb{P}(E)$.*

Proposition 1.8 can then be immediately extended to the vector bundle case by taking $L = \mathcal{O}_E(1), X = \mathbb{P}(E), Y = \mathbb{P}(f^\star E)$:

Proposition 1.10. — *Let X, Y be compact complex manifolds, let $f : Y \to X$ be an analytic map and let E be a vector bundle over X. Then E nef implies $f^\star E$ nef, and the converse is true if f is surjective and has equidimensional fibres.*

It will be useful to have a differential geometric criterion for nefness directly over X. First recall a few basic properties of hermitian differential geometry (see e.g. Griffiths [Gr69]).

Proposition 1.11. — *If (E, h) is a hermitian vector bundle, the Chern curvature tensor*

$$\Theta_h(E) = \frac{i}{2\pi} D_{E,h}^2 = i \sum_{\substack{1\le j,k\le n \\ 1\le\lambda,\mu\le r}} a_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e_\lambda^\star \otimes e_\mu$$

is a hermitian (1,1)-form with values in $\mathrm{Hom}(E, E)$; here $(z_1, \ldots, z_n)$ are local coordinates on X and $(e_\lambda)_{1\le\lambda\le r}$ is a local orthonormal frame of E.

(i) *We say that (E, h) is semipositive in Griffiths' sense and write $\Theta_h(E) \ge 0$ if $\Theta_h(E)(\xi \otimes t) = \sum a_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu \ge 0$ for every $\xi \in T_x X$, $v \in E_x$, $x \in X$. We write $\Theta_h(E) > 0$ in case there is strict inequality for $\xi \ne 0, v \ne 0$.*

(ii) *If (F, h') is another hermitian vector bundle, then*

$$\Theta_{h\otimes h'}(E \otimes F) = \Theta_h(E) \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \Theta_{h'}(F).$$

(iii) *If $0 \to F \to E \to Q \to 0$ is an exact sequence and if the metrics of F, Q are induced by the metric of E, the orthogonal splitting $E \simeq F \oplus Q$ yields*

$$D_E = \begin{pmatrix} D_F & -\beta^\star \\ \beta & D_Q \end{pmatrix}, \quad \Theta(E) = \begin{pmatrix} \Theta(F) - \beta^\star \wedge \beta & -D^{1,0}\beta^\star \\ D^{0,1}\beta & \Theta(Q) - \beta \wedge \beta^\star \end{pmatrix}$$

where β is the (1,0) second fundamental form with values in $\mathrm{Hom}(F, Q)$.

(iv) *As follows easily from (iii), we have $\Theta(Q) = \Theta(E)_{|Q} + \beta \wedge \beta^\star \ge \Theta(E)_{|Q}$ in Griffiths' sense, where $\Theta(E)_{|Q}$ is a short-hand notation for the block of $\Theta(E)$ that lies in $\mathrm{Hom}(Q, Q)$.*

Theorem 1.12. — *Let ω be a fixed hermitian metric on X. A vector bundle E on X is nef if and only if the symmetric powers $S^m E$ satisfy the following property:*

There is a sequence of hermitian metrics h_m on $S^m E$ such that for every $\varepsilon > 0$ and $m \geq m_0(\varepsilon)$

$$\Theta_{h_m}(S^m E) \geq -m\varepsilon\omega \otimes \mathrm{Id}_{S^m E}$$

in the sense of Griffiths.

Proof. If $\pi : \mathbb{P}(E) \to X$ is the projection, there is a surjective morphism $\pi^\star S^m E \to \mathcal{O}_E(m)$. Suppose first that h_m exists and satisfies the above inequality. Let $\widetilde{h}_m$ be the quotient metric on $\mathcal{O}_E(m)$. By 1.11 (iv) we get

$$\Theta_{\widetilde{h}_m}(\mathcal{O}_E(m)) \geq \pi^\star\Theta_{h_m}(S^m E)_{|\mathcal{O}_E(m)} \geq -m\varepsilon\pi^\star\omega \otimes \mathrm{Id}_{\mathcal{O}_E(m)},$$

i.e.

$$\Theta_{\widetilde{h}_m^{1/m}}(\mathcal{O}_E(1)) \geq -\varepsilon\pi^\star\omega.$$

This implies that $\mathcal{O}_E(1)$ is nef. Conversely, suppose that $\mathcal{O}_E(1)$ is nef and let ω' be a hermitian metric on $\mathbb{P}(E)$. Then for every $\varepsilon > 0$ there is a hermitian metric $\widetilde{h}_\varepsilon$ on $\mathcal{O}_E(1)$ such that

$$\Theta_{\widetilde{h}_\varepsilon}(\mathcal{O}_E(1)) \geq -\varepsilon\omega'.$$

Let H be a fixed hermitian metric on E and let $\widetilde{H}$ be the induced metric on $\mathcal{O}_E(1)$. Then $\Theta_{\widetilde{H}}(\mathcal{O}_E(1))$ is positive along the fibres of π, so there exists a constant $C > 0$ such that

$$\Theta_{\widetilde{H}}(\mathcal{O}_E(1)) + C\pi^\star\omega \geq C^{-1}\omega'$$

on $\mathbb{P}(E)$. We consider the barycenter metric $\widehat{h}_\varepsilon = \widetilde{h}_\varepsilon^{1-C\varepsilon}\widetilde{H}^{C\varepsilon}$ on $\mathcal{O}_E(1)$. Then

$$\begin{aligned}
\Theta_{\widehat{h}_\varepsilon}(\mathcal{O}_E(1)) &= (1 - C\varepsilon)\,\Theta_{\widetilde{h}_\varepsilon}(\mathcal{O}_E(1)) + C\varepsilon\Theta_{\widetilde{H}}(\mathcal{O}_E(1)) \\
&\geq -(1 - C\varepsilon)\,\varepsilon\omega' + C\varepsilon\left(C^{-1}\omega' - C\pi^\star\omega\right) \\
&\geq C\varepsilon^2\omega' - C^2\varepsilon\pi^\star\omega \\
&> -C^2\varepsilon\pi^\star\omega.
\end{aligned}$$

By theorem 4.1 of [De91] applied with $v = -C^2\varepsilon\omega$, we can therefore find a metric $h_{m,\varepsilon}$ on $S^m E$ such that for $m \geq \mu_0(\varepsilon)$ large

$$\Theta_{h_{m,\varepsilon}}(S^m E) \geq -m\left(C^2 + 1\right)\varepsilon\omega \otimes \mathrm{Id}_{S^m E}.$$

This implies theorem 1.12 by taking $h_m = h_{m,\varepsilon_m}$ with a sequence ε_m converging to 0 so slowly that $\mu_0(\varepsilon_m) \leq m$. □

Corollary 1.13. — *Let $\Gamma^a E$ be the irreducible tensor representation of $Gl(E)$ of highest weight $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$, with $a_1 \geq \ldots \geq a_r \geq 0$. Then there are hermitian metrics h_a on $\Gamma^a E$ such that for every $\varepsilon > 0$*

$$\Theta_{h_a}(\Gamma^a E) \geq -|a|\varepsilon\omega \otimes \mathrm{Id}_{\Gamma^a E}$$

for $|a| := \sum a_j \geq A(\varepsilon)$.

Proof. (Compare with [Ha66]). The representation $\Gamma^a E$ is a direct summand in $S^{a_1}E \otimes \ldots \otimes S^{a_r}E$. Let h_m be a metric on $S^m E$ such that

$$\Theta_{h_m}(S^m E) \geq -m\varepsilon_m\omega \otimes \mathrm{Id}_{S^m E}$$

with $\lim \varepsilon_m = 0$ and let $\widetilde{h}_a$ be the quotient metric of $h_{a_1} \otimes \ldots \otimes h_{a_r}$ on $\Gamma^a E$. Then by 1.11 (ii) and (iv) we get

$$\Theta_{h_a}(\Gamma^a E) \geq -\sum a_j \varepsilon_{a_j} \omega \otimes \mathrm{Id}_{\Gamma^a E}$$

and it is clear that the ratio $\sum a_j \varepsilon_{a_j}/|a|$ tends to 0 when $|a| = \sum a_j$ tends to $+\infty$. □

We can now prove the basic properties of nef vector bundles. In the case of ample vector bundles over algebraic varieties, these properties are well known by [Ha66]; the nef case then follows essentially by passing to the limit if the base X is algebraic (see [CP91] for details).

Proposition 1.14. — *Let E, F be holomorphic vector bundles.*

(i) *If E and F are nef, then $E \otimes F$ is nef.*

(ii) *If E is nef, any tensor power $\Gamma^a E$ with $a_1 \geq \ldots \geq a_r \geq 0$ is nef.*

(iii) *If some symmetric power $S^m E$ is nef ($m > 0$), then E is nef.*

Proof.
(i) There is a direct sum decomposition

$$S^m(E \otimes F) = \bigoplus_{\substack{|a|=m \\ |b|=m}} n_{a,b} \Gamma^a E \otimes \Gamma^b F$$

into irreducible $Gl(E) \times Gl(F)$ modules of degree (m, m) (the exact formula is not needed, except that the weights a, b are non negative). We take the metric $\widetilde{h}_{a,E} \otimes \widetilde{h}_{b,F}$ on each summand. Corollary 1.13 and property 1.11 (ii) imply

$$\Theta\left(S^m(E \otimes F)\right) \geq -2m\varepsilon\omega \otimes \mathrm{Id}_{S^m(E \otimes F)}$$

for $m \geq A(\varepsilon)$ (note that each summand contributes in a different block of the curvature matrix and that $|a|, |b| \geq A(\varepsilon)$).

(ii) The nefness of $\Gamma^a E$ is obtained in the same way, by observing that $S^m(\Gamma^a E)$ is a direct sum of tensor powers $\Gamma^b E$ with $b \geq 0$ and $|b| = m|a|$.

(iii) The pull-back $\pi^\star S^m E$ of $S^m E$ to $\mathbb{P}(E)$ is nef by Prop. 1.10. Its quotient $\mathcal{O}_E(m)$ is therefore nef by Prop. 1.15 (i) below. Hence $\mathcal{O}_E(1) = \mathcal{O}_E(m)^{1/m}$ is nef. □

Proposition 1.15. — *Let $0 \to F \to E \to Q \to 0$ be an exact sequence of holomorphic vector bundles. Then*

(i) *E is nef $\Rightarrow$ Q is nef.*

(ii) *F, Q nef $\Rightarrow$ E is nef.*

(ii) *E nef, $(\det Q)^{-1}$ nef $\Rightarrow$ F nef.*

Proof.

(i) Apply theorem 1.12 and take the quotient metric of $S^m E$ on $S^m Q$. Then

$$\Theta\left(S^m Q\right) \geq \Theta\left(S^m E\right)_{|S^m Q} \geq -m\varepsilon\omega \otimes \mathrm{Id}_{S^m Q}$$

for $m \geq m_0(\varepsilon)$.

(ii) Fix once for all a C^∞ splitting $E = F \oplus Q$. Then we have a C^∞ splitting

$$\Phi : S^m E \xrightarrow{\simeq} \bigoplus_{a+b=m} S^a F \otimes S^b Q.$$

By assumption, there are metrics h'_a on $S^a F$ and h''_b on $S^b Q$ such that

$$\Theta_{h'_a}\left(S^a F\right) \geq -a\varepsilon_a \omega \otimes \mathrm{Id}_{S^a F}, \quad \Theta_{h''_b}\left(S^b Q\right) \geq -b\eta_b \omega \otimes \mathrm{Id}_{S^b Q}$$

with $\lim\limits_{a\to+\infty} \varepsilon_a = \lim\limits_{b\to+\infty} \eta_b = 0$. We define hermitian metrics $h_{m,\rho}$ on $S^m E$ by

$$h_{m,\rho} = \Phi^\star \bigoplus_{a+b=m} \rho^{-2b} h'_a \otimes h''_b\ ,$$

where $\rho > 0$ is a small rescaling factor. Changing slightly our viewpoint, we replace the initial splitting Φ by the splitting Φ_ρ obtained by multiplying the $S^a F \otimes S^b Q$ component by ρ^{-b}. Then

$$h_{m,\rho} = \Phi_\rho^\star \bigoplus_{a+b=m} h'_a \otimes h''_b\ .$$

With respect to Φ_ρ, the Chern connection $D_{h_{m,\rho}}$ on $S^m E$ takes the form

$$D_{h_{m,\rho}} = \begin{pmatrix} D_{h'_m \otimes h''_0} & \cdots & -\rho^{k-j}\beta^\star_{k,j} \\ \vdots & D_{h'_{m-j}\otimes h''_j} & \vdots \\ \rho^{j-k}\beta_{j,k} & \cdots & D_{h'_0 \otimes h''_m} \end{pmatrix}$$

where $\beta_{j,k}$ is a (1,0)-form with values in

$$\mathrm{Hom}(S^{m-k}F \otimes S^k Q, S^{m-j}F \otimes S^j Q), \quad j > k$$

and is independent of ρ. To check this, recall the formula $\beta^\star_{j,k} = -\pi_k \circ \overline{\partial}\pi^\star_j$ where π_j is the orthogonal projection of $S^m E$ onto $S^{m-j}F \otimes S^j Q$ given by Φ_1 , and $\pi^\star_j : S^{m-j}F \otimes S^j Q \to S^m E$ is the adjoint (and right inverse) of π_j. By definition of Φ_ρ, the projection π_k is replaced by $\rho^{-k}\pi_k$, thus $\pi^\star_j$ is replaced by $\rho^j \pi^\star_j$ and $\beta^\star_{j,k}$ by $\rho^{j-k}\beta^\star_{j,k}$, as desired. By computing the matrix of $D^2_{h_{m,\rho}}$, we get

$$\Theta_{h_{m,\rho}}\left(S^m E\right) = \bigoplus_{0\leq j\leq m} \pi^\star_j \circ \Theta_{h'_{m-j}\otimes h''_j} \circ \pi_j + O(\rho),$$

therefore

$$\Theta_{h_{m,\rho}}\left(S^m E\right) \geq -\left(\max_{0\leq j\leq m}\left((m-j)\varepsilon_{m-j} + j\eta_j\right) + O(\rho)\right)\omega \otimes \mathrm{Id}_{S^m E}$$

where $O(\rho)$ comes from the contribution of the second fundamental form $\rho^{j-k}\beta_{j,k}$, $j > k$. For $\rho = \rho_m$ small this contribution is negligible. On the other hand, it is easy to see that

$$\lim_{m\to+\infty} \frac{1}{m} \max_{0\leq j\leq m} ((m-j)\varepsilon_{m-j} + j\eta_j) = 0,$$

thus E is nef.

(iii) We have $F = \Lambda^{s-1}F^\star \otimes \det F$ where s is the rank of F. By dualizing and taking the $s-1$ exterior product in the exact sequence, we get a surjection

$$\Lambda^{s-1}E^\star \to \Lambda^{s-1}F^\star = F \otimes (\det F)^{-1}.$$

We multiply by $\det F = \det E \otimes (\det Q)^{-1}$ to get a surjection

$$\Lambda^{r-(s-1)}E \otimes (\det Q)^{-1} \to F$$

where $r = \operatorname{rank}(E)$. Now the left hand side is nef by Prop. 1.14 and the assumption that $(\det Q)^{-1}$ is nef, so its quotient F is nef. $\square$

The next proposition concerns the behaviour of holomorphic sections in the dual of a nef vector bundle.

PROPOSITION 1.16. — *Let E be a nef vector bundle over a connected compact n-fold X and let $\sigma \in H^0(X, E^\star)$ be a non zero section. Then σ does not vanish anywhere.*

Proof. By Gauduchon [Ga77], we know that any conformal class of hermitian metrics on X contains a special metric with $\partial\overline{\partial}\left(\omega^{n-1}\right) = 0$. Let h_m be hermitian metrics as in theorem 1.12 and let $h_m^\star$ be the dual metric on $S^mE^\star$. Then the closed (1,1)-current

$$T_m = \frac{i}{2\pi}\partial\overline{\partial}\frac{1}{m}\log \parallel \sigma^m \parallel^2_{h_m^\star}$$

satisfies

$$T_m \geq -\frac{1}{m}\frac{\langle \Theta_{h_m^\star}\left(S^mE^\star\right)\sigma^m, \sigma^m\rangle_{h_m^\star}}{\parallel \sigma^m \parallel^2_{h_m^\star}}.$$

For $m \geq m_0(\varepsilon)$ we have

$$-\Theta_{h_m^\star}\left(S^mE^\star\right) = {}^t\Theta_{h_m}\left(S^mE\right) \geq -m\varepsilon\omega \otimes \mathrm{Id}_{S^mE^\star}.$$

Hence $T_m \geq -\varepsilon\omega$ for $m \geq m_0(\varepsilon)$. Now, if ω is a Gauduchon metric

$$\int_X (T_m + \varepsilon\omega) \wedge \omega^{n-1} = \varepsilon \int_X \omega^n$$

because T_m is $\partial\overline{\partial}$-exact. This implies that $T_m + \varepsilon\omega$ and T_m converge weakly to zero as $m \to +\infty$. If $\sigma(x) = 0$ at some point $x \in X$, the Lelong number $\nu(T_m, x)$ would be at least 1 (and equal to the vanishing order of σ at x). Thus by [S74], the Lelong number of any weak limit $T = \lim T_m$ would satisfy $\nu(T, x) \geq 1$, a contradiction. $\square$

We end this section with the following important analogue of Corollary 1.5 in the vector bundle case.

Definition 1.17. — *We say that a holomorphic vector bundle E is numerically flat if both E and $E^\star$ are nef (equivalently if E and $(\det E)^{-1}$ are nef).*

Theorem 1.18. — *Suppose that X is Kähler. Then a holomorphic vector bundle E over X is numerically flat if and only if E admits a filtration*

$$\{0\} = E_0 \subset E_1 \subset \ldots \subset E_p = E$$

by vector subbundles such that the quotients E_k/E_{k-1} are hermitian flat, i.e. given by unitary representations $\pi_1(X) \to U(r_k)$.

It should be observed that E itself need not be hermitian flat as example 1.7 shows.

Corollary 1.19. — *If X is Kähler and E is numerically flat, then all Chern classes $c_k(E)$ vanish.*

Proof. Note that the "if" part of the statement is clear by Prop. 1.15 (ii). The proof of the "only if" part is based on two deep facts: first the Uhlenbeck-Yau theorem asserting the existence of Hermite-Einstein metrics on stable vector bundles, secondly Kobayashi's flatness criterion derived from Lübke's inequality on Chern classes.

Since $\det E$ is numerically flat, Cor. 1.5 implies that $c_1(E) = c_1(\det E) = 0$. Fix a Kähler metric ω on X.

Step 1. Let $\mathcal{F}$ be a subsheaf of $\mathcal{O}(E)$ of rank p and let $\det \mathcal{F} = (\Lambda^p \mathcal{F})^{\star\star}$ be its determinant: $\det \mathcal{F}$ is a reflexive rank 1 sheaf, so it is locally free and we have an injection of sheaves

$$\det \mathcal{F} \hookrightarrow \Lambda^p E.$$

This morphism can be seen as a (non necessarily injective!) morphism of bundles. We know by 1.14 (ii) that $\Lambda^p E^\star$ is nef, so there is a sequence h_m of hermitian metrics on $S^m(\Lambda^p E^\star)$ such that

$$\Theta_{h_m}\left(S^m\left(\Lambda^p E^\star\right)\right) \geq -m\varepsilon\omega \otimes \mathrm{Id}$$

for $m \geq m_0(\varepsilon)$. We equip $\det \mathcal{F}$ with the possibly singular hermitian metric induced as a quotient of h_m by the non necessarily surjective morphism

$$S^m\left(\Lambda^p E^*\right) \to \left(\det \mathcal{F}\right)^{-m}.$$

Then 1.1 (iv) implies in the sense of currents

$$\Theta_{\widetilde{h}_m}\left((\det \mathcal{F})^{-1}\right) \geq \frac{1}{m}\Theta_m\left(S^m(\Lambda^p E^\star)\right)_{|(\det \mathcal{F})^{-m}} \geq -\varepsilon\omega$$

for $m \geq m_0(\varepsilon)$ (one cannot conclude from this that $(\det \mathcal{F})^{-1}$ is nef because $\widetilde{h}_m$ is singular). Therefore $c_1(\mathcal{F}) = c_1(\det \mathcal{F})$ satisfies the cohomology inequality

$$c_1(\mathcal{F}) \cdot \{\omega\}^{n-1} = \int_X -\Theta_{\widetilde{h}_m}\left((\det \mathcal{F})^{-1}\right) \wedge \omega^{n-1} \leq 0.$$

If $c_1(\mathcal{F})\cdot\{\omega\}^{n-1}=0$, the same arguments as in the proof of Proposition 1.16 show that $\lim\Theta_{\widetilde{h}_m}((\det\mathcal{F})^{-1})=0$ in the weak topology, in particular $c_1(\mathcal{F})=0$ and $\det\mathcal{F}$ is hermitian flat. In that case, Prop. 1.16 implies that $\det\mathcal{F}\to\Lambda^pE$ has no zeros: consider the section given by $\mathcal{O}\to\Lambda^pE\otimes(\det\mathcal{F})^{-1}$ and observe that the dual of $\Lambda^pE\otimes(\det\mathcal{F})^{-1}$ is nef.

Step 2. Now take a subsheaf $\mathcal{F}\subset\mathcal{O}(E)$ of minimal rank $p>0$ with $c_1(\mathcal{F})\cdot\{\omega\}^{n-1}=0$. After replacing $\mathcal{F}$ by $\mathcal{F}^{\star\star}$ we can assume that $\mathcal{F}$ is reflexive. The discussion made in Step 1 shows that $\det\mathcal{F}$ is a line subbundle of Λ^pE. The elementary lemma 1.20 below implies that $\mathcal{F}$ itself is locally free and is a subbundle of E. By definition, $\mathcal{F}$ cannot have any subsheaf of degree≥ 0 and of positive rank $<p$. Therefore $\mathcal{F}$ is ω-stable and by Uhlenbeck-Yau [UY86] $\mathcal{F}$ possesses a Hermite-Einstein metric. Since $\mathcal{F}^\star$ is a quotient of $E^\star$, 1.15 (i) shows that $\mathcal{F}^\star$ is nef and therefore $\mathcal{F}$ is numerically flat because $\det\mathcal{F}$ is flat. The inequalities for Chern classes which will be proved in Cor. 2.6 (independently of Cor. 1.19 !) imply $c_2(\mathcal{F})\cdot\{\omega\}^{n-2}=0$, because $c_1(\mathcal{F})=0$. The Kobayashi-Lübke flatness criterion ([Ko87], Th. 4.11) finally shows that $\mathcal{F}$ is hermitian flat. We set $E_1=\mathcal{F}$ and conclude by induction on the rank, since E/E_1 is again numerically flat by Prop. 1.15. □

LEMMA 1.20. — *Let E be a vector bundle over a complex manifold M and let $\mathcal{F}$ be a reflexive subsheaf of $\mathcal{O}(E)$ such that the induced bundle morphism $\det\mathcal{F}\to\Lambda^pE$ is injective,where $p=\operatorname{rank}\mathcal{F}$. Then $\mathcal{F}$ is locally free and is a subbundle of E.*

Proof. We are indebted to F.-O. Schreyer for the simple and elegant argument given below.

Since the question is purely local, we can assume that M is a ball in $\mathbb{C}^n$ and that $E=\mathcal{O}^r$ is a trivial vector bundle. As $\mathcal{F}$ is reflexive, $\mathcal{F}$ is locally free on the complement $M\setminus Z$ of some analytic subset Z of codimension≥ 3. As $\det\mathcal{F}\to\Lambda^pE$ is injective, $\mathcal{F}_{|M\setminus Z}$ is actually a subbundle of $E_{|M\setminus Z}$, so we get a holomorphic map into the Grassmannian $G(p,r)$ of p-planes of $\mathbb{C}^n$:

$$M\setminus Z\to G(p,r),\quad x\to\mathcal{F}_x.$$

Now the composition with the Plücker embedding

$$G(p,r)\to P(\Lambda^p\mathbb{C}^r),\quad S\mapsto\Lambda^pS$$

is precisely the map $M\setminus Z\to P(\Lambda^pE)$, $x\mapsto\det\mathcal{F}_x$. This map extends across Z by the assumption that $\det\mathcal{F}\to\Lambda^pE$ is everywhere injective. Hence the map $x\mapsto\mathcal{F}_x$ extends as a holomorphic map $\Phi:M\to G(p,r)$. By construction $\mathcal{F}_{|M\setminus Z}$ is contained in the pull-back $\Phi^\star S$ of the tautological rank p bundle on $G(p,r)$. Therefore $\mathcal{F}\subset\Phi^\star S$ with equality on $M\setminus Z$, and $\Phi^\star S$ is a vector subbundle of E. The sheaves $\operatorname{Hom}(\mathcal{F},\Phi^\star S)$ and $\operatorname{Hom}(\Phi^\star S,\mathcal{F})$ are themselves reflexive, so the identity homomorphism extends across Z into an isomorphism. Hence $\mathcal{F}=\Phi^\star S$ is a subbundle of E. □

Remark 1.21. — It would be interesting to know whether Theorem 1.18 and Cor. 1.19 are still true in the non Kähler case. In fact Li-Yau [LY87] proved the existence of Hermite-Einstein metrics also in that case, assuming that ω is a Gauduchon metric. The difficulty is with the Fulton-Lazarsfeld inequalities for Chern classes, which we can only prove for Kähler manifolds (the hard Lefschetz theorem is used, see §2). $\square$

2. Inequalities for Chern classes

The goal of this section is to show that the Fulton-Lazarsfeld inequalities [FL83] for Chern classes of ample vector bundles still hold for nef vector bundles over compact Kähler manifolds. The proof follows basically the same lines, but the arguments involving tensor products with ample line bundles must be replaced by arguments using the Kähler metric. This gain in generality will fortunately be accompanied by a substantial simplification of the technique (covering tricks are eliminated). The first step is an extension of the Bloch-Gieseker inequalities [BG71].

Proposition 2.1. — *Let X be a compact complex n-fold possessing a Kähler metric ω and let E be a nef vector bundle of rank r over X. Then for every $\lambda \geq 0$ we have*

$$\sum_{k=0}^{n} \binom{r \vee n-k}{n-k} \lambda^{n-k} \int_X c_k(E) \wedge \omega^{n-k} \geq 0$$

where $r \vee n = \max(r, n)$. In particular $\int_X c_n(E) \geq 0$ and more generally

$$\int_X c_k(E) \wedge \omega^{n-k} \geq 0$$

if the integral vanishes for higher values of k.

Proof. We first observe that for every $\lambda > 0$

$$\xi_\lambda = c_1(\mathcal{O}_E(1)) + \lambda \pi^\star\{\omega\}$$

is a Kähler class on $\mathbb{P}(E)$. Indeed, the construction of $\widehat{h}_\varepsilon$ in the proof of Th. 1.12 shows that

$$\Theta_{\widehat{h}_\varepsilon}(\mathcal{O}_E(1)) + \lambda \pi^\star \omega > 0$$

for $\varepsilon > 0$ small. We can of course suppose that $r \geq n$, otherwise we replace E by $E \oplus \mathcal{O}^{n-r}$. Then $\xi_0 = c_1(\mathcal{O}_E(1))$ satisfies the relation

$$\xi_0^r - \pi^\star c_1(E) \xi_0^{r-1} + \ldots + (-1)^n \pi^\star c_n(E) \xi_0^{r-n} = 0.$$

It follows that $\xi_\lambda = \xi_0 + \lambda \pi^\star\{\omega\}$ satisfies

$$\xi_\lambda^r - \pi^\star a_1(\lambda) \xi_\lambda^{r-1} + \ldots + (-1)^n \pi^\star a_n(\lambda) \xi_\lambda^{r-n} = 0,$$

where

$$a_p(\lambda) = \sum_{k=0}^{p} \binom{r-k}{p-k} \lambda^{p-k} c_k(E) \cdot \{\omega\}^{p-k}$$

($a_p(\lambda)$ is just the p-th elementary symmetric function in $\rho_j + \lambda\{\omega\}$, where $\rho_1, \ldots, \rho_r$ are the Chern roots). If the inequality of Prop. 2.1 does not hold, there is $\lambda \geq 0$ such that $a_n(\lambda) < 0$. Since $a_n(\lambda) \sim \binom{r}{n}\{\omega\}^n \lambda^n$ tends to $+\infty$ when $\lambda \to +\infty$, there would exist $\lambda > 0$ such that $a_n(\lambda) = 0$. Suppose that λ is chosen in this way and consider the cohomology class

$$\alpha = \xi_\lambda^{n-1} - \pi^\star a_1(\lambda)\xi_\lambda^{n-2} + \ldots + (-1)^{n-1}\pi^\star a_{n-1}(\lambda) \in H^{2n-2}(\mathbb{P}(E), \mathbb{R}).$$

Then $\alpha \neq 0$, for the projection formula implies

$$\pi_\star\left(\alpha \cdot \xi_\lambda^{r-n}\right) = \pi_\star\left(\xi_\lambda^{r-1}\right) = 1 \in H^0(X, \mathbb{R}).$$

On the other hand $\alpha \cdot \xi_\lambda^{r+1-n} = 0$ by our choice of λ. This contradicts the hard Lefschetz theorem. $\square$

Corollary 2.2. — *If Y is a d-dimensional analytic subset of X with $d \geq r$, then $\int_Y c_r(E) \wedge \omega^{d-r} \geq 0$.*

Proof. Let $\widetilde{X} \to X$ be an embedded resolution of singularities of Y in X (Hironaka [Hi64], Bierstone-Milman [BM90]), and let $f : \widetilde{Y} \to X$ be the induced map to X. Then $f^\star E$ is nef over $\widetilde{Y}$ and $\widetilde{Y}$ is Kähler, so

$$\int_{\widetilde{Y}} c_r(f^\star E) \wedge \widetilde{\omega}^{d-r} \geq 0$$

for every Kähler metric $\widetilde{\omega}$ on $\widetilde{Y}$ (clearly the integrals corresponding to degrees greater than $r = \operatorname{rank} E$ vanish, so Prop. 2.1 can be applied). Replacing $\widetilde{\omega}$ by $\widetilde{\omega}_\varepsilon = f^\star\omega + \varepsilon\widetilde{\omega}$ with $\varepsilon \to 0$, we get

$$\int_Y c_r(E) \wedge \omega^{d-r} = \int_{\widetilde{Y}} c_r(f^\star E) \wedge f^\star\omega^{d-r} \geq 0. \qquad \square$$

The second step is a semipositivity statement for the intersection with cone classes. Let $\overline{E} = P(E \oplus \mathcal{O})$ be the compactification of E by the hyperplane at infinity and let Z_E be the zero section of E. If C is a cone in E, that is, a $\mathbb{C}^\star$-invariant analytic subset in the total space of E, we denote by $\overline{C} = P(C \oplus \mathcal{O})$ its closure in $\overline{E}$. There are well defined cohomology classes

$$\{Z_E\} \in H^{2r}(\overline{E}, \mathbb{R}), \quad \{\overline{C}\} \in H^{2p}(\overline{E}, \mathbb{R})$$

with $p = \operatorname{codim} C$, which can be seen for instance as the cohomology classes of the associated closed positive currents $[Z_E]$, $[\overline{C}]$. We are interested in computing the intersection class

$$\{Z_E\} \cdot \{\overline{C}\}$$

in $H^{2(r+p)}(\overline{E}, \mathbb{R})$ (note that $\dim \overline{E} = r + n$). Following [FL83], we introduce the tautological quotient vector bundle $Q = \pi^\star(E \oplus \mathcal{O})/\mathcal{O}_{\overline{E}}(-1)$ of rank r on $\overline{E}$. Then we have $c_r(Q) = \{Z_E\}$ since the canonical section of Q (=image of $0 \oplus 1$ in Q) admits Z_E as its transversal zero locus in $\overline{E}$. In this way, by using the fact that $\overline{E}$ is Kähler, Cor. 2.2 implies

Proposition 2.3. — *If E is nef, then for every cone C in E of dimension $d = r + n - p \geq r$, we have*

$$\{Z_E\} \cdot \{\overline{C}\} \cdot \{\pi^\star \omega^{d-r}\} = \int_{\overline{C}} c_r(Q) \wedge \pi^\star \omega^{d-r} \geq 0.$$

The third and final step is an application of the determinantal formula of Kempf-Laksov [KL74]. First recall a few notations about Schur polynomials and Schubert calculus. Denote by $\Lambda(k, r)$ the set of all partitions of k by non negative integers $\leq r$. Thus an element $a \in \Lambda(k, r)$ is a sequence

$$r \geq a_1 \geq a_2 \geq \ldots \geq a_r \geq 0 \quad \text{with} \sum a_j = k.$$

Each $a \in \Lambda(k, r)$ gives rise to a Schur polynomial $P_a \in \mathbb{Z}[c_1, \ldots, c_r]$ of weighted degree $2k$ (with $\deg c_i = 2i$), defined as the $r \times r$ determinant

$$P_a(c) = \det(c_{a_i - i + j})_{1 \leq i, j \leq r}$$

where by convention $c_0 = 1$ and $c_i = 0$ if $i \notin [0, r]$. For every vector bundle E, the Kempf-Laksov formula expresses $P_a(c(E))$ as the class of a cone intersection.

Formula 2.4. — *Let E be a complex vector bundle of rank r on a compact differentiable manifold M. Let V be a complex vector space of dimension $k + r$ and let V_M be the trivial vector bundle $M \times V$. Fix a flag of subspaces $\{0\} \subset V_1 \subset \ldots \subset V_k \subset V$ with $\dim V_i = r + i - a_i$. Consider the cone $\Omega_a(E) \subset \mathrm{Hom}(V_M, E)$ whose fibre over $x \in M$ consists of all $u \in \mathrm{Hom}(V, E_x)$ such that*

$$\dim(\ker u \cap V_i) \geq i.$$

Then $\Omega_a(E)$ is a locally trivial cone bundle over M of codimension $k = \sum a_i$ in $\mathrm{Hom}(V_M, E)$ and we have

$$P_a(c(E)) = \pi_\star \Big(\{Z_{\mathrm{Hom}(V_M, E)}\} \cdot \{\overline{\Omega_a(E)}\} \Big)$$

where $\pi_\star$ denotes the integration along the fibres of $\pi : \overline{\mathrm{Hom}(V_M, E)} \to M$.

Proof. Observe first that the current $[\overline{\Omega_a(E)}]$ is well defined, although $\overline{\Omega_a(E)}$ is singular and M is not analytic; indeed $\overline{\Omega_a(E)}$ is locally the product of an open set in the base with an analytic set in the fibre of $\mathrm{Hom}(V_M, E)$. Formula 2.4 is proved in [KL74] in the context of schemes and non singular projective varieties. Since we allow ourselves to deal with C^∞ objects, the proof becomes in fact much easier. First observe that the formula is functorial with respect to base change. Now, every C^∞ vector bundle is generated by a finite number N of sections, hence there is a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & Q \\ \downarrow & & \downarrow \\ M & \longrightarrow & G(N-r, N) \end{array}$$

where Q is the universal quotient vector bundle of rank r over the Grassmannian. Hence it is sufficient to check the formula for $E = Q$. In that case the formula is simply a sophisticated formulation of Giambelli's formula for Schubert cycles in $G(N-r, N)$ (see e.g. [GH78], p.206 and 415). $\square$

A combination of formula 2.4 with Prop. 2.3 gives

$$\int_X P_a(c(E)) \wedge \omega^{n-k} = \{Z_{\mathrm{Hom}(V_M,E)}\} \cdot \{\overline{\Omega_a(E)}\} \cdot \{\pi^\star \omega^{n-k}\} \geq 0$$

when E is nef. More generally, the argument of Cor. 2.2 implies:

THEOREM 2.5. — *Let E be a nef vector bundle over a compact complex manifold X equipped with a Kähler metric ω. Then for any Schur polynomial P_a of degree $2k$ and any analytic subset $Y \subset X$ of dimension $d \geq k$ we have*

$$\int_Y P_a(c(E)) \wedge \omega^{d-k} \geq 0. \qquad \square$$

Even in the case of projective algebraic manifolds, this result is new because ω^{d-k} need not be representable by an algebraic cycle. Interesting examples of Schur polynomials are

$$P_{k00\ldots}(c) = \begin{vmatrix} c_k & c_{k+1} & c_{k+2} & \ldots & \ldots \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \ldots & & 1 \end{vmatrix} = c_k,$$

$$P_{11\ldots10\ldots0}(c) = \begin{vmatrix} c_1 & c_2 & c_3 & \ldots & c_k \\ 1 & c_1 & c_2 & & \vdots \\ 0 & 1 & c_1 & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & c_1 & c_2 \\ 0 & \ldots & 0 & 1 & c_1 \end{vmatrix} = \text{Segre class } s_k,$$

$$P_{k-1,10\ldots0}(c) = \begin{vmatrix} c_{k-1} & c_k \\ 1 & c_1 \end{vmatrix} = c_1 c_{k-1} - c_k.$$

The Schur polynomials P_a with $a \in \Lambda(k,r)$ form a basis of the $\mathbb{Q}$-vector space of polynomials of weighted degree $2k$. A Polynomial

$$P = \sum \lambda_a P_a$$

is said to be numerically positive if all coefficients λ_a are ≥ 0. It is well known in the theory of Schur polynomials that a product of numerically positive polynomials is again numerically positive (see [FL83] or [Md79] p.68 for a purely combinatorial proof based on the Littlewood-Richardson rule). In particular

$$c_1^k - c_k = \sum_{j=2}^{k} c_1^{k-j}(c_1 c_{j-1} - c_j) \geq 0,$$

i.e. $c_k \leq c_1^k$. Similarly, any Chern monomial $c_I = c_1^{i_1} \ldots c_r^{i_r}$ of degree $2k$ satisfies $0 \leq c_I \leq c_1^k$. By Th. 2.5 we thus get:

Corollary 2.6. — *If E is nef, any Chern monomial c_I of degree $2k$ satisfies*

$$0 \leq \int_X c_I(E) \wedge \omega^{n-k} \leq \int_X c_1(E)^k \wedge \omega^{n-k}.$$

Corollary 2.7. — *If E is nef and $c_1(E)^n = 0$, all Chern polynomials $P(c(E))$ of degree $2n$ vanish.*

3. Compact manifolds with nef tangent bundles. Structure of the Albanese map (Kähler case)

3.A. Some examples

The first examples are produced by

Proposition 3.1. — *Every homogeneous compact complex manifold has a nef tangent bundle.*

Indeed, if X is homogeneous, the Killing vector fields generate T_X, so T_X is a quotient of a trivial vector bundle. In particular, we get the following

Examples 3.2 (homogeneous case). —

(i) Rational homogeneous manifolds: $\mathbb{P}_n$, flag manifolds, quadrics Q_n, and more generally quotients $X = G/P$ of a simply connected complex Lie group G by a parabolic subgroup P
(all projective algebraic with K_X^{-1} ample).

(ii) Tori $\mathbb{C}^n/\Lambda$
(Kähler, possibly non algebraic).

(iii) Hopf manifolds $(\mathbb{C}^n \setminus \{0\})/H$ where H is a discrete group of homotheties
(non Kähler for $n \geq 2$).

(iv) Iwasawa manifolds G/Λ where G is the group of unipotent upper triangular $p \times p$ matrices and Λ the subgroup of matrices with entries in the ring of

integers of some imaginary quadratic field, e.g. $\mathbb{Z}[i]$
(non Kähler if $p \geq 3$, although T_X is trivial). $\square$

We must remark at this point that not all manifolds X with nef tangent bundles are homogeneous, the automorphism group may even be discrete:

Example 3.3. — Let $\Gamma = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, $\operatorname{Im}\tau > 0$, be an elliptic curve. Consider the quotient space $X = (\Gamma \times \Gamma \times \Gamma)/G$ where $G = \{1, g_1, g_2, g_1g_2\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (z_1 + \tfrac{1}{2}, -z_2, -z_3), \\ g_2(z_1, z_2, z_3) &= (-z_1, z_2 + \tfrac{1}{2}, -z_3 + \tfrac{1}{2}), \\ g_1g_2(z_1, z_2, z_3) &= (-z_1 + \tfrac{1}{2}, -z_2 + \tfrac{1}{2}, z_3 + \tfrac{1}{2}). \end{aligned}$$

Then G acts freely, so X is smooth. It is clear also that T_X is nef (in fact T_X is unitary flat). Since the pull-back of T_X to $\Gamma \times \Gamma \times \Gamma$ is trivial, we easily conclude that T_X has no sections, thanks to the change of signs in g_1, g_2, g_1g_2. Therefore the automorphism group $\operatorname{Aut}(X)$ is discrete. The same argument shows that $H^0(X, \Omega_X^1) = 0$.

Similar examples can be found in higher dimensions: for instance, one can replace one or several of the factors Γ in $\Gamma \times \Gamma \times \Gamma$ by an arbitrary torus, the translation by $\frac{1}{2}$ being replaced by a translation with an element in the half-lattice. In this way we get examples of Kähler non algebraic manifolds with T_X nef and $H^0(X, \Omega_X^1) = 0$ for all $n = \dim X \geq 4$, possibly of algebraic dimension 0 when $n \geq 6$ (by Prop. 7.3 below, such non algebraic examples do not exist in dimension ≤ 3). $\square$

The above example suggests that we should try to classify manifolds with nef tangent bundles only up to finite étale covers. In fact Prop. 1.10 immediately implies:

Proposition 3.4. — *Let X be a compact complex manifold and let $\widetilde{X}$ be a finite étale cover of X. Then T_X is nef if and only if $T_{\widetilde{X}}$ is nef.*

Example 3.5. — Let X be the ruled surface $\mathbb{P}(E)$ over the elliptic curve $\Gamma = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ defined in 1.7. Then the relative tangent bundle of $\mathbb{P}(E) \to \Gamma$ (=relative anticanonical line bundle) is $\pi^\star(\det E^\star) \otimes \mathcal{O}_E(2) \simeq \mathcal{O}_E(2)$ and T_Γ is trivial, so T_X is nef. Moreover X is almost homogeneous, with automorphisms induced by

$$(x, z_1, z_2) \mapsto (x + a, z_1 + b, z_2), \quad (a, b) \in \mathbb{C}^2$$

and has a single closed orbit equal to the curve $\{z_2 = 0\}$. Here, no finite étale cover of X can be homogeneous, otherwise $K_X^{-1} = \mathcal{O}_E(2)$ would be semipositive. Observe that no power of K_X^{-1} is generated by sections, although K_X^{-1} is nef. $\square$

3.B. General properties

Let X be a compact complex n-dimensional manifold. We collect here a few remarkable facts which hold under the assumption that T_X is nef.

PROPOSITION 3.6 [De91]. — *Suppose that T_X is nef. If X belongs to the Fujiki class $\mathcal{C}$ (class of manifolds bimeromorphic to Kähler manifolds), then X is Kähler. In particular, if X is Moishezon, then X is projective algebraic.*

An algebraic proof of 3.6 for the Moishezon case will be given in §4. The next proposition was also proved in [CP91] in the algebraic case.

PROPOSITION 3.7. — *If T_X is nef, then the line bundle $\mathcal{O}(D)$ is nef for every effective divisor D in X.*

Proof. Propositions 3.6 and 3.7 are obtained in [De91] as consequences of a general regularization theorem for closed positive currents. Since 3.7 is quite elementary we indicate here briefly the argument.

Let $\sigma \in H^0(X, \mathcal{O}(D))$ be a section with divisor D. Then for k larger than the maximum vanishing order of σ on X, the k-jet section $J^k\sigma \in H^0(X, J^k\mathcal{O}(D))$ has no zeros. Therefore, there is an injection $\mathcal{O} \to J^k\mathcal{O}(D)$ and a dual surjection

$$\big(J^k\mathcal{O}(D)\big)^\star \otimes \mathcal{O}(D) \to \mathcal{O}(D).$$

Now, we observe that $J^k\mathcal{O}(D)$ has a filtration whose graded bundle is $\bigoplus_{0\le p\le k} S^pT^\star_X \otimes \mathcal{O}(D)$, so $(J^k\mathcal{O}(D))^\star \otimes \mathcal{O}(D)$ has a dual filtration with graded bundle $\bigoplus_{0\le p\le k} S^pT_X$. By 1.14 (ii) and 1.15 (ii), we conclude that $(J^k\mathcal{O}(D))^\star \otimes \mathcal{O}(D)$ is nef, so its quotient $\mathcal{O}(D)$ is nef by 1.15 (i). □

A combination of 3.6 and 3.7 leads to the following result.

PROPOSITION 3.8. — *Let X be a Kähler n-fold such that T_X is nef. If X admits a nef line bundle L such that $c_1(L)^n \neq 0$, then X is projective algebraic and L is ample.*

Proof. Let ω be a Kähler metric on X. Then for every $\varepsilon > 0$ there is a hermitian metric h_ε on L such that $\Theta_{h_\varepsilon}(L) \ge -\varepsilon\omega$. We apply the holomorphic Morse inequalities proved in [De85]: let $X(h_\varepsilon, q)$ be the open set of points $x \in X$ where $\Theta_{h_\varepsilon}(L)$ has q negative eigenvalues and $n-q$ positive ones, and let $X(h_\varepsilon, \le q)$ be the union of $X(h_\varepsilon, j)$ for $j \le q$. On $X(h_\varepsilon, q)$ we have

$$0 \le \frac{1}{n!}(-1)^q\Theta_{h_\varepsilon}(L)^n \le \frac{1}{q!(n-q)!}\varepsilon^q\omega^q \wedge (\Theta_{h_\varepsilon}(L) + \varepsilon\omega)^{n-q}\,.$$

Indeed, $\frac{1}{n!}\Theta_{h_\varepsilon}(L)^n/\omega^n$ is the product $\lambda_1 \ldots \lambda_n$ of the eigenvalues of $\Theta_{h_\varepsilon}(L)$, which satisfy

$$-\varepsilon \le \lambda_1 \le \ldots \le \lambda_q < 0 < \lambda_{q+1} \le \ldots \le \lambda_n \text{ on } X(h_\varepsilon, q).$$

Our inequality then follows from the obvious bound

$$|\lambda_1 \ldots \lambda_n| \leq \varepsilon^q (\lambda_{q+1}+\varepsilon) \ldots (\lambda_n+\varepsilon).$$

By integration over X, we infer from this that

$$\left| \int_{X(h_\varepsilon, \geq 2)} \Theta_{h_\varepsilon}(L)^n \right| \leq \sum_{q=2}^{n} \frac{n!}{q!(n-q)!} \varepsilon^q \{\omega\}^q \cdot (c_1(L)+\varepsilon\{\omega\})^{n-q} = O(\varepsilon^2).$$

Hence

$$\int_{X(h_\varepsilon, \leq 1)} \Theta_{h_\varepsilon}(L)^n = c_1(L)^n - \int_{X(h_\varepsilon, \geq 2)} \Theta_{h_\varepsilon}(L)^n > 0$$

for $\varepsilon > 0$ small enough. By [De85], it follows that X is Moishezon and $\kappa(L) = n$, and so X is projective algebraic thanks to Proposition 3.6. It remains to show that L is ample. Let A be a very ample smooth divisor. The exact sequence

$$0 \to H^0(X, \mathcal{O}(kL-A)) \to H^0(X, \mathcal{O}(kL)) \to H^0(A, \mathcal{O}(kL)_{|A})$$

shows that $\mathcal{O}(kL-A)$ has sections for k large: in fact since L is nef and big we have $h^0(X, \mathcal{O}(kL)) \geq ck^n$, while $h^0(A, \mathcal{O}(kL)_{|A}) \leq ck^{n-1}$. Hence there is an effective divisor D such that $\mathcal{O}(kL) = \mathcal{O}(A+D)$. Proposition 3.7 implies that D is nef, therefore $A+D$ and L are ample. □

3.C. Structure of the Albanese map (Kähler case)

We now investigate the structure of the Albanese map when X is Kähler. Let $q = h^1(X, \mathcal{O}_X) = h^0(X, \Omega_X^1)$ be the irregularity of X and let $u_1, \ldots, u_q$ be a basis of holomorphic 1-forms. Since X is Kähler, the forms u_j are always closed. The Albanese map is classically defined as follows: consider the map

$$\widetilde{X} \to \mathbb{C}^q, \quad x \mapsto \left(\int_{x_0}^{x} u_j \right)_{1 \leq j \leq q}$$

where $\widetilde{X}$ is the universal covering of X and $x_0 \in \widetilde{X}$ a given point. Then the orbit of x_0 under $\pi_1(X)$ is mapped onto a lattice $\Lambda \subset \mathbb{C}^q$ (by Hodge theory). Therefore the map $\widetilde{X} \to \mathbb{C}^q$ factorizes into a map

$$\alpha : X \to A(X) := \mathbb{C}^q/\Lambda,$$

which is precisely the Albanese map. By construction $d\alpha = (u_1, \ldots, u_q)$ at every point. The following fact was already remarked in [CP91].

Proposition 3.9. — *Let X be a Kähler manifold with T_X nef. Then the Albanese map α is a submersion onto the Albanese torus $A(X)$ and the relative tangent bundle to α is nef. The fibres are connected and have nef tangent bundles.*

Proof. If we had rank $(d\alpha(x)) < q$ at some point $x \in X$, there would exist a non zero linear combination $u = \lambda_1 u_1 + \ldots + \lambda_q u_q$ with $u(x) = 0$. Since T_X is nef,

$u \in H^0(X, T^\star_X)$ would contradict Prop. 1.16. Therefore α is a submersion. The relative tangent bundle sequence

$$0 \longrightarrow T_{X/A(X)} \longrightarrow T_X \xrightarrow{d\alpha} \alpha^\star T_{A(X)} \longrightarrow 0$$

in which $T_{A(X)}$ is trivial shows by 1.15 (iii) that $T_{X/A(X)}$ is nef. This implies in particular that the fibres have nef tangent bundles. If the fibres were not connected, their connected components would give rise to a factorization $\alpha : X \to Y \to A(X)$ where $X \to Y$ has connected fibres and $Y \to A(X)$ is a finite étale cover. Then Y is a complex torus, so the universal property of the Albanese map gives a factorization $X \to A(X) \to Y$ where $A(X) \to Y$ is again étale. But then the fibres of $X \to Y$ would not be connected, contradiction. $\square$

The crucial step in the proof of our structure theorem is contained in the following proposition (we denote here by $c_k(X) = c_k(T_X)$ the Chern classes of X).

PROPOSITION 3.10. — *Let X be a Kähler manifold with T_X nef. One of the two following situations occurs:*

(i) $c_1(X)^n > 0$.
Then X is a Fano manifold (that is, a projective algebraic manifold with K_X^{-1} ample) and $\chi(X, \mathcal{O}_X) = 1$.

(ii) $c_1(X)^n = 0$.
Then $\chi(X, \mathcal{O}_X) = 0$ and X admits a non trivial holomorphic p-form of odd degree. Moreover, there is a finite étale cover $\widetilde{X}$ of X such that

$$q(\widetilde{X}) := h^0(X, \Omega^1_{\widetilde{X}}) > 0.$$

Proof. Since $c_1(X) = c_1(K_X^{-1})$ and $K_X^{-1} = \Lambda^n T_X$ is nef, we certainly have $c_1(X)^n \geq 0$.

(i) If $c_1(X)^n > 0$, Prop. 3.8 shows that X is projective algebraic and that K_X^{-1} is ample. Then the Kodaira vanishing theorem implies

$$H^q(X, \mathcal{O}_X) = H^q(X, K_X \otimes K_X^{-1}) = 0 \quad \text{for}\, q \geq 1,$$

therefore $\chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$.

(ii) If $c_1(X)^n = c_1(T_X)^n = 0$, Corollary 2.7 implies that all Chern polynomials $P(c_1(X), \ldots, c_n(X))$ of degree $2n$ vanish. Hence the Hirzebruch-Riemann-Roch formula

$$\chi(X, \mathcal{O}_X) = \int_X \mathrm{ch}(\mathcal{O}_X) \cdot \mathrm{Todd}(T_X)$$

shows that $\chi(X, \mathcal{O}_X) = \sum (-1)^p h^p(X, \mathcal{O}_X) = 0$. Since $h^0(X, \mathcal{O}_X) = 1$, it follows necessarily that $h^0(X, \Omega^p_X) = h^p(X, \mathcal{O}_X) \geq 1$ for some odd degree p.

Let $u \in H^0(X, \Omega^p_X)$ be a non zero p-form. We consider the contraction morphism

$$\Lambda^{p-1} T_X \xrightarrow{S} \Omega^1_X, \quad S(v) = v \lrcorner\, u.$$

We claim that S has constant rank (as a bundle morphism). In fact, if r is the generic rank of S, then

$$\Lambda^r S : \Lambda^r\left(\Lambda^{p-1}T_X\right) \to \Omega^r_X$$

can be seen as a non zero holomorphic section of

$$\left(\Lambda^r(\Lambda^{p-1}T_X) \otimes \Lambda^r T_X\right)^\star,$$

and $\Lambda^r(\Lambda^{p-1}T_X) \otimes \Lambda^r T_X$ is nef by Prop. 1.14. Therefore $\Lambda^r S$ has no zeros and so the rank of S is equal to r at every point (note that $r > 0$ since $u \neq 0$). Now $E = \operatorname{Im} S$ is a rank r subbundle of Ω^1_X which is a quotient of $\Lambda^{p-1}T_X$. By Prop. 1.14 (ii) and 1.15 (i) we conclude that E is nef; since $E^\star$ is a quotient of $(\Omega^1_X)^\star = T_X$, we also infer that $E^\star$ is nef.

Finally, Th. 1.18 implies that E admits a hermitian flat subbundle E_1 of positive rank $r_1 \leq r$, given by some unitary representation $\rho : \pi_1(X) \to U(r_1)$. We would like E_1 to have a finite monodromy group $G := \rho(\pi_1(X))$, so that the pull-back of E_1 by some finite étale cover $\pi : \widetilde{X} \to X$ gives rise to a trivial subbundle $\pi^\star E_1 \subset \pi^\star\Omega^1_X = \Omega^1_{\widetilde{X}}$. A well-known theorem of Tits [Ti72] asserts that every subgroup G of a linear group either contains a non abelian free subgroup or a solvable subgroup of finite index. The first case of Tits' alternative is not possible here: in fact, Th. 1.1 of [DPS92] shows, since K_X^{-1} is nef, that $\pi_1(X)$ has subexponential growth and cannot contain a non abelian free subgroup. Therefore G contains a solvable subgroup Γ of finite index. Let

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_N = \{1\}$$

be the series of derived subgroups $\Gamma_{i+1} = [\Gamma_i, \Gamma_i]$. If G is finite, we have already seen that there is a trivial finite étale pull-back $\pi^\star E_1 \subset \Omega^1_{\widetilde{X}}$ and so $H^0(\widetilde{X}, \Omega^1_{\widetilde{X}}) \neq 0$. Otherwise G and Γ are infinite. Let i be the smallest index such that Γ_i/Γ_{i+1} is infinite. Then G/Γ, $\Gamma/\Gamma_1, \ldots, \Gamma_{i-1}/\Gamma_i$ are finite, and so is G/Γ_i. Therefore $\Pi_i = \rho^{-1}(\Gamma_i)$ is of finite index in $\pi_1(X)$ and gives rise to a finite étale cover $\widetilde{X} \to X$ with $\pi_1(\widetilde{X}) = \Pi_i$. The representation ρ induces a surjective homomorphism

$$H_1(\widetilde{X}, \mathbb{Z}) = \Pi_i/[\Pi_i, \Pi_i] \longrightarrow \Gamma_i/[\Gamma_i, \Gamma_i] = \Gamma_i/\Gamma_{i+1}.$$

Hence $H_1(\widetilde{X}, \mathbb{Z})$ is infinite, $\widetilde{X}$ has non zero first Betti number b_1 and

$$h^0(\widetilde{X}, \Omega^1_{\widetilde{X}}) = \tfrac{1}{2}b_1 > 0,$$

as desired. □

Remark 3.11. — Example 3.3 shows that $H^0(X, \Omega^1_X)$ is not necessarily different from 0 in the second case (ii) of Prop. 3.10. In this example, it is easily verified that

$$h^0(X, \mathcal{O}_X) = 1, \quad h^0(X, \Omega^1_X) = h^0(X, \Omega^2_X) = 0, \quad h^0(X, \Omega^3_X) = 1. \qquad \square$$

The proof of the structure theorem proceeds by induction on the dimension. In the induction step, we need to compute the irregularity of the fibres of the Albanese map. For convenience, we introduce the notations

$$q(X) = h^1(X, \mathcal{O}_X), \quad \widetilde{q}(X) = \sup q(\widetilde{X})$$

where the supremum is taken over all finite étale covers $\widetilde{X} \to X$. The following topological result generalizes and makes more precise Prop. 2.8 of [CP91].

PROPOSITION 3.12. — *Let X, Y be compact Kähler manifolds and let $g : X \to Y$ be a smooth fibration with connected fibres. If F denotes any fibre of g, then*

(i) $q(X) \leq q(Y) + q(F)$,

(ii) $\widetilde{q}(X) \leq \widetilde{q}(Y) + \widetilde{q}(F)$.

(ii) *Suppose that Y is a complex torus and that the fibres F have the following properties: $\pi_1(F)$ contains an abelian subgroup of finite index, and for every finite étale cover $\widetilde{F}$ of F the Albanese map $\widetilde{F} \to A(\widetilde{F})$ has constant rank. Then*

$$\widetilde{q}(X) = \widetilde{q}(Y) + \widetilde{q}(F).$$

Proof. **(i)** The Leray spectral sequence of the constant sheaf $\mathbb{R}_X$ over X satisfies

$$\begin{aligned} E_2^{s,t} &= H^s(Y, R^t g_\star \mathbb{R}_X), \\ E_r^{s,t} &\Rightarrow H^{s+t}(X, \mathbb{R}). \end{aligned}$$

Since the fibres of g are connected and since g is differentiably a locally trivial fibre bundle, we have $R^0 g_\star \mathbb{R}_X = \mathbb{R}_Y$, while $R^1 g_\star \mathbb{R}_X$ is a locally constant real vector bundle of rank $2q$, $q = q(F)$, containing $R^1 g_\star \mathbb{Z}_X$ as a lattice; in particular the monodromy group is contained in $SL_{2q}(\mathbb{Z})$. Therefore

$$\dim H^0(Y, R^1 g_\star \mathbb{R}_X) \leq 2q(F),$$

the equality occuring if and only if the locally constant system $R^1 g_\star \mathbb{Z}_X$ is trivial, i.e. $\pi_1(X)$ acts trivially on $H^1(F, \mathbb{Z})$ by monodromy. We know by Blanchard [Bl56] that the Leray spectral sequence always degenerates in E_2 for Kähler fibrations. Hence

$$\begin{aligned} 2q(X) &= \dim H^1(X, \mathbb{R}) \\ &= \dim H^1(Y, \mathbb{R}) + \dim H^0(Y, R^1 g_\star \mathbb{R}_X) \\ &\leq 2q(Y) + 2q(F), \end{aligned}$$

with equality if and only if $\pi_1(X) \to \mathrm{Aut}(H^1(F, \mathbb{Z}))$ is trivial.

(ii) The homotopy exact sequence of the fibration yields

$$\pi_2(Y) \to \pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to 1.$$

Any finite étale cover $\widetilde{X}$ of X corresponds to a subgroup $\pi_1(\widetilde{X}) \subset \pi_1(X)$ of finite index. If we take the image of $\pi_1(\widetilde{X})$ in $\pi_1(Y)$ and its inverse image in $\pi_1(F)$, we obtain subgroups of finite indices, corresponding to étale covers $\widetilde{Y}$ and $\widetilde{F}$. Moreover $\pi_2(Y) = \pi_2(\widetilde{Y})$ is mapped onto the kernel of $\pi_1(\widetilde{F}) \to \pi_1(\widetilde{X})$, so we get a smooth fibration $\widetilde{X} \to \widetilde{Y}$ with fibre $\widetilde{F}$. By (i) this implies

$$q(\widetilde{X}) \leq q(\widetilde{Y}) + q(\widetilde{F}) \leq \widetilde{q}(Y) + \widetilde{q}(F).$$

Inequality (ii) follows by taking the supremum over all $\widetilde{X}$.

(iii) Since $\pi_1(F)$ is finitely generated, the additional hypotheses in (iii) imply that $\pi_1(F)$ contains a free finitely generated abelian subgroup Λ of finite index. Take a finite étale cover $\widetilde{F}$ of F with $\pi_1(\widetilde{F}) = \Lambda$, so that $q(\widetilde{F}) = \frac{1}{2}\,\mathrm{rank}\,\Lambda = \widetilde{q}(F)$. It is easy to see that Λ has only finitely many conjugates by $\mathrm{Aut}(\pi_1(F))$: if $\varphi \in \mathrm{Aut}(\pi_1(F))$ and $N = [\pi_1(F) : \Lambda]$, then $\Lambda \cap \varphi(\Lambda)$ has index at most $[\pi_1(F) : \varphi(\Lambda)] = N$ in Λ, so $\Lambda \cap \varphi(\Lambda) \supset N\Lambda$. Therefore $\Lambda' = \bigcap_{\varphi \in \mathrm{Aut}(\pi_1(F))} \varphi(\Lambda)$ is such that $N\Lambda \subset \Lambda' \subset \Lambda$, has finite index in $\pi_1(F)$ and is left invariant by the monodromy $\pi_1(Y) \to \mathrm{Aut}(\pi_1(F))$. This gives rise to a semidirect product $G' = \Lambda' \ltimes \pi_1(Y)$ and to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda' & \longrightarrow & G' & \longrightarrow & \pi_1(Y) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \downarrow = & & \\ 0 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(Y) & \longrightarrow & 0 \end{array}$$

where the injections have finite index (note that $\pi_2(Y) = 0$). The upper row corresponds to a Kähler fibration $X' \to Y$ whose fibre F' covers $\widetilde{F}$. We check below in Steps (iv-vii) that the monodromy action

$$\pi_1(Y) \to \mathrm{Aut}(H^1(F', \mathbb{Z})) \simeq \mathrm{Aut}(\Lambda'^\star)$$

has a finite image, and therefore there is a subgroup $\widetilde{H} \subset \pi_1(Y)$ of finite index which has trivial action. We set $\widetilde{G} = \Lambda' \ltimes \widetilde{H} \subset G'$ and obtain in this way a fibration $\widetilde{X} \to \widetilde{Y}$ with fibre $\widetilde{F} = F'$, such that the monodromy is trivial. This implies

$$\widetilde{q}(X) \geq q(\widetilde{X}) = q(\widetilde{Y}) + q(\widetilde{F}) = \widetilde{q}(Y) + \widetilde{q}(F)$$

as desired.

(iv) It is sufficient to show, under the hypotheses of (iii) and after changing (X, Y, F) into (X', Y, F'), that

$$\pi_1(Y) \to \mathrm{Aut}(H^1(F, \mathbb{Z}))$$

has a finite image, i.e. that the monodromy group of $R^1 g_\star \mathbb{R}_X$ is finite. For this, we define a euclidean structure on $R^1 g_\star \mathbb{R}_X$ as follows: if u, v are local sections of $R^1 g_\star \mathbb{R}_X$ near a point $y \in Y$ corresponding to classes $u, v \in H^1(F_y, \mathbb{R})$, Hodge theory shows that one can write $u = Re(u')$, $v = Re(v')$ for some uniquely defined holomorphic 1-forms u', v' on F_y. Then we set

$$\langle u, v \rangle_y = Re \int_{F_y} iu' \wedge \overline{v}' \wedge \omega^{p-1}, \quad p = \dim F_y,$$

where ω is a Kähler metric on X. If the euclidean structure can be proved to be locally flat, then the monodromy group is contained in $SL_{2q}(\mathbb{Z}) \cap O_{2q}(\langle\ ,\ \rangle)$ which is a finite group.

(v) Consider the relative Albanese factorization

$$X \to A(X/Y) \to Y$$

of the fibration: $A(X/Y)$ is a smooth torus fibration over Y whose fibres are the Albanese tori of the fibres F_y; the existence of such a relative Albanese map is proved in [Fj83] and [Cm85] in a much more general context, but is very easy to check here since everything is smooth. We prove later that $A(X/Y) \to Y$ is a holomorphically locally trivial torus bundle over Y. Then if U is a small neighborhood of a point $y \in Y$ and $g^{-1}(U)$ its inverse image in X, there are well defined holomorphic maps

$$g^{-1}(U) \to A(g^{-1}(U)/U) \simeq A(F_y) \times U \xrightarrow{pr_2} A(F_y)$$

over U. This shows that every holomorphic 1-form on F_y can be extended to a d-closed holomorphic 1-form on $g^{-1}(U)$. Therefore in the above definition of $\langle u, v\rangle_y$ we may suppose that u', v' are d-closed holomorphic 1-forms on $g^{-1}(U)$. Stokes' formula then shows that $\langle u, v\rangle_y$ does not depend on $y \in U$, as desired.

(vi) We first check that the total space $A(X/Y)$ is Kähler. In fact, for every fibre F the image $\alpha(F)$ in the Albanese torus $A(F)$ is by assumption a submanifold S of $A(F)$. The tangent spaces $T_x S$ at various points $x \in S$ generate the vector space $T_{A(F)}$, otherwise S would be contained in a subtorus of $A(F)$ (this is of course impossible by the universal property of the Albanese map). Therefore there is an integer N such that the map

$$\begin{aligned} F^N &\longrightarrow A(F) \\ (x_1, \ldots, x_N) &\longmapsto \alpha(X_1) + \sum_{j=2}^{N} (\alpha(x_j) - \alpha(x_1)) \end{aligned}$$

is a submersion. Doing this fibre by fibre and using the compactness of Y, we find a N-th power fibered product of X over Y and a submersion

$$X \times_Y X \times_Y \ldots \times_Y X \to A(X/Y).$$

The fibered product is a closed submanifold of X^N, thus Kähler, and therefore its submersive image $A(X/Y)$ is Kähler by pushing forward an appropriate power ω^p of a Kähler metric.

(vii) By (vi), the local triviality of $A(X/Y) \to Y$ is a consequence of the fact that any smooth Kähler fibration $T \to \mathbb{C}$ by complex tori is trivial (join any two points of Y by a line and take the pull-back of $A(X/Y) \to Y$ to the line). In fact any such fibration $T \to \mathbb{C}$ is given by a map from $\mathbb{C}$ to the Siegel domain D parametrizing the moduli space of complex tori which are polarized by a Kähler form ω with given periods. It is known by general results of Griffiths and Schmid (see [GS69], Cor.9.4) that there are no non constant holomorphic maps $\mathbb{C} \to D$. In our case the argument is so simple that we can explain briefly the details. Let $\mathbb{C} \ni z \mapsto P(z)$ be the $n \times 2n$ holomorphic matrix representing a lattice basis corresponding to the torus T_z. If $H(z)$ is the positive hermitian matrix representing the Kähler class $\omega_{|T_z}$, the periods of ω are given by

$$\mathrm{Im}\left({}^t P H \overline{P}\right) = Q$$

where Q is a real symplectic $2n \times 2n$ matrix with constant coefficients. Let U be an invertible real $2n \times 2n$ matrix such that ${}^t UQU$ is the standard symplectic matrix and set $PU = (AB)$ where A, B are holomorphic $n \times n$ matrices. We may assume (possibly after changing U) that $A(z)$ is generically invertible, and thus that $A(z) = I$ by changing the basis of $\mathbb{C}^n$ via $A(z)^{-1}$. We then get a meromorphic matrix B on $\mathbb{C}$ such that

$$\mathrm{Im}\begin{pmatrix} I \\ {}^tB \end{pmatrix} H(IB) = {}^tUQU = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

i.e. ${}^tB = B$ and $\mathrm{Im}\, B = H^{-1}$ positive definite. The second condition clearly implies that B must be constant by Liouville's theorem, therefore the lattice matrix P is constant. □

Remark 3.13. — A careful examination of the proof shows that the equality $\widetilde{q}(X) = \widetilde{q}(Y) + \widetilde{q}(F)$ holds under the following much more general hypotheses:

(iii') X, Y are Kähler, the boundary map $\pi_2(Y) \to \pi_1(F)$ is zero, $\pi_1(F)$ contains an abelian (or polycyclic) subgroup of finite index and Y contains a subvariety S with $\pi_1(S) \simeq \pi_1(Y)$, such that any two generic points in the universal covering $\widehat{S}$ can be joined through a chain of holomorphic images $\mathbb{C} \to \widehat{S}$.

The proof of Prop. 3.12 actually shows that the monodromy group of $\pi_1(Y)$ in $\mathrm{Aut}(H^1(F, \mathbb{Z}))$ is finite and that the relative Albanese fibration $A(X/Y) \to Y$ is locally trivial over S.

The constant rank assumption made in (iii) is not necessary because we always get in (vi) a surjective map $X \times_Y \ldots \times_Y X \to A(X/Y)$, in particular $A(X/Y)$ always admits a Kähler current with poles. This shows that the polarization used in (vii) exists for almost all fibres T_z of the relative Albanese space $A(X/Y)$, after pulling back everything to $\mathbb{C}$ by $\mathbb{C} \to \widehat{S} \to S \to Y$. □

We are now ready to prove our main theorem on the structure of the Albanese map.

Theorem 3.14. — *Let X be a compact Kähler manifold with nef tangent bundle T_X. Let $\widetilde{X}$ be a finite étale cover of maximum irregularity $q = q(\widetilde{X}) = \widetilde{q}(X)$. Then*

(i) $\pi_1(\widetilde{X}) \simeq \mathbb{Z}^{2q}$.

(ii) *The Albanese map $\alpha : \widetilde{X} \to A(\widetilde{X})$ is a smooth fibration over a q-dimensional torus with nef relative tangent bundle.*

(iii) *The fibres F of α are Fano manifolds with nef tangent bundles (i.e. K_F^{-1} is ample and T_F is nef).*

Corollary 3.15. — *As an abstract group, the fundamental group $\pi_1(X)$ is an extension of a finite group by $\mathbb{Z}^{2q}$.*

Proof. We prove 3.14 and 3.15 simultaneously by induction on $n = \dim X$. Suppose that the result is known in dimension $< n$. Then $\widetilde{X} \to A(\widetilde{X})$ satisfies 3.14 (ii) by

Prop. 3.9. In particular the fibres F have nef tangent bundle, and the induction hypothesis for 3.15 implies that $\pi_1(F)$ contains an abelian subgroup of finite index. We can now apply Prop. 3.12 (iii) to conclude that $\widetilde{q}(F) = 0$. Hence the alternative 3.10 (ii) cannot occur for F and so F is a Fano manifold. Property 3.14 (iii) is thus proved. Finally it is well known [Ko61] that Fano manifolds are simply connected. The homotopy exact sequence of the Albanese fibration shows that $\pi_1(\widetilde{X}) \simeq \pi_1(A(\widetilde{X})) \simeq \mathbb{Z}^{2q}$. There is a composition of finite étale covers $\widetilde{\widetilde{X}} \to \widetilde{X} \to X$ such that $\widetilde{\widetilde{X}}$ is a Galois cover of X (take $\pi_1(\widetilde{\widetilde{X}})$ to be the normal subgroup of finite index in $\pi_1(X)$ obtained by taking the intersection of all conjugates of $\pi_1(\widetilde{X})$). Again $\pi_1(\widetilde{\widetilde{X}}) \simeq \mathbb{Z}^{2q}$ as a subgroup of finite index in $\pi_1(X) \simeq \mathbb{Z}^{2q}$, and 3.15 follows. $\square$

REMARK 3.16. — In order to complete the classification of Kähler manifolds with nef tangent bundles, it would remain essentially to understand the case when X is a Fano manifold. This case is by far the most difficult. It is conjectured in [CP91] that every Fano manifold with nef tangent bundle is rational homogeneous.

3.D. Numerical flatness of the Albanese map

In this subsection, we want to analyze more precisely how the fibres of the Albanese map can vary over the base torus. The Albanese map appears in fact to be extremely rigid. We will show, at least when X is projective, that the fibres can be embedded "in a numerically flat way" in a projectivized numerically flat vector bundle over the Albanese torus. We start with the special case of projective bundles.

EXAMPLE 3.17. — Let E be a holomorphic vector bundle of rank r over a complex torus Y and let $X = \mathbb{P}(E)$. Then the natural projection $\alpha : X \to Y$ coincides with the Albanese map. The relative and Euler exact sequences for the tangent bundles give

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow \alpha^\star T_Y \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}_X \longrightarrow \alpha^\star E^\star \otimes \mathcal{O}_E(1) \longrightarrow T_{X/Y} \longrightarrow 0.$$

By Prop.1.15 it follows immediately from this that $X = \mathbb{P}(E)$ has a nef tangent bundle as soon as E is numerically flat. This is true more generally if E is projectively flat in the sense that $E \otimes (\det E)^{-1/r}$ is numerically flat (i.e. $G := S^r E \otimes (\det E)^{-1}$ is numerically flat). To see this, observe that

$$\begin{aligned} S^r\left(\alpha^\star E^\star \otimes \mathcal{O}_E(1)\right) &= \alpha^\star S^r E^\star \otimes \mathcal{O}_E(r) \\ &= \alpha^\star G^\star \otimes \mathcal{O}_E(r) \otimes \alpha^\star(\det E)^{-1} \end{aligned}$$

and that $\mathcal{O}_E(r) \otimes \alpha^\star(\det E)^{-1}$ is a quotient of $\alpha^\star G$; hence $\alpha^\star E^\star \otimes \mathcal{O}_E(1)$ is nef by 1.14 (iii). The converse is also true and was proved in [CP91] in the algebraic case:

Proposition 3.18. — *Let E be a holomorphic vector bundle of rank r over a torus. Then $T_{\mathbb{P}(E)}$ is nef if and only if $E \otimes (\det E)^{-1/r}$ is numerically flat.*

Proof. If $T_{\mathbb{P}(E)}$ is nef, the above exact sequences imply that $\alpha^\star E^\star \otimes \mathcal{O}_E(1)$ is nef (note that T_Y is trivial). In particular

$$\det(\alpha^\star E^\star \otimes \mathcal{O}_E(1))^{\frac{1}{r}} = (\alpha^\star(\det E)^\star \otimes \mathcal{O}_E(r))^{\frac{1}{r}} = \mathcal{O}_{E\otimes(\det E)^{-1/r}}(1)$$

is nef over $\mathbb{P}(E)$. Theorem 1.12 then shows that $E \otimes (\det E)^{-1/r}$ is nef (this formal tensor product is not really a vector bundle over Y but the proof of 1.12 together with 1.14 (ii) implies that $S^r E \otimes (\det E)^{-1}$ is nef). Since $S^r E \otimes (\det E)^{-1}$ has trivial determinant, we conclude that $S^r E \otimes (\det E)^{-1}$ is numerically flat. □

We now suppose for the rest of this subsection that X is projective. It is then easy to show that the numerical dimension $\nu(K_X^{-1})$ coincides with the fibre dimension of the Albanese map of $\widetilde{X}$ (if L is a nef line bundle, $\nu(L)$ is by definition the largest integer m such that $c_1(L)^m \neq 0$).

Proposition 3.19. — *Let X be a projective manifold with nef tangent bundle. Then*

$$\nu(K_X^{-1}) = \dim X - \widetilde{q}(X).$$

Proof. Replacing X with a suitable finite étale covering, we can suppose that $q(X) = \widetilde{q}(X) = q$. Set $n = \dim X$ and let $\alpha : X \to A(X)$ be the Albanese map. Since α is surjective and the fibres are Fano manifolds, we have $c_1(X)^{n-q} \neq 0$, hence $m := \nu(-K_X) \geq n - q$. Assume that $m > n - q$. This implies $c_1(X)^{n-q+1} \neq 0$. Hence the Kawamata-Viehweg vanishing theorem [Ka82, Vi82] applied to K_X^{-1} gives

$$H^i(X, \mathcal{O}_X) = 0 \quad \text{for } i \geq q\,.$$

Since $q = \dim A(X)$ we have $H^0(X, \Omega_X^q) \neq 0$, contradicting $H^q(X, \mathcal{O}_X) = 0$. □

We now suppose that X is a projective manifold with T_X nef and such that $q(X) = \widetilde{q}(X)$. Then K_X^{-1} is relatively ample along the fibres of $\alpha : X \longrightarrow A(X)$ since the fibres are Fano varieties. The Kodaira vanishing theorem implies that $R^q\alpha_\star(K_X^{-m}) = 0$ for all $m, q \geq 1$. Therefore $\alpha_\star(K_X^{-m})$ is a locally free sheaf, which we can identify with a vector bundle E_m over $A(X)$. For $m \geq m_0$ large, each fibre F has a very ample pluricanonical line bundle K_F^{-m} and we thus get a pluricanonical embedding of F into $\mathbb{P}\big(H^0(F, K_F^{-m})\big)$. By compactness of the base, the integer m_0 can be chosen independent of F. In this way we obtain a relative embedding over $A(X)$:

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "\alpha"'] & & \mathbb{P}(E_m) \arrow[dl, "\pi"] \\ & A(X) & \end{tikzcd}$$

for all $m \geq m_0$. Set

$$V_{m,p} = \pi_\star\big(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}(E_m)}(p)\big) \subset \pi_\star\big(\mathcal{O}_{\mathbb{P}(E_m)}(p)\big) = \mathcal{O}(S^p E_m).$$

For $p \geq p_0(m)$ large enough, the relative sections of $\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}(E_m)}(p)$ define X in the fibres of $\mathbb{P}(E_m)$ as a transversal intersection of hypersurfaces of degree p, and we have $R^q\pi_\star\big(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}(E_m)}(p)\big) = 0$ for all $q \geq 1$. In particular $V_{m,p}$ is a locally free subsheaf of $\mathcal{O}(S^p E_m)$.

Theorem 3.20. — *With the above notations and hypotheses, the locally free sheaves $E_m = \alpha_\star(K_X^{-m})$ and $V_{m,p} = \pi_\star\big(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}(E_m)}(p)\big)$ are numerically flat.*

We first need some general results about the nefness of direct images of nef line bundles by smooth morphisms.

Lemma 3.21. — *Let $\varphi : X \longrightarrow Y$ be a smooth morphism of projective manifolds and let L be a nef and relatively ample line bundle on X.*

(i) *Let G be a very ample line bundle over Y and let $m = \dim Y$. Then*

$$\varphi_\star(K_{X/Y} \otimes L) \otimes K_Y \otimes G^{m+1}$$

is generated by sections.

(ii) *Suppose that Y has the following property: there exists a holomorphic self-map $\theta : Y \to Y$ and an ample line bundle H over Y such that $\theta^\star H \otimes H^{-2}$ is nef. Then $\varphi_\star(K_{X/Y} \otimes L)$ is nef over Y.*

Proof. First observe that the sheaf $\varphi_\star(K_{X/Y} \otimes L)$ is locally free, because we have $R^q\varphi_\star(K_{X/Y} \otimes L) = 0$ for $q \geq 1$ by the Kodaira vanishing theorem.

(i) The assertion means that every section σ of

$$K_{X/Y} \otimes L \otimes \varphi^\star(K_Y \otimes G^{m+1}) = K_X \otimes L \otimes \varphi^\star(G^{m+1})$$

along a fibre $F = \varphi^{-1}(x_0)$ can be extended to all of X. Let $\mu : \widetilde{X} \to X$ (resp. $\nu : \widetilde{Y} \to Y$) be the blow-up of X with center F (resp. of Y at x_0) and let $D \subset \widetilde{X}$ (resp. $E \subset \widetilde{Y}$) be the exceptional divisor. Then there is a map $\widetilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ extending φ to the blow-ups, i.e. such that $\nu \circ \widetilde{\varphi} = \varphi \circ \mu$, and we have $D = \widetilde{\varphi}^\star E$, $K_{\widetilde{X}} = \mu^\star K_X \otimes \mathcal{O}((m-1)D)$. Now,

$$B := \mu^\star\big(L \otimes \varphi^\star(G^{m+1})\big) \otimes \mathcal{O}(-mD)$$

is easily seen to be ample on $\widetilde{X}$; indeed $\mu^\star L$ is nef and $\widetilde{\varphi}$-ample, while $\mu^\star\varphi^\star(G^{m+1}) \otimes \mathcal{O}(-mD)$ is the pull-back by $\widetilde{\varphi}$ of $\nu^\star(G^{m+1}) \otimes \mathcal{O}(-mE)$ which is ample on $\widetilde{Y}$ (note that $\nu^\star G^p \otimes \mathcal{O}(-E)$ is ample for p large and generated by sections for $p = 1$ thanks to the very ampleness of G, and make an interpolation). By the Kodaira vanishing theorem and the Leray spectral sequence we get

$$\begin{aligned} 0 = H^1(\widetilde{X}, K_{\widetilde{X}} \otimes B) &= H^1\big(\widetilde{X}, \mu^\star(K_X \otimes L \otimes \varphi^\star(G^{m+1})) \otimes \mathcal{O}(-D)\big) \\ &= H^1\big(X, K_X \otimes L \otimes \varphi^\star(G^{m+1}) \otimes \mathcal{I}_F\big). \end{aligned}$$

This implies the desired extension property.

(ii) By replacing if necessary H by some power, we may assume that

$$H \otimes (K_Y \otimes G^{m+1})^{-1}$$

is nef. Then (i) implies that $\varphi_\star(K_{X/Y} \otimes L) \otimes H$ is nef. Observe that the line bundles G and H can be chosen independent of X, L and φ. Let X_p be the fibered product $X\ {}_\varphi\!\times_{\theta^p} Y$, so that we have a commutative diagram

$$\begin{array}{ccc} X_p & \xrightarrow{\Theta_p} & X \\ \varphi_p \downarrow & & \downarrow \varphi \\ Y & \xrightarrow{\theta^p} & Y \end{array}$$

Set $L_p = \Theta_p^\star L$. An obvious base change yields

$$(\theta^p)^\star\big(\varphi_\star(K_{X/Y} \otimes L)\big) = (\varphi_p)_\star\big(K_{X_p/Y} \otimes L_p\big).$$

By (i) applied to $\varphi_p : X_p \to Y$, we infer that $(\theta^p)^\star\big(\varphi_\star(K_{X/Y} \otimes L)\big) \otimes H$ is nef. The assumption on θ implies $(\theta^p)^\star H \geq H^{2^p}$, hence

$$(\theta^p)^\star\big(\varphi_\star(K_{X/Y} \otimes L) \otimes H^{-2^p}\big)$$

is nef, and therefore so is $\big(\varphi_\star(K_{X/Y} \otimes L)\big) \otimes H^{-2^p}$ by Proposition 1.10. Since this result holds for arbitrarily large p, we conclude that $\varphi_\star(K_{X/Y} \otimes L)$ is nef (one can avoid fractional powers by taking symmetric powers $S^{2^p}(\bullet)$; note also that the equidimensionality of the fibres of the base change is not needed in the projective case; we will anyway only use the equidimensional case later). $\square$

Proof of Theorem 3.20. The existence of a self-map $\theta : Y \to Y$ as in Lemma 3.21 (ii) is clear when Y is a torus (take θ to be the isogeny $\theta(x) = 2x$) or when Y is a projective space $\mathbb{P}^N$ (take $\theta[z_0, \ldots, z_N] = [z_0^2, \ldots, z_N^2]$). Since $K_X^{-1} = K_{X/A(X)}^{-1}$ is nef and relatively ample, Lemma 3.21 (ii) shows that

$$E_m = \alpha_\star(K_X^{-m}) = \alpha_\star(K_{X/A(X)}^{-m})$$

is a nef vector bundle. Now, since $R^q\alpha_\star(K_X^{-m}) = 0$ for $q \geq 1$, the Riemann-Roch-Grothendieck formula yields (see [BS58]):

$$\mathrm{Ch}(E_m) = \alpha_\star\big(\mathrm{Ch}(K_X^{-1}) \cdot \mathrm{Todd}(T_{X/A(X)})\big).$$

In particular, we see that $c_1(E_m)$ is the direct image (integration along the fibres) of a polynomial of degree $\dim F+1$ in the Chern classes of X. All these polynomials must vanish by Proposition 3.19 and Corollary 2.6. Therefore $c_1(E_m) = 0$; the nefness of E_m then shows that E_m is numerically flat. Now, take the tensor product of the exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}(E_m)} \longrightarrow j_\star\mathcal{O}_X \longrightarrow 0$$

by $\mathcal{O}_{\mathbb{P}(E_m)}(p)$ and compute the direct image by π. We have by construction $j^\star\mathcal{O}_{\mathbb{P}(E_m)}(1) = K_X^{-m}$ and so

$$\pi_\star\big(j_\star\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}(E_m)}(p)\big) = \alpha_\star(K_X^{-mp}) = E_{mp}.$$

This gives rise to an exact sequence

$$0 \longrightarrow V_{m,p} \longrightarrow S^p E_m \longrightarrow E_{mp} \longrightarrow 0,$$

and we conclude that $V_{m,p}$ is numerically flat by Proposition 1.15. $\square$

REMARK 3.22. — We expect Proposition 3.19 and Theorem 3.20 to be true also in the Kähler case. This would require a Kähler version of the Kawamata-Viehweg vanishing theorem, at least in our special situation. The rest of the proof would be almost unchanged, except that we would have to replace very ample line bundles in the proof of Lemma 3.21 by high multiples of a Kähler form on Y (and invoke the results of [De91] on the almost generation of nef vector bundles over arbitrary compact complex manifolds).

4. Moishezon manifolds with nef tangent bundles

In this section we show in a purely algebraic way that Moishezon manifolds with nef tangent bundles are projective. This is just the algebraic part of 3.6. Moreover we give another proof that any compact Kähler n-fold X with T_X nef and $K_X^n \neq 0$ is automatically projective and even Fano.

THEOREM 4.1. — *Let X be a Moishezon manifold with T_X nef. Then X is projective.*

Proof. Let $\pi : \widehat{X} \to X$ be a sequence of blow-up's such that $\widehat{X}$ is a projective manifold. Let $\widehat{\mathcal{L}}$ be very ample on $\widehat{X}$ and put $\mathcal{L} = \pi_\star(\widehat{\mathcal{L}})^{\star\star}$. Since $H^0(X, \mathcal{L}) \neq 0$ and since every effective divisor on X is nef by [CP91] or by 3.7, $\mathcal{L}$ is nef. Since $\kappa(X, \mathcal{L}) = \dim X$, $\mathcal{L}$ is also big. As K_X^{-1} is nef, we deduce nefness of $\mathcal{L} \otimes K_X$, hence by the base point free theorem, $\mathcal{L}^m$ is generated by sections for some $m \gg 0$ (the base point free theorem also holds for Moishezon manifolds by exactly the same proof, see e.g. [KMM85]). We obtain a morphism $\phi : X \to Y$ to a normal projective variety Y and it is sufficient to prove that ϕ is an isomorphism. Assume the contrary and let E be the degeneracy set ($\mathcal{L}$ being big, ϕ is a modification). By substituting Y by local hypersurface sections, we may assume that E is mapped to a point y_0, i.e. E is exceptional. Thus we substitute X by a lower-dimensional manifold containing E whose tangent bundle is still nef (near E). Now we equip E with a complex structure whose conormal sheaf $N^\star_{E/X}$ is (weakly) positive, i.e. $\mathcal{O}_{\mathbb{P}(N^\star_{E/X})}(1)$ is ample ([An82], [Pe82]). We find some $m \in \mathbb{N}$ (the "minimal vanishing order") and a canonical generically non zero map

$$\alpha : N^\star_{E/X|\,\mathrm{red}\ E} \to S^m \Omega^1_{X|\,\mathrm{red}\ E},$$

by associating to a function the homogeneous part of degree m in the power series expansion. Now we take a general curve $C \subset \mathrm{red}\ E$, which may be assumed to be smooth (otherwise normalize). Let $S = N^\star_{E/X|C}/\mathrm{torsion}$. Then $\alpha_C : S \to S^m \Omega^1_{X|C}$

is non zero. Since $\mathbb{P}(S) \subset \mathbb{P}(N^\star_{E/X|C})$ and since

$$\mathcal{O}_{\mathbb{P}(S)}(1) \simeq \mathcal{O}_{\mathbb{P}(N^\star_{E/X|C})}(1)_{|\mathbb{P}(S)},$$

S is positive, i.e. ample. But this contradicts the nefness of T_X (for this consider $\det S \to \Lambda^r S^m \Omega^1{}_{X|C}$, $r = \mathrm{rk}(S)$). $\square$

We will use 4.1 to prove

THEOREM 4.2. — *Let X be a compact Kähler manifold of dimension n. Assume that T_X is nef and that $K_X^n \neq 0$. Then X is Fano, in particular projective.*

Proof. Since K_X^{-1} is nef and $(K_X^{-1})^n > 0$ by our assumption, it follows from [De85] that X must be Moishezon, compare the proof of 3.8. By 4.1, X is projective. By the base point free theorem K_X^{-m} is generated by sections for some $m \gg 0$. Let $\varphi : X \to Y$ be the map defined by $|K_X^{-m}|$. φ is birational, and if K_X^{-1} is not ample, φ is not finite. Since K_X^{-1} is φ-nef, we can apply [Ka91, appendix] to produce a rational curve C contracted by φ. T_X being nef, C deforms to fill up X,which gives a contradiction since $K_X \cdot C' = 0$ if and only if C' is contracted by φ. Hence K_X^{-1} is ample, i.e. X is Fano. $\square$

5. Two structure Theorems

In this section we prove two general theorems on the structure of projective manifolds with nef tangent bundles. In particular we prove a criterion when X is Fano.

PROPOSITION 5.1. — *Let X be a complex manifold, Y a reduced locally irreducible complex space and $\varphi : X \to Y$ a holomorphic surjective map. Assume $\dim Y < \dim X$ and T_X to be φ-nef, i.e. T_X is nef on every curve contracted by φ. Assume for a fixed $y_0 \in Y$ that every component of $\varphi^{-1}(y_0)$ of dimension $\dim \varphi^{-1}(y_0)$ is compact. Let $F \subset \varphi^{-1}(y_0)$ be such a component of maximal dimension (with the canonical structure). Then:*

(i) red F *is smooth.*

(ii) $N^\star_{\mathrm{red}\ F/X}$ *is topologically trivial up to tensoring with a torsion line bundle.*

(iii) $N^\star_{\mathrm{red}\ F/X}$ *is numerically flat.*

Proof. We let $F' = \varphi^{-1}(y_0)$ be equipped with the canonical fibre structure. Let $F_0 = \mathrm{red}\ F$. We prove the proposition by induction on $n = \dim X$. For $n = 1$ we have nothing to prove. So assume 5.1 to be proved for $\dim X < n$.

1. First case: $\dim F = \dim F' \leq n - 2$.

Take $f_{y_0} \in \mathcal{O}_{Y,y_0}$ general with $f(y_0) = 0$. Then $\{f = 0\}$ will be a locally irreducible reduced hypersurface in Y (possibly shrink Y). Let $Y_1 = \{f = 0\}$ and $X_1 = \{\varphi^\star(f) = 0\}$, with reduced structure. Let $\widetilde{X}_1 \subset X$ be the analytic preimage of Y_1, i.e. defined ideal-theoretically by $\varphi^\star(f)$. F is (set-theoretically) still contained in X_1 since $\dim F \leq n-2$. Then the conormal sheaf satisfies

$$N^\star_{\widetilde{X}_1/X|F_0} \simeq \mathcal{O}_{F_0}.$$

Since $\widetilde{X}_1 = kX_1$ as divisors on X, we conclude

$$N^{\star k}_{X_1/X|F_0} \simeq \mathcal{O}_{F_0}.$$

So $N^\star_{X_1/X|F_0}$ is a torsion line bundle. Now consider the exact sequence

$$0 \to N^\star_{X_1/X|F_0} \xrightarrow{\alpha} \Omega^1_{X|F_0} \to \Omega^1_{X_1|F_0} \to 0.$$

Since T_X is nef, α must have constant rank, otherwise $S^k\Omega^1_{X|F_0}$ would have a section with zeros which is impossible by [CP91, 2.12] or by 1.16. Hence $\Omega^1_{X_1|F_0}$ is locally free, consequently X_1 must be smooth near F_0. By shrinking we may assume X_1 to be smooth everywhere. By induction we conclude smoothness of F_0. Moreover the exact sequence

$$0 \to N^\star_{X_1/X|F_0} \to N^\star_{F_0/X} \to N^\star_{F_0/X_1} \to 0$$

together with induction proves that $N^\star_{F_0/X}$ is topologically trivial up to tensoring with a torsion line bundle. Since $N_{F_0/X}$ is nef as a quotient of $T_{X|F_0}$ and $\det N_{F_0/X}$ is topological trivial we conclude that $N_{F_0/X}$ is numerically flat.

2. Second case: $\dim F = n-1$.

In this case we even don't need induction. Since F' carries the full fibre structure $N^\star_{F'/X}$ is generated by global sections. Choose k maximal, such that the k-th infinitesimal neighborhood $F_{0,k}$ is contained in F'. Let

$$\beta : N^\star_{F'/X|F_0} \to N^\star_{F_{0,k}/X|F_0} = N^{\star k}_{F_0/X}$$

be the canonical map. Then we conclude the existence of $s \in H^0(N^\star_{F'/X})$, such that $\beta(s) \neq 0$. Now consider the exact sequence

$$0 \to N^\star_{F_0/X} \to \Omega^1_{X|F_0} \to \Omega^1_{F_0} \to 0.$$

Since $T_{X|F_0}$ is nef, $\beta(s)$ cannot have zeros (pass to S^k). So $N^{\star k}_{F_0/X} \simeq \mathcal{O}_{F_0}$ and moreover F_0 is smooth. $\square$

Theorem 5.2. — *Let X be a projective manifold with T_X nef, let $\varphi : X \to Y$ be the contraction of an extremal ray* ([Mo82], [KMM85]). *Then Y and φ are smooth.*

Proof. Of course we may assume $\dim Y > 0$, moreover by [CP91] we know that φ cannot be a modification, hence $\dim Y < \dim X$. We let $F' = \varphi^{-1}(y)$ be a fibre of φ equipped with the full fibre structure. It is sufficient to prove that F' is smooth with $\dim F' = \dim X - \dim Y$. This follows easily from [Fi76, p.100] since by our assumption $\Omega^1_{X/Y}$ is locally free of rank $= \dim X - \dim Y$. So let $F \subset F'$ be a component of maximal dimension in F', with induced structure. By 5.1, red F is smooth and $N^\star_{\mathrm{red}\ F/X}$ is topologically trivial up to tensoring with a torsion line bundle. So $\det N^\star \equiv 0$ (numerically). By the adjunction formula and the fact that $-K_X$ is φ-ample (φ being a Mori contraction), $K_{\mathrm{red}\ F}$ is negative, i.e. $F_0 = \mathrm{red}\ F$ is Fano. By 5.1 $N^\star_{F_0/X}$ is nef as well as its dual. By 1.18, $N^\star_{F_0/X}$ has a filtration by hermitian flat quotient vector bundles. F_0 being simply connected, all these are trivial, hence $N^\star_{F_0/X}$ is trivial. Now $N_{F_0/X}$ is generated by sections and

$$H^1(N_{F_0/X}) = 0,$$

so the deformations of F_0 fill up X [Gr66]. Since $\dim \varphi(F_0) = 0$, all deformations $F_{0,t}$ fulfill

$$\dim \varphi(F_{0,t}) = 0$$

as well (consider $(F_{0,t} \cdot \varphi^\star(L^k))$, L ample on Y). Hence $\dim F_0 = \dim X - \dim Y$ and moreover $F = F_0$ as complex spaces. So F is smooth. □

Remark 5.3. — Instead of assuming φ to be a contraction of an extremal ray in 5.2, it is sufficient to assume:

(i) X to be a compact manifold,

(ii) Y to be a normal compact complex space which is a subspace of a Kähler manifold.

(iii) K_X to be negative on every fibre of φ, $\varphi_\star(\mathcal{O}_X) = \mathcal{O}_Y$. □

We now prove 3.10 in the algebraic case.

Theorem 5.4. — *Let X be a projective manifold with T_X nef. Assume $c_1(X)^n = 0$ with $n = \dim X$. Then $\widetilde{q}(X) > 0$.*

Proof. We proceed by induction on n. For $n = 1$ we have nothing to show. Now let X be of dimension n. If $K_X \equiv 0$ we are done since X is covered by an abelian variety. So assume K_X not to be nef. Then there is a contraction of an extremal ray, say $\varphi : X \to Y$. Y cannot be a point, because then X would be Fano. So by 5.1 Y and φ are smooth and $0 < \dim Y < n$. Thus T_Y is nef again. Let $m = \dim Y$. If $c_1(Y)^m \neq 0$ then Y is Fano by 4.2, hence $\chi(\mathcal{O}_Y) = 1$, hence $\chi(\mathcal{O}_X) = 1$ contradicting $c_1(X)^n = 0$ (which implies $\chi(\mathcal{O}_X) = 0$ by 2.7 and Riemann-Roch). So $c_1(Y)^m = 0$. Then $\widetilde{q}(Y) > 0$ by induction, hence $\widetilde{q}(X) > 0$. □

Corollary 5.5. — *Let X be a projective manifold with T_X nef. Then X is Fano if and only if $\widetilde{q}(X) = 0$, respectively if and only if $c_1(X)^n \neq 0$ ($n = \dim X$).*

6. Surfaces with nef tangent bundles

In this section we give a classification of all (non necessarily Kähler) surfaces whose tangent bundles are nef. In the algebraic case this has been carried out in [CP91]:

THEOREM 6.1. — *Let X be a smooth projective surface and assume T_X to be nef. Then X is minimal and is exactly one of the surfaces in the following list:*

(1) *X is abelian*;

(2) *X is hyperelliptic*;

(3) $X = \mathbb{P}_2$;

(4) $X = \mathbb{P}_1 \times \mathbb{P}_1$;

(5) $X = \mathbb{P}(E)$, *where E is a rank 2-vector bundle on an elliptic curve C with either*

   (a) $E = \mathcal{O} \oplus L$, $L \in \mathrm{Pic}^0(C)$, *or*

   (b) *E is given by a non split extension $0 \to \mathcal{O} \to E \to L \to 0$ with $L = \mathcal{O}$ or* $\deg L = 1$.

In the non algebraic case the classification is as follows:

THEOREM 6.2. — *Let X be a smooth non algebraic compact surface with T_X nef. Then X is minimal and is exactly one of the surfaces in the following list:*

(1) *X is a torus*;

(2) *X is a Kodaira surface*;

(3) *X is a Hopf surface.*

For the convenience of the reader we recall the Kodaira classification of non algebraic surfaces, cf. [BPV84] or Kodaira's original papers. Let X be a minimal non algebraic surface. Then the algebraic dimension $a(X)$ is either one or zero.

If $a(X) = 1$, then either

(a) $\kappa(X) = 1$ and X is elliptic or

(b) $\kappa(X) = 0$ and X is a torus or a K3-surface or a Kodaira surface or

(c) $\kappa(X) = -\infty$ and X is a Hopf surface.

If $a(X) = 0$, then either

(a) $\kappa(X) = 0$ and X is a torus or a K3-surface or

(b) $\kappa(X) = -\infty$ and X is said to be of type VII.

Recall that a Hopf surface is a compact surface whose universal cover is $\mathbb{C}^2 \setminus \{0\}$, a K3-surface is a surface with $K_X \simeq \mathcal{O}$ and $b_1(X) = 0$ and a Kodaira surface is a surface with $\kappa(X) = 0$ and $b_1(X)$ odd.

The proof of 6.2 splits into several steps according to this classification.

Proposition 6.3. — *Let X be a compact surface with T_X nef and $a(X) = 1$. Then X is exactly one of the surfaces in the following list:*

(1) *X is a torus;*

(2) *X is a Kodaira surface;*

(3) *X is a Hopf surface.*

Proof. By [CP91, 2.1] X is minimal. Let $f : X \to C$ be the algebraic reduction. The map f is an elliptic fibration over a smooth curve C. The genus $g(C)$ is bounded by 1 since otherwise X would admit a holomorphic 1-form with zeros contradicting [CP91, 2.1]. Using the classification of singular fibres of f (cf. [BPV84]) and [CP91, 2.1] the only possible singular fibres are multiples of smooth elliptic curves.

First observe that $\kappa(X) \leq 0$: clearly $\kappa(X) \leq 1$ and if $\kappa(X) = 1$, X would admit an elliptic fibration over an elliptic curve C (observe $c_1^2 = c_2 = 0$). Since $X \to C$ must have a singular fibre (a multiple elliptic curve) we obtain again a holomorphic 1-form with zeros.

(α) $\kappa(X) = 0$.

Since $0 = c_1(X)^2 \geq c_2(X)$ and T_X is nef we get $c_2(X) = 0$. Therefore X cannot be a K3-surface and by surface classification X is a torus or a Kodaira surface. Hence it remains to be shown that Kodaira surfaces have nef tangent bundles. A primary Kodaira surfaces X is an elliptic fibre bundle over an elliptic curve C with $b_1(X) = 3$. By the exact sequence

$$0 \to T_{X/C} \to T_X \to f^\star T_C \to 0$$

and the equality $K_X = \mathcal{O}_X$, we deduce that T_X is nef by 1.14 (observe $\omega_{X/C} \simeq \mathcal{O}_X$ since $\omega_X \simeq \mathcal{O}_X$). A non primary (often called secondary) Kodaira surface admits a primary Kodaira surface as a finite étale cover and hence has nef tangent bundle.

(β) $\kappa(X) = -\infty$. Then X is a Hopf surface by classification. Therefore X is an elliptic fibre bundle over $\mathbb{P}_1$ (via algebraic reduction).By the exact sequence

$$0 \to T_{X/\mathbb{P}_1} \to T_X \to f^\star T_{\mathbb{P}_1} \to 0$$

it is enough to show that $T_{X/\mathbb{P}_1}$ is trivial. For this purpose consider the line bundle $f_\star(\omega_{X/\mathbb{P}_1})$. Since f is smooth we have $\deg f_\star(\omega_{X/\mathbb{P}_1}) = 0$ (cf. [BPV84]) and hence $f_\star(\omega_{X/\mathbb{P}_1}) \simeq \mathcal{O}_{\mathbb{P}_1}$. This gives a section of $\omega_{X/\mathbb{P}_1}$ which has no zeros (restrict to the fibres of f).

Proposition 6.4. — *Let X be a compact surface with $a(X) = 0$. Then T_X is nef precisely if X is a Hopf surface or a torus.*

Proof. Assume first that T_X is nef. By the same argument as in 6.3 X cannot be a K3-surface. Hence by classification X is a torus or of type VII. Assume now that X is of type VII. Since T_X is nef, $c_1(X)^2 = 0$. Moreover $\chi(\mathcal{O}_X) = 0$ and therefore we get by Riemann-Roch that $c_2(X) = 0$. This implies $b_2(X) = 0$. By [Bo83] and [LYZ90] X is either a Hopf surface or an Inoue surface.

Every Hopf surface can be covered by a primary Hopf surface, i.e. a Hopf surface such that $\pi_1(X) \simeq \mathbb{Z}$. By [Kd66b], a primary Hopf surface is isomorphic to the quotient of $\mathbb{C}^2 \setminus \{0\}$ by an infinite cyclic group of automorphisms generated either by a linear automorphism

$$g(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2), \qquad 0 < |\alpha_1| \le |\alpha_2| < 1,$$

or by a polynomial automorphism of the type

$$g(z_1, z_2) = (\lambda^p z_1 + z_2^p, \lambda z_2), \qquad 0 < |\lambda| < 1, \quad p \in \mathbb{N} \setminus \{0\}.$$

In the first case the directions $\partial/\partial z_1$, $\partial/\partial z_2$ are invariant and give rise to a splitting $T_X = L_1 \oplus L_2$ into two line bundles. Set $c = \log|\alpha_2|/\log|\alpha_1|$ and fix a positive integer k. Then L_1, L_2 can be equipped with hermitian metrics associated to the plurisubharmonic weights:

$$\left(|z_1|^{2kc} + |z_2|^{2k}\right)^{-\frac{1}{2kc}} |dz_1|^2, \quad \text{resp.} \quad \left(|z_1|^{2kc} + |z_2|^{2k}\right)^{-\frac{1}{2k}} |dz_2|^2.$$

These metrics are of class C^2 for k large and have semipositive curvature. Hence T_X is Griffiths semipositive (with a metric of class C^∞ only when $c \in \mathbb{Q}$ and $kc \in \mathbb{N}$).

In the second case, only the first direction $\partial/\partial z_1$ is invariant and we get instead an exact sequence

$$0 \to L_1 \to T_X \to L_2 \to 0.$$

Observe that the metric defined on the pull-back of L_1, L_2 to $\mathbb{C}^2 \setminus \{0\}$ by

$$\left(\varepsilon^2|z_1|^2 + |z_2|^{2p}\right)^{-1} |dz_1|^2, \qquad \text{resp.} \quad \left(\varepsilon^2|z_1|^2 + |z_2|^{2p}\right)^{-1/p} |dz_2|^2$$

is almost invariant, since the inequality $2\varepsilon|z_1||z_2^p| \le \varepsilon^2|z_1|^2 + |z_2|^{2p}$ implies

$$\begin{aligned}\left(|\lambda|^{2p} - \varepsilon|\lambda|^p\right)\left(\varepsilon^2|z_1|^2 + |z_2|^{2p}\right) &\le \varepsilon^2|\lambda^p z_1 + z_2^p|^2 + |\lambda z_2|^{2p} \\ &\le \left(|\lambda|^{2p} + \varepsilon|\lambda|^p + \varepsilon^2\right)\left(\varepsilon^2|z_1|^2 + |z_2|^{2p}\right).\end{aligned}$$

We use a partition of unity to glue the weight $w_1(z_1, z_2) = \log\left(\varepsilon^2|z_1|^2 + |z_2|^{2p}\right)$ with its transform w_2 by the automorphism on two overlapping spherical shells, so as to obtain a well defined metric on X. The resulting weight function w can be made almost plurisubharmonic, with an arbitrary small negative part $O(\varepsilon)$ in its complex Hessian: apply e.g. Lemma 3.5 of [De91] to

$$w = \frac{1}{k}\log\left(\theta_1^2 e^{kw_1} + \theta_2^2 e^{kw_2}\right), \quad k \sim \frac{1}{\varepsilon},$$

after adding bump functions of order of magnitude $O(\varepsilon)$ to w_1 and w_2 so that $C_1 = C_2 = 0$. Therefore L_1, L_2 and T_X are nef vector bundles by 1.15(ii).

The Inoue surfaces S_M are constructed as follows (see [In74]). Let M be a matrix in $SL_3(\mathbb{Z})$ admitting one real eigenvalue $\alpha > 1$ and two complex conjugate eigenvalues $\beta \neq \overline{\beta}$. Let (a_1, a_2, a_3) be a real eigenvector of M corresponding to α and let (b_1, b_2, b_3) be an eigenvector of M corresponding to β. Then $X = S_M$ is the quotient of $\mathcal{H} \times \mathbb{C}$ (where $\mathcal{H}$ is the upper half plane in $\mathbb{C}$) by the group of affine automorphisms generated by

$$g_0(w, z) = (\alpha w, \beta z),$$
$$g_i(w, z) = (w + a_i, z + b_i), \quad i = 1, 2, 3.$$

It is clear that the directions $\partial/\partial w$ and $\partial/\partial z$ are left invariant by the automorphisms. Hence they give rise to a splitting $T_X = L_w \oplus L_z$. The line bundles L_w, L_z can be equipped respectively with the invariant hermitian metrics

$$(\operatorname{Im} w)^{-2} \, |dw|^2, \quad (\operatorname{Im} w) \, |dz|^2$$

(note that $\alpha|\beta|^2 = \det M = 1$). Their curvature forms are

$$\Theta(L_w) = \frac{i}{2\pi} \partial\overline{\partial} \log(\operatorname{Im} w)^2 = -\frac{i}{4\pi} \frac{dw \wedge d\overline{w}}{(\operatorname{Im} w)^2} \leq 0,$$
$$\Theta(L_z) = \frac{i}{2\pi} \partial\overline{\partial} \log(\operatorname{Im} w)^{-1} = \frac{i}{8\pi} \frac{dw \wedge d\overline{w}}{(\operatorname{Im} w)^2} \geq 0.$$

Since L_w is seminegative and not flat (use e.g. a Gauduchon metric to see this), T_X cannot be nef.

The Inoue surfaces $X = S^+_{N,p,q,r;t}$ are defined as the quotient of $\mathcal{H} \times \mathbb{C}$ by the group of affine automorphisms generated by

$$g_0(w, z) = (\alpha w, z + t),$$
$$g_i(w, z) = (w + a_i, z + b_i w + c_i), \quad i = 1, 2,$$
$$g_3(w, z) = (w, z + \frac{b_1 a_2 - b_2 a_1}{r}),$$

where (a_1, a_2) and (b_1, b_2) are the eigenvectors of some matrix $N \in SL_2(\mathbb{Z})$ admitting real eigenvalues $\alpha > 1$, $\frac{1}{\alpha}$. Moreover $t \in \mathbb{C}$ and $p, q, r \, (r \neq 0)$ are integers, while (c_1, c_2) depends on $(a_i, b_i), p, q, r$ (see [In74]). The vector field $\partial/\partial z$ is invariant by the automorphisms, thus we get an exact sequence

$$0 \to \mathcal{O}_X \to T_X \to L_w \to 0.$$

The quotient line bundle L_w can be equipped with the hermitian metric $(\operatorname{Im} w)^{-2}|dw|^2$ which has seminegative curvature but is not flat. Hence T_X cannot be nef.

Finally, the Inoue surfaces of type $S^-_{N,p,q,r}$ possess a 2:1 étale cover of the type $S^+_{N_1,p_1,q_1,r_1;0}$ and it follows that their tangent bundles cannot be nef.

Since every Inoue surface is of type S_M, S^+ or S^-, the proof of Prop. 6.4 is finished. □

7. Kähler 3-folds with nef tangent bundles

Due to the complexity of the classification in the non Kähler case we will restrict ourselves now to Kähler manifolds. In the 3-dimensional case we can give a complete classification. In the algebraic case this has been carried out in [CP91].

THEOREM 7.1. — *Let X be a projective 3-fold. Then T_X is nef if and only if X is up to finite étale cover one of the manifolds in the following list:*

(1) $X = \mathbb{P}_3$;

(2) $X = Q_3$, *the 3-dimensional quadric*;

(3) $X = \mathbb{P}_1 \times \mathbb{P}_2$;

(4) $X = \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$;

(5) $X = \mathbb{P}(T_{\mathbb{P}_2})$;

(6) $X = \mathbb{P}(E)$, *with E a numerically flat rank 3 bundle over an elliptic curve C* ;

(7) $X = \mathbb{P}(E) \times_C \mathbb{P}(F)$, *with E, F numerically flat rank 2 bundles over an elliptic curve C* ;

(8) $X = \mathbb{P}(E)$, *with E a numerically flat rank 2 bundle over an abelian surface*;

(9) $X =$ *abelian variety.*

In the non algebraic case the classification is as follows.

THEOREM 7.2. — *Let X be a non algebraic 3-dimensional compact Kähler manifold. Then T_X is nef if and only if X is up to finite étale cover either a torus or of the form $\mathbb{P}(E)$, where E is a numerically flat rank 2-bundle over a 2-dimensional torus.*

The proof is divided into several steps. First we treat the case that $q(X) = 0$.

PROPOSITION 7.3. — *Let X be a compact Kähler 3-fold with T_X nef and $q(X) = 0$. Then X is projective.*

Proof. By a well-known result of Kodaira it is enough to show that $H^2(X, \mathcal{O}_X) = 0$. Assume that $H^2(X, \mathcal{O}_X) \neq 0$. This gives non zero sections of Ω^2_X. These sections have no zeros since $\Lambda^2 T_X$ is nef by 1.12. This provides us with an exact sequence

$$0 \to \mathcal{O}_X \to \Omega^2_X \to V \to 0,$$

where V is a rank 2-bundle on X. Tensoring this sequence by K_X^{-1} gives a sequence

$$0 \to K_X^{-1} \to T_X \to W \to 0$$

with $\det W \cong \mathcal{O}_X$. Since W is nef all Chern classes of W vanish. In particular $c_2(X) = 0$. By Riemann-Roch we have $\chi(\mathcal{O}_X) = \frac{1}{24} c_1(X) c_2(X) = 0$. On the other hand $\chi(\mathcal{O}_X) = 1 + h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X)$. But $h^3(\mathcal{O}_X) = h^0(K_X) \leq 1$, $-K_X$ being nef. Therefore we obtain $h^2(\mathcal{O}_X) \leq 0$, a contradiction. $\square$

We should remark that 7.3 does not extend to arbitrary dimension. But it does extend if we assume $\widetilde{q}(X) = 0$, see 3.10, 3.11. For the case of positive irregularity we need

LEMMA 7.4. — *Let X be a 3-dimensional compact Kähler manifold with T_X nef and $q(X) > 0$. Then the Albanese map $\alpha : X \to A(X)$ is smooth and surjective and the fibres have nef tangent bundles. If $\widetilde{q}(X) = q(X)$, any fibre F of α satisfies $\widetilde{q}(F) = 0$.*

Proof. Special case of 3.9 and 3.12. □

Proof of Theorem 7.2. Let T_X be nef. By 7.3 we have $q(X) \geq 1$. Let $\alpha : X \to A$ be the Albanese map. By 7.4 the map α is smooth and surjective. We may assume that $\dim A \leq 2$ and $\widetilde{q}(X) = q(X)$.

First we treat the case $\dim A = 1$. Let F be a fibre of α. By 7.4 we have $\widetilde{q}(F) = 0$ and T_F is nef. By 6.1 and 6.2 the fibre F is either $\mathbb{P}_2$ or $\mathbb{P}_1 \times \mathbb{P}_1$. Hence X is either a $\mathbb{P}_2$-bundle or a $\mathbb{P}_1 \times \mathbb{P}_1$-bundle over an elliptic curve. Therefore X is algebraic (use [CP91], 7.2 in the case of $\mathbb{P}_1 \times \mathbb{P}_1$).

Now let $\dim A = 2$. Arguing as in the first case X is a $\mathbb{P}_1$-bundle over A. Then after finite étale cover X is of the form $\mathbb{P}(E)$, E a flat 2-bundle over a 2-dimensional torus. This can be proved as in ([CP91], 8.2) with the following modification: instead of quoting Donaldson we use the existence of a Hermite-Einstein metric on a stable vector bundle over a Kähler manifold, due to Uhlenbeck and Yau [UY86]. □

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