L^2 Vanishing Theorems for Positive Line Bundles and Adjunction Theory

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0. Introduction

Transcendental methods of algebraic geometry have been extensively studied since a very long time, starting with the work of Abel, Jacobi and Riemann in the nineteenth century. More recently, in the period 1940-1970, the work of Hodge, Hirzebruch, Kodaira, Atiyah revealed still deeper relations between complex analysis, topology, PDE theory and algebraic geometry. In the last ten years, gauge theory has proved to be a very efficient tool for the study of many important questions: moduli spaces, stable sheaves, non abelian Hodge theory, low dimensional topology ...

Our main purpose here is to describe a few analytic tools which are useful to study questions such as linear series and vanishing theorems for algebraic vector bundles. One of the early success of analytic methods in this context is Kodaira's use of the Bochner technique in relation with the theory of harmonic forms, during the decade 1950-60. The idea is to represent cohomology classes by harmonic forms and to prove vanishing theorems by means of suitable a priori curvature estimates. The prototype of such results is the Akizuki-Kodaira-Nakano theorem (1954): if X is a nonsingular projective algebraic variety and L is a holomorphic line bundle on X with positive curvature, then $H^q(X, \Omega_X^p \otimes L) = 0$ for $p+q > \dim X$ (throughout the paper we set $\Omega^p_X = \Lambda^p T^\star_X$ and $K_X = \Lambda^n T^\star_X$, $n = \dim X$, viewing these objects either as holomorphic bundles or as locally free $\mathcal{O}_X$-modules). It is only much later that an algebraic proof of this result has been proposed by Deligne-Illusie, via characteristic p methods, in 1986.

A refinement of the Bochner technique used by Kodaira led about ten years later to fundamental L^2 estimates due to Hörmander [Hör65], concerning solutions of the Cauchy-Riemann operator. Not only vanishing theorems are proved, but more precise information of a quantitative nature is obtained about solutions of $\overline{\partial}$-equations. The best way of expressing these L^2 estimates is to use a geometric setting first considered by Andreotti-Vesentini [AV65]. More explicitly, suppose that we have a holomorphic line bundle L is equipped with a hermitian metric of weight $e^{-2\varphi}$, where φ is a (locally defined) plurisubharmonic function; then explicit bounds on the L^2 norm $\int_X |f|^2 e^{-2\varphi}$ of solutions is obtained. The result is still more useful if the plurisubharmonic weight φ is allowed to have singularities. Following Nadel [Nad89], one defines the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ to be the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally summable. Then $\mathcal{I}(\varphi)$ is a coherent algebraic sheaf over X and $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$ for all $q \geq 1$ if the curvature of L is positive (as a current). This important result can be seen as a generalization of the Kawamata-Viehweg vanishing theorem ([Kaw82], [Vie82]), which is one of the cornerstones of higher dimensional algebraic geometry (especially of Mori's minimal model program).

In the dictionary between analytic geometry and algebraic geometry, the ideal $\mathcal{I}(\varphi)$ plays a very important role, since it directly converts an analytic object into an algebraic one, and, simultaneously, takes care of the singularities in a very efficient way. Another analytic tool used to deal with singularities is the theory of positive currents introduced by Lelong [Lel57]. Currents can be seen as generalizations of algebraic cycles, and many classical results of intersection theory still apply to currents. The concept of Lelong number of a current is the analytic analogue of the concept of multiplicity of a germ of algebraic variety. Intersections of cycles correspond to wedge products of currents (whenever these products are defined). A convenient measure of local positivity of a holomorphic line can be defined in this context: the *Seshadri constant* of a line bundle at a point is the largest possible Lelong number for a singular metric of positive curvature assuming an isolated singularity at the given point (see [Dem90]). Seshadri constants can also be given equivalent purely algebraic definitions. We refer to Ein-Lazarsfeld [EL92] and Ein-Küchle-Lazarsfeld [EKL94] for very interesting new results concerning Seshadri constants.

One of our main motivations has been the study of the following conjecture of Fujita: if L is an ample (i.e. positive) line bundle on a projective n-dimensional algebraic variety X, then $K_K + (n+2)L$ is very ample. A major result obtained by Reider [Rei88] is a proof of the Fujita conjecture in the case of surfaces (the case of curves is easy). Reider's approach is based on Bogomolov's inequality for stable vector bundles and the results obtained are almost optimal. Unfortunately, it seems difficult to extend Reider's original method to higher dimensions. In the analytic approch, which works for arbitrary dimensions, one tries to construct a suitable (singular) hermitian metric on L such that the the ideal $\mathcal{I}(\varphi)$ has a given 0-dimensional subscheme of X as its zero variety. As we showed in [Dem93b], this can be done essentially by solving a complex Monge-Ampère equation

$$(\mathrm{i}d'd''\varphi)^n = \text{linear combination of Dirac measures},$$

via the Aubin-Calabi-Yau theorem ([Aub78], [Yau78]). The solution φ then assumes logarithmic poles and the difficulty is to force the singularity to be an isolated pole; this is the point where intersection theory of currents is useful. In this way, we can prove e.g. that $2K_X + L$ is very ample under suitable numerical conditions for L. Alternative algebraic techniques have been developed recently by Kollár [Kol92], Ein-Lazarsfeld [EL93], Fujita [Fuj93] and [Siu94a, b]. The basic idea is to apply the Kawamata-Viehweg vanishing theorem, and to use the Riemann-Roch formula instead of the Monge-Ampère equation. The proofs proceed with careful inductions on dimension, together with an analysis of the base locus of the linear systems involved. Although the results obtained in low dimensions are slightly more precise than with the analytic method, it is still not clear whether the range of applicability of the methods are exactly the same. Because it fits well with our approach, we have included here a simple algebraic method due to Y.T. Siu [Siu94a], showing that $2K_X + mL$ is very ample for $m \geq 2 + \binom{3n+1}{n}$.

Our final concern in these notes is a proof of the effective Matsusaka big theorem obtained by [Siu93]. Siu's result is the existence of an effective value m_0 depending only on the intersection numbers L^n and $L^{n-1} \cdot K_X$, such that mL is very ample for $m \geq m_0$. The basic idea is to combine results on the very ampleness of $2K_X + mL$ together with the theory of holomorphic Morse inequalities ([Dem85b]). The Morse inequalities are used to construct sections of $m'L - K_X$ for m' large. Again this step can be made algebraic (following suggestions by F. Catanese and R. Lazarsfeld), but the analytic formulation apparently has a wider range of applicability.

These notes are essentially written with the idea of serving as an analytic toolbox for algebraic geometry. Although efficient algebraic techniques exist, our feeling is that the analytic techniques are very flexible and offer a large variety of guidelines for more algebraic questions (including applications to number theory which are not discussed here). We made a special effort to use as little prerequisites and to be as self-contained as possible; hence the rather long preliminary sections dealing with basic facts of complex differential geometry. The reader wishing to have a presentation of the algebraic approach to vanishing theorems and linear series is referred to the excellent notes written by R. Lazarsfeld [Laz93]. In the last years, there has been a continuous and fruitful interplay between the algebraic and analytic viewpoints on these questions, and I have greatly benefitted from observations and ideas contained in the works of J. Kollár, L. Ein, R. Lazarsfeld and Y.T. Siu. I would like to thank them for their interest in my work and for their encouragements.

1. Preliminary Material

1.A. Dolbeault Cohomology and Sheaf Cohomology

Let X be a $\mathbb{C}$-analytic manifold of dimension n. We denote by $\Lambda^{p,q}T^\star_X$ the bundle of differential forms of bidegree (p,q) on X, i.e., differential forms which can be written as

$$u = \sum_{|I|=p,\,|J|=q} u_{I,J}dz_I \wedge d\overline{z}_J.$$

Here $(z_1,\ldots,z_n)$ denote arbitrary local holomorphic coordinates, $I=(i_1,\ldots,i_p)$, $J=(j_1,\ldots,j_q)$ are multiindices (increasing sequences of integers in the range $[1,\ldots,n]$, of lengths $|I|=p$, $|J|=q$), and

$$dz_I := dz_{i_1}\wedge\ldots\wedge dz_{i_p}, \qquad d\overline{z}_J := d\overline{z}_{j_1}\wedge\ldots\wedge d\overline{z}_{j_q}.$$

Let $\mathcal{E}^{p,q}$ be the sheaf of germs of complex valued differential (p,q)-forms with C^∞ coefficients. Recall that the exterior derivative d splits as $d=d'+d''$ where

$$d'u = \sum_{|I|=p,\,|J|=q,1\le k\le n} \frac{\partial u_{I,J}}{\partial z_k} dz_k\wedge dz_I\wedge d\overline{z}_J,$$

$$d''u = \sum_{|I|=p,\,|J|=q,1\le k\le n} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k\wedge dz_I\wedge d\overline{z}_J$$

are of type $(p+1,q)$, $(p,q+1)$ respectively. The well-known Dolbeault-Grothendieck lemma asserts that any d''-closed form of type (p,q) with $q>0$ is locally d''-exact (this is the analogue for d'' of the usual Poincaré lemma for d, see e.g. Hörmander 1966). In other words, the complex of sheaves $(\mathcal{E}^{p,\bullet},d'')$ is exact in degree $q>0$; in degree $q=0$, $\operatorname{Ker} d''$ is the sheaf Ω^p_X of germs of holomorphic forms of degree p on X.

More generally, if F is a holomorphic vector bundle of rank r over X, there is a natural d'' operator acting on the space $C^\infty(X,\Lambda^{p,q}T^\star_X\otimes F)$ of smooth (p,q)-forms with values in F; if $s=\sum_{1\le\lambda\le r}s_\lambda e_\lambda$ is a (p,q)-form expressed in terms of a local holomorphic frame of F, we simply define $d''s := \sum d''s_\lambda\otimes e_\lambda$, observing that the holomorphic transition matrices involved in changes of holomorphic frames do not affect the computation of d''. It is then clear that the Dolbeault-Grothendieck lemma still holds for F-valued forms. For every integer $p=0,1,\ldots,n$, the *Dolbeault Cohomology* groups $H^{p,q}(X,F)$ are defined to be the cohomology groups of the complex of global (p,q) forms (graded by q):

$$H^{p,q}(X,F) = H^q\big(C^\infty(X,\Lambda^{p,\bullet}T^\star_X\otimes F)\big). \tag{1.1}$$

Now, let us recall the following fundamental result from sheaf theory (De Rham-Weil isomorphism theorem): let $(\mathcal{L}^\bullet,d)$ be a resolution of a sheaf $\mathcal{A}$ by acyclic sheaves, i.e. a complex of sheaves $(\mathcal{L}^\bullet,\delta)$ such that there is an exact sequence of sheaves

$$0\longrightarrow\mathcal{A}\xrightarrow{\ j\ }\mathcal{L}^0\xrightarrow{\ \delta^0\ }\mathcal{L}^1\longrightarrow\cdots\longrightarrow\mathcal{L}^q\xrightarrow{\ \delta^q\ }\mathcal{L}^{q+1}\longrightarrow\cdots\ ,$$

and $H^s(X,\mathcal{L}^q)=0$ for all $q\ge 0$ and $s\ge 1$. Then there is a fonctorial isomorphism

$$H^q\big(\Gamma(X,\mathcal{L}^\bullet)\big) \longrightarrow H^q(X,\mathcal{A}). \tag{1.2}$$

We apply this to the following situation: let $\mathcal{E}(F)^{p,q}$ be the sheaf of germs of C^∞ sections of $\Lambda^{p,q}T^\star_X \otimes F$, Then $(\mathcal{E}(F)^{p,\bullet}, d'')$ is a resolution of the locally free $\mathcal{O}_X$-module $\Omega^p_X \otimes \mathcal{O}(F)$ (Dolbeault-Grothendieck lemma), and the sheaves $\mathcal{E}(F)^{p,q}$ are acyclic as modules over the soft sheaf of rings $\mathcal{C}^\infty$. Hence by (1.2) we get

(1.3) Dolbeault Isomorphism Theorem (1953)**.** *For every holomorphic vector bundle F on X, there is a canonical isomorphism*

$$H^{p,q}(X,F) \simeq H^q(X,\Omega^p_X \otimes \mathcal{O}(F)). \qquad \square$$

If X is projective algebraic and F is an algebraic vector bundle, Serre's GAGA theorem [Ser56] shows that the algebraic sheaf cohomology group $H^q(X,\Omega^p_X\otimes\mathcal{O}(F))$ computed with algebraic sections over Zariski open sets is actually isomorphic to the analytic cohomology group. These results are the most basic tools to attack algebraic problems via analytic methods. Another important tool is the theory of plurisubharmonic functions and positive currents originated by K. Oka and P. Lelong in the decades 1940-1960.

1.B. Plurisubharmonic Functions

Plurisubharmonic functions have been introduced independently by Lelong and Oka in the study of holomorphic convexity. We refer to [Lel67, 69] for more details.

(1.4) Definition. *A function $u : \Omega \longrightarrow [-\infty,+\infty[$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic* (*psh for short*) *if*

a) *u is upper semicontinuous* ;

b) *for every complex line $L \subset \mathbb{C}^n$, $u_{\restriction\Omega\cap L}$ is subharmonic on $\Omega\cap L$, that is, for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$ with $|\xi| < d(a,\complement\Omega)$, the function u satisfies the mean value inequality*
$$u(a) \le \frac{1}{2\pi}\int_0^{2\pi} u(a+e^{\mathrm{i}\theta}\,\xi)\,d\theta.$$

The set of psh functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

We list below the most basic properties of psh functions. They all follow easily from the definition.

(1.5) Basic properties.

a) Every function $u \in \mathrm{Psh}(\Omega)$ is subharmonic, namely it satisfies the mean value inequality on euclidean balls or spheres:
$$u(a) \le \frac{1}{\pi^n r^{2n}/n!}\int_{B(a,r)} u(z)\,d\lambda(z)$$

for every $a \in \Omega$ and $r < d(a, \complement\Omega$. Either $u \equiv -\infty$ or $u \in L^1_{\rm loc}$ on every connected component of Ω.

b) For any decreasing sequence of psh functions $u_k \in \mathrm{Psh}(\Omega)$, the limit $u = \lim u_k$ is psh on Ω.

c) Let $u \in \mathrm{Psh}(\Omega)$ be such that $u \not\equiv -\infty$ on every connected component of Ω. If (ρ_ε) is a family of smoothing kernels, then $u \star \rho_\varepsilon$ is C^∞ and psh on

$$\Omega_\varepsilon = \big\{x \in \Omega\,;\ d(x, \complement\Omega) > \varepsilon\big\},$$

the family $(u \star \rho_\varepsilon)$ is increasing in ε and $\lim_{\varepsilon\to 0} u \star \rho_\varepsilon = u$.

d) Let $u_1, \ldots, u_p \in \mathrm{Psh}(\Omega)$ and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_p)$ is increasing in each t_j. Then $\chi(u_1, \ldots, u_p)$ is psh on Ω. In particular $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(e^{u_1} + \cdots + e^{u_p})$ are psh on Ω. □

(1.6) Lemma. *A function $u \in C^2(\Omega, \mathbb{R})$ is psh on Ω if and only if the hermitian form $Hu(a)(\xi) = \sum_{1\le j,k\le n} \partial^2 u/\partial z_j \partial\overline{z}_k(a)\, \xi_j \overline{\xi}_k$ is semipositive at every point $a \in \Omega$.*

Proof. This is an easy consequence of the following standard formula

$$\frac{1}{2\pi}\int_0^{2\pi} u(a + e^{\mathrm{i}\theta}\,\xi)\,d\theta - u(a) = \frac{2}{\pi}\int_0^1 \frac{dt}{t}\int_{|\zeta|<t} Hu(a + \zeta\xi)(\xi)\,d\lambda(\zeta),$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}$. Lemma 1.6 is a strong evidence that plurisubharmonicity is the natural complex analogue of linear convexity. □

For non smooth functions, a similar characterization of plurisubharmonicity can be obtained by means of a regularization process.

(1.7) Theorem. *If $u \in \mathrm{Psh}(\Omega)$, $u \not\equiv -\infty$ on every connected component of Ω, then for all $\xi \in \mathbb{C}^n$*

$$Hu(\xi) = \sum_{1\le j,k\le n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\, \xi_j \overline{\xi}_k \in \mathcal{D}'(\Omega)$$

is a positive measure. Conversely, if $v \in \mathcal{D}'(\Omega)$ is such that $Hv(\xi)$ is a positive measure for every $\xi \in \mathbb{C}^n$, there exists a unique function $u \in \mathrm{Psh}(\Omega)$ which is locally integrable on Ω and such that v is the distribution associated to u. □

In order to get a better geometric insight of this notion, we assume more generally that u is a function on a complex n-dimensional manifold X. If $\Phi : X \to Y$ is a holomorphic mapping and if $v \in C^2(Y, \mathbb{R})$, we have $d'd''(v \circ \Phi) = \Phi^\star d'd'' v$, hence

$$H(v \circ \Phi)(a, \xi) = Hv\big(\Phi(a), \Phi'(a).\xi\big).$$

In particular Hu, viewed as a hermitian form on T_X, does not depend on the choice of coordinates $(z_1, \ldots, z_n)$. Therefore, the notion of psh function makes sense on any complex manifold. More generally, we have

(1.8) Proposition. *If $\Phi : X \longrightarrow Y$ is a holomorphic map and $v \in \mathrm{Psh}(Y)$, then $v \circ \Phi \in \mathrm{Psh}(X)$.* □

(1.9) Example. It is a standard fact that $\log|z|$ is psh (i.e. subharmonic) on $\mathbb{C}$. Thus $\log|f| \in \mathrm{Psh}(X)$ for every holomorphic function $f \in H^0(X, \mathcal{O}_X)$. More generally

$$\log\big(|f_1|^{\alpha_1} + \cdots + |f_q|^{\alpha_q}\big) \in \mathrm{Psh}(X)$$

for every $f_j \in H^0(X, \mathcal{O}_X)$ and $\alpha_j \geq 0$ (apply Property 1.5 d with $u_j = \alpha_j \log|f_j|$). We will be especially interested in the singularities obtained at points of the zero variety $f_1 = \ldots = f_q = 0$, when the α_j are rational numbers. □

(1.10) Definition. *A psh function $u \in \mathrm{Psh}(X)$ will be said to have analytic singularities if u can be written locally as*

$$u = \frac{\alpha}{2} \log\big(|f_1|^2 + \cdots + |f_N|^2\big) + v,$$

where $\alpha \in \mathbb{R}_+$, v is a locally bounded function and the f_j are holomorphic functions. If X is algebraic, we say that u has algebraic singularities if u can be written as above on sufficiently small Zariski open sets, with $\alpha \in \mathbb{Q}_+$ and f_j algebraic.

We then introduce the ideal $\mathcal{J} = \mathcal{J}(u/\alpha)$ of germs of holomorphic functions h such that $|h| \leq Ce^{u/\alpha}$ for some constant C, i.e.

$$|h| \leq C\big(|f_1| + \cdots + |f_N|\big).$$

This is a globally defined ideal sheaf on X, locally equal to the integral closure $\overline{\mathcal{I}}$ of the ideal sheaf $\mathcal{I} = (f_1, \ldots, f_N)$, thus $\mathcal{J}$ is coherent on X. If $(g_1, \ldots, g_{N'})$ are local generators of $\mathcal{J}$, we still have

$$u = \frac{\alpha}{2} \log\big(|g_1|^2 + \cdots + |g_{N'}|^2\big) + O(1).$$

If X is projective algebraic and u has analytic singularities with $\alpha \in \mathbb{Q}_+$, then u automatically has algebraic singularities. From an algebraic point of view, the singularities of u are in 1:1 correspondence with the "algebraic data" $(\mathcal{J}, \alpha)$. Later on, we will see another important method for associating an ideal sheaf to a psh function.

(1.11) Exercise. Show that the above definition of the integral closure of an ideal $\mathcal{I}$ is equivalent to the following more algebraic definition: $\overline{\mathcal{I}}$ consists of all germs h satisfying an integral equation

$$h^d + a_1 h^{d-1} + \ldots + a_{d-1} h + a_d = 0, \qquad a_k \in \mathcal{I}^k.$$

Hint. One inclusion is clear. To prove the other inclusion, consider the normalization of the blow-up of X along the (non necessarily reduced) zero variety $V(\mathcal{I})$. □

1.C. Positive Currents

The reader can consult [Fed69] for a more thorough treatment of current theory. Let us first recall a few basic definitions. A *current* of degree q on an oriented differentiable manifold M is simply a differential q-form Θ with distribution coefficients. The space of currents of degree q over M will be denoted by $\mathcal{D}'^q(M)$. Alternatively, a current of degree q can be seen as an element Θ in the dual space $\mathcal{D}'_p(M) := \big(\mathcal{D}^p(M)\big)'$ of the space $\mathcal{D}^p(M)$ of smooth differential forms of degree $p = \dim M - q$ with compact support; the duality pairing is given by

$$(1.12)\qquad \langle \Theta, \alpha\rangle = \int_M \Theta\wedge\alpha, \qquad \alpha\in\mathcal{D}^p(M).$$

A basic example is the *current of integration* $[S]$ over a compact oriented submanifold S of M:

$$(1.13)\qquad \langle [S], \alpha\rangle = \int_S \alpha, \qquad \deg\alpha = p = \dim_{\mathbb{R}} S.$$

Then $[S]$ is a current with measure coefficients, and Stokes' formula shows that $d[S] = (-1)^{q-1}[\partial S]$, in particular $d[S]=0$ if S has no boundary. Because of this example, the integer p is said to be the dimension of Θ when $\Theta\in\mathcal{D}'_p(M)$. The current Θ is said to be *closed* if $d\Theta = 0$.

On a complex manifold X, we have similar notions of bidegree and bidimension; as in the real case, we denote by

$$\mathcal{D}'^{p,q}(X) = \mathcal{D}'_{n-p,n-q}(X), \qquad n = \dim X,$$

the space of currents of bidegree (p,q) and bidimension $(n-p,n-q)$ on X. According to [Lel57], a current Θ of bidimension (p,p) is said to be (*weakly*) *positive* if for every choice of smooth $(1,0)$-forms $\alpha_1,\ldots,\alpha_p$ on X the distribution

$$(1.14)\qquad \Theta\wedge \mathrm{i}\alpha_1\wedge\overline{\alpha}_1\wedge\ldots\wedge\mathrm{i}\alpha_p\wedge\overline{\alpha}_p \quad \text{is a positive measure.}$$

(1.15) Exercise. If Θ is positive, show that the coefficients $\Theta_{I,J}$ of Θ are complex measures, and that, up to constants, they are dominated by the trace measure

$$\sigma_\Theta = \Theta\wedge\frac{1}{p!}\beta^p = 2^{-p}\sum\Theta_{I,I}, \qquad \beta = \frac{\mathrm{i}}{2}d'd''|z|^2 = \frac{\mathrm{i}}{2}\sum_{1\le j\le n} dz_j\wedge d\overline{z}_j,$$

which is a positive measure.
Hint. Observe that $\sum\Theta_{I,I}$ is invariant by unitary changes of coordinates and that the (p,p)-forms $\mathrm{i}\alpha_1\wedge\overline{\alpha}_1\wedge\ldots\wedge\mathrm{i}\alpha_p\wedge\overline{\alpha}_p$ generate $\Lambda^{p,p}T^\star_{\mathbb{C}^n}$ as a $\mathbb{C}$-vector space. □

A current $\Theta = \mathrm{i}\sum_{1\le j,k\le n}\Theta_{jk}dz_j\wedge dz_k$ of bidegree $(1,1)$ is easily seen to be positive if and only if the complex measure $\sum\lambda_j\overline{\lambda}_k\Theta_{jk}$ is a positive measure for every n-tuple $(\lambda_1,\ldots,\lambda_n)\in\mathbb{C}^n$.

(1.16) Example. If u is a (not identically $-\infty$) psh function on X, we can associate with u a (closed) positive current $\Theta = \mathrm{i}\partial\overline{\partial}u$ of bidegree $(1,1)$. Conversely, every closed positive current of bidegree $(1,1)$ can be written under this form on any open subset $\Omega \subset X$ such that $H^2_{DR}(\Omega, \mathbb{R}) = H^1(\Omega, \mathcal{O}) = 0$, e.g. on small coordinate balls (exercise to the reader). $\square$

It is not difficult to show that a product $\Theta_1 \wedge \ldots \wedge \Theta_q$ of positive currents of bidegree $(1,1)$ is positive whenever the product is well defined (this is certainly the case if all Θ_j but one at most are smooth; much finer conditions will be discussed in Section 2).

We now discuss another very important example of closed positive current. In fact, with every closed analytic set $A \subset X$ of pure dimension p is associated a current of integration

$$\langle [A], \alpha \rangle = \int_{A_{\mathrm{reg}}} \alpha, \qquad \alpha \in \mathcal{D}^{p,p}(X), \tag{1.17}$$

obtained by integrating over the regular points of A. In order to show that (1.17) is a correct definition of a current on X, one must show that A_{reg} has locally finite area in a neighborhood of A_{sing}. This result, due to [Lel57] is shown as follows. Suppose that 0 is a singular point of A. By the local parametrization theorem for analytic sets, there is a linear change of coordinates on $\mathbb{C}^n$ such that all projections

$$\pi_I : (z_1, \ldots, z_n) \mapsto (z_{i_1}, \ldots, z_{i_p})$$

define a finite ramified covering of the intersection $A \cap \Delta$ with a small polydisk Δ in $\mathbb{C}^n$ onto a small polydisk Δ_I in $\mathbb{C}^p$. Let n_I be the sheet number. Then the p-dimensional area of $A \cap \Delta$ is bounded above by the sum of the areas of its projections counted with multiplicities, i.e.

$$\mathrm{Area}(A \cap \Delta) \le \sum n_I \mathrm{Vol}(\Delta_I).$$

The fact that $[A]$ is positive is also easy. In fact

$$\mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p = |\det(\alpha_{jk})|^2\, \mathrm{i}w_1 \wedge \overline{w}_1 \wedge \ldots \wedge \mathrm{i}w_p \wedge \overline{w}_p$$

if $\alpha_j = \sum \alpha_{jk} dw_k$ in terms of local coordinates $(w_1, \ldots, w_p)$ on A_{reg}. This shows that all such forms are ≥ 0 in the canonical orientation defined by $\mathrm{i}w_1 \wedge \overline{w}_1 \wedge \ldots \wedge \mathrm{i}w_p \wedge \overline{w}_p$. More importantly, Lelong [Lel57] has shown that $[A]$ is d-closed in X, even at points of A_{sing}. This last result can be seen today as a consequence of the Skoda-El Mir extension theorem. For this we need the following definition: a *complete pluripolar* set is a set E such that there is an open covering (Ω_j) of X and psh functions u_j on Ω_j with $E \cap \Omega_j = u_j^{-1}(-\infty)$. Any (closed) analytic set is of course complete pluripolar (take u_j as in Example 1.9).

(1.18) Theorem (Skoda [Sko81], El Mir [EM84], Sibony [Sib85])**.** *Let E be a closed complete pluripolar set in X, and let Θ be a closed positive current on $X \smallsetminus E$ such that the coefficients $\Theta_{I,J}$ of Θ are measures with locally finite mass near E. Then the trivial extension $\widetilde{\Theta}$ obtained by extending the measures $\Theta_{I,J}$ by 0 on E is still closed on X.*

Lelong's result $d[A]=0$ is obtained by applying the Skoda-El Mir theorem to $\Theta=[A_{\rm reg}]$ on $X\smallsetminus A_{\rm sing}$.

Proof of Theorem 1.18. The statement is local on X, so we may work on a small open set Ω such that $E\cap\Omega=v^{-1}(-\infty)$, $v\in\mathrm{Psh}(\Omega)$. Let $\chi:\mathbb{R}\to\mathbb{R}$ be a convex increasing function such that $\chi(t)=0$ for $t\le -1$ and $\chi(0)=1$. By shrinking Ω and putting $v_k=\chi(k^{-1}v\star\rho_{\varepsilon_k})$ with $\varepsilon_k\to 0$ fast, we get a sequence of functions $v_k\in\mathrm{Psh}(\Omega)\cap C^\infty(\Omega)$ such that $0\le v_k\le 1$, $v_k=0$ in a neighborhood of $E\cap\Omega$ and $\lim v_k(x)=1$ at every point of $\Omega\smallsetminus E$. Let $\theta\in C^\infty([0,1])$ be a function such that $\theta=0$ on $[0,1/3]$, $\theta=1$ on $[2/3,1]$ and $0\le\theta\le 1$. Then $\theta\circ v_k=0$ near $E\cap\Omega$ and $\theta\circ v_k\to 1$ on $\Omega\smallsetminus E$. Therefore $\tilde\Theta=\lim_{k\to+\infty}(\theta\circ v_k)\Theta$ and

$$d'\tilde\Theta=\lim_{k\to+\infty}\Theta\wedge d'(\theta\circ v_k)$$

in the weak topology of currents. It is therefore sufficient to verify that $\Theta\wedge d'(\theta\circ v_k)$ converges weakly to 0 (note that $d''\tilde\Theta$ is conjugate to $d'\tilde\Theta$, thus $d''\tilde\Theta$ will also vanish).

Assume first that $\Theta\in\mathcal{D}'^{n-1,n-1}(X)$. Then $\Theta\wedge d'(\theta\circ v_k)\in\mathcal{D}'^{n,n-1}(\Omega)$, and we have to show that

$$\langle\Theta\wedge d'(\theta\circ v_k),\overline\alpha\rangle=\langle\Theta,\theta'(v_k)d'v_k\wedge\overline\alpha\rangle\underset{k\to+\infty}{\longrightarrow}0,\qquad\forall\alpha\in\mathcal{D}^{1,0}(\Omega).$$

As $\gamma\mapsto\langle\Theta,\mathrm{i}\gamma\wedge\overline\gamma\rangle$ is a non-negative hermitian form on $\mathcal{D}^{1,0}(\Omega)$, the Cauchy-Schwarz inequality yields

$$\big|\langle\Theta,\mathrm{i}\beta\wedge\overline\gamma\rangle\big|^2\le\langle\Theta,\mathrm{i}\beta\wedge\overline\beta\rangle\ \langle\Theta,\mathrm{i}\gamma\wedge\overline\gamma\rangle,\qquad\forall\beta,\gamma\in\mathcal{D}^{1,0}(\Omega).$$

Let $\psi\in\mathcal{D}(\Omega)$, $0\le\psi\le 1$, be equal to 1 in a neighborhood of $\mathrm{Supp}\,\alpha$. We find

$$\big|\langle\Theta,\theta'(v_k)d'v_k\wedge\overline\alpha\rangle\big|^2\le\langle\Theta,\psi\mathrm{i}d'v_k\wedge d''v_k\rangle\ \langle\Theta,\theta'(v_k)^2\mathrm{i}\alpha\wedge\overline\alpha\rangle.$$

By hypothesis $\int_{\Omega\smallsetminus E}\Theta\wedge\mathrm{i}\alpha\wedge\overline\alpha<+\infty$ and $\theta'(v_k)$ converges everywhere to 0 on Ω, thus $\langle\Theta,\theta'(v_k)^2\mathrm{i}\alpha\wedge\overline\alpha\rangle$ converges to 0 by Lebesgue's dominated convergence theorem. On the other hand

$$\begin{aligned}&\mathrm{i}d'd''v_k^2=2v_k\,\mathrm{i}d'd''v_k+2\mathrm{i}d'v_k\wedge d''v_k\ge 2\mathrm{i}d'v_k\wedge d''v_k,\\&2\langle\Theta,\psi\mathrm{i}d'v_k\wedge d''v_k\rangle\le\langle\Theta,\psi\mathrm{i}d'd''v_k^2\rangle.\end{aligned}$$

As $\psi\in\mathcal{D}(\Omega)$, $v_k=0$ near E and $d\Theta=0$ on $\Omega\smallsetminus E$, an integration by parts yields

$$\langle\Theta,\psi\mathrm{i}d'd''v_k^2\rangle=\langle\Theta,v_k^2\mathrm{i}d'd''\psi\rangle\le C\int_{\Omega\smallsetminus E}\|\Theta\|<+\infty$$

where C is a bound for the coefficients of $\mathrm{i}d'd''\psi$. Thus $\langle\Theta,\psi\mathrm{i}d'v_k\wedge d''v_k\rangle$ is bounded, and the proof is complete when $\Theta\in\mathcal{D}'^{n-1,n-1}$.

In the general case $\Theta\in\mathcal{D}'^{p,p}$, $p<n$, we simply apply the result already proved to all positive currents $\Theta\wedge\gamma\in\mathcal{D}'^{n-1,n-1}$ where $\gamma=\mathrm{i}\gamma_1\wedge\overline\gamma_1\wedge\ldots\wedge\mathrm{i}\gamma_{n-p-1},\wedge\overline\gamma_{n-p-1}$ runs over a basis of forms of $\Lambda^{n-p-1,n-p-1}T^\star_\Omega$ with constant coefficients (Lemma 1.4). Then we get $d(\tilde\Theta\wedge\gamma)=d\tilde\Theta\wedge\gamma=0$ for all such γ, hence $d\tilde\Theta=0$. □

(1.19) Corollary. *Let Θ be a closed positive current on X and let E be a complete pluripolar set. Then $\mathbf{1}_E\Theta$ and $\mathbf{1}_{X\smallsetminus E}\Theta$ are closed positive currents. In fact, $\widetilde{\Theta} = \mathbf{1}_{X\smallsetminus E}\Theta$ is the trivial extension of $\Theta_{\restriction X\smallsetminus E}$ to X, and $\mathbf{1}_E\Theta = \Theta - \widetilde{\Theta}$.* □

As mentioned above, any current $\Theta = \mathrm{i}d'd''u$ associated with a psh function u is a closed positive $(1,1)$-current. In the special case $u = \log|f|$ where $f \in H^0(X, \mathcal{O}_X)$ is a non zero holomorphic function, we have the important

(1.20) Lelong-Poincaré equation. *Let $f \in H^0(X, \mathcal{O}_X)$ be a non zero holomorphic function, $Z_f = \sum m_j Z_j$, $m_j \in \mathbb{N}$, the zero divisor of f and $[Z_f] = \sum m_j[Z_j]$ the associated current of integration. Then*

$$\frac{\mathrm{i}}{\pi}\partial\overline{\partial}\log|f| = [Z_f].$$

Proof (sketch). It is clear that $\mathrm{i}d'd''\log|f| = 0$ in a neighborhood of every point $x \notin \operatorname{Supp}(Z_f) = \bigcup Z_j$, so it is enough to check the equation in a neighborhood of every point of $\operatorname{Supp}(Z_f)$. Let A be the set of singular points of $\operatorname{Supp}(Z_f)$, i.e. the union of the pairwise intersections $Z_j \cap Z_k$ and of the singular loci $Z_{j,\mathrm{sing}}$; we thus have $\dim A \le n-2$. In a neighborhood of any point $x \in \operatorname{Supp}(Z_f) \smallsetminus A$ there are local coordinates $(z_1, \ldots, z_n)$ such that $f(z) = z_1^{m_j}$ where m_j is the multiplicity of f along the component Z_j which contains x and $z_1 = 0$ is an equation for Z_j near x. Hence

$$\frac{\mathrm{i}}{\pi}d'd''\log|f| = m_j\frac{\mathrm{i}}{\pi}d'd''\log|z_1| = m_j[Z_j]$$

in a neighborhood of x, as desired (the identity comes from the standard formula $\frac{\mathrm{i}}{\pi}d'd''\log|z| =$ Dirac measure δ_0 in $\mathbb{C}$). This shows that the equation holds on $X \smallsetminus A$. Hence the difference $\frac{\mathrm{i}}{\pi}d'd''\log|f| - [Z_f]$ is a closed current of degree 2 with measure coefficients, whose support is contained in A. By Exercise 1.21, this current must be 0, for A has too small dimension to carry its support (A is stratified by submanifolds of real codimension ≥ 4). □

(1.21) Exercise. Let Θ be a current of degree q on a real manifold M, such that both Θ and $d\Theta$ have measure coefficients ("normal current"). Suppose that $\operatorname{Supp}\Theta$ is contained in a real submanifold A with $\operatorname{codim}_{\mathbb{R}} A > q$. Show that $\Theta = 0$.
Hint: Let $m = \dim_{\mathbb{R}} M$ and let $(x_1, \ldots, x_m)$ be a coordinate system in a neighborhood Ω of a point $a \in A$ such that $A \cap \Omega = \{x_1 = \ldots = x_k = 0\}$, $k > q$. Observe that $x_j\Theta = x_j d\Theta = 0$ for $1 \le j \le k$, thanks to the hypothesis on supports and on the normality of Θ, hence $dx_j \wedge \Theta = d(x_j\Theta) - x_j d\Theta = 0$, $1 \le j \le k$. Infer from this that all coefficients in $\Theta = \sum_{|I|=q}\Theta_I dx_I$ vanish. □

We now recall a few basic facts of slicing theory (the reader will profitably consult [Fed69] and [Siu74] for further developments). Let $\sigma : M \to M'$ be a submersion of smooth differentiable manifolds and let Θ be a *locally flat* current on M, that is, a current which can be written locally as $\Theta = U + dV$ where U, V have L^1_{loc} coefficients. It is a standard fact (see Federer) that every current Θ such that both Θ and $d\Theta$ have measure coefficients is locally flat; in particular, closed positive currents are locally flat. Then, for almost every $x'\in M'$, there is a well defined *slice* $\Theta_{x'}$, which is the current on the fiber $\sigma^{-1}(x')$ defined by

$$\Theta_{x'} = U_{\restriction\sigma^{-1}(x')} + dV_{\restriction\sigma^{-1}(x')}.$$

The restrictions of U, V to the fibers exist for almost all x' by the Fubini theorem. The slices $\Theta_{x'}$ are currents on the fibers with the same degree as Θ (thus of dimension $\dim\Theta-\dim(\text{fibers})$). Of course, every slice $\Theta_{x'}$ coincides with the usual restriction of Θ to the fiber if Θ has smooth coefficients. By using a regularization $\Theta_\varepsilon=\Theta\star\rho_\varepsilon$, it is easy to show that the slices of a closed positive current are again closed and positive: in fact $U_{\varepsilon,x'}$ and $V_{\varepsilon,x'}$ converge to $U_{x'}$ and $V_{x'}$ in $L^1_{\rm loc}(\sigma^{-1}(x'))$, thus $\Theta_{\varepsilon,x'}$ converges weakly to $\Theta_{x'}$ for almost every x'. Now, the basic slicing formula is

$$\int_M \Theta\wedge\alpha\wedge\sigma^\star\beta=\int_{x'\in M'}\Big(\int_{x''\in\sigma^{-1}(x')}\Theta_{x'}(x'')\wedge\alpha_{\restriction\sigma^{-1}(x')}(x'')\Big)\beta(x') \tag{1.22}$$

for every smooth form α on M and β on M', such that α has compact support and $\deg\alpha=\dim M-\dim M'-\deg\Theta$, $\deg\beta=\dim M'$. This is an easy consequence of the usual Fubini theorem applied to U and V in the decomposition $\Theta=U+dV$, if we identify locally σ with a projection map $M=M'\times M''\to M'$, $x=(x',x'')\mapsto x'$, and use a partitition of unity on the support of α.

To conclude this section, we discuss De Rham and Dolbeault cohomology theory in the context of currents. A basic observation is that the Poincaré and Dolbeault-Grothendieck lemma still hold for currents. Namely, if $(\mathcal{D}'^q,d)$ and $(\mathcal{D}'(F)^{p,q},d'')$ denote the complex of sheaves of degree q currents (resp. of (p,q)-currents with values in a holomorphic vector bundle F), we still have De Rham and Dolbeault sheaf resolutions

$$0\to\mathbb{R}\to\mathcal{D}'^\bullet,\qquad 0\to\Omega^p_X\otimes\mathcal{O}(F)\to\mathcal{D}'(F)^{p,\bullet}.$$

Hence we get canonical isomorphisms

$$\begin{aligned} H^q_{\rm DR}(M,\mathbb{R})&=H^q\big((\Gamma(M,\mathcal{D}'^\bullet),d)\big),\\ H^{p,q}(X,F)&=H^q\big((\Gamma(X,\mathcal{D}'(F)^{p,\bullet}),d'')\big).\end{aligned} \tag{1.23}$$

In other words, we can attach a cohomology class $\{\Theta\}\in H^q_{\rm DR}(M,\mathbb{R})$ to any closed current Θ of degree q, resp. a cohomology class $\{\Theta\}\in H^{p,q}(X,F)$ to any d''-closed current of bidegree (p,q). Replacing if necessary every current by a smooth representative in the same cohomology class, we see that there is a well defined cup product given by the wedge product of differential forms

$$\begin{aligned} H^{q_1}(M,\mathbb{R})\times\ldots\times H^{q_m}(M,\mathbb{R})&\longrightarrow H^{q_1+\ldots+q_m}(M,\mathbb{R}),\\ (\{\Theta_1\},\ldots,\{\Theta_1\})&\longmapsto\{\Theta_1\}\wedge\ldots\wedge\{\Theta_m\}.\end{aligned}$$

In particular, if M is a compact oriented variety and $q_1+\ldots+q_m=\dim M$, there is a well defined intersection number

$$\{\Theta_1\}\cdot\{\Theta_2\}\cdot\,\cdots\,\cdot\{\Theta_m\}=\int_M\{\Theta_1\}\wedge\ldots\wedge\{\Theta_m\}.$$

However, as we will see in the next section, the pointwise product $\Theta_1\wedge\ldots\wedge\Theta_m$ need not exist in general.

2. Lelong Numbers and Intersection Theory

2.A. Multiplication of Currents and Monge-Ampère Operators

Let X be a n-dimensional complex manifold. We set

$$d^c = \frac{1}{2\mathrm{i}\pi}(d' - d'').$$

It follows in particular that d^c is a real operator, i.e. $\overline{d^c u} = d^c\overline{u}$, and that $dd^c = \frac{\mathrm{i}}{\pi}d'd''$. Although not quite standard, the $1/2\mathrm{i}\pi$ normalization is very convenient for many purposes, since we may then forget the factor π or 2π almost everywhere (e.g. in the Lelong-Poincaré equation (1.20)).

Let u be a psh function and let Θ be a closed positive current on X. Our desire is to define the wedge product $dd^c u \wedge \Theta$ even when neither u nor Θ are smooth. In general, this product does not make sense because $dd^c u$ and Θ have measure coefficients and measures cannot be multiplied; see Kiselman [Kis84] for interesting counterexamples. Even in the algebraic setting considered here, multiplication of currents is not always possible: suppose e.g. that $\Theta = [D]$ is the exceptional divisor of a blow-up in a surface; then $D\cdot D = -1$ cannot be the cohomology class of a closed positive current $[D]^2$. Assume however that u is a *locally bounded* psh function. Then the current $u\Theta$ is well defined since u is a locally bounded Borel function and Θ has measure coefficients. According to Bedford-Taylor [BT82] we define

$$dd^c u \wedge \Theta = dd^c(u\Theta)$$

where $dd^c(\ \ )$ is taken in the sense of distribution theory.

(2.1) Proposition. *If u is a locally bounded psh function, the wedge product $dd^c u \wedge \Theta$ is again a closed positive current.*

Proof. The result is local. Use a convolution $u_\nu = u \star \rho_{1/\nu}$ to get a decreasing sequence of smooth psh functions converging to u. Then write

$$dd^c(u\Theta) = \lim_{\nu\to+\infty} dd^c(u_\nu\Theta) = dd^c u_\nu \wedge \Theta$$

as a weak limit of closed positive currents. Observe that $u_\nu\Theta$ converges weakly to $u\Theta$ by Lebesgue's monotone convergence theorem. □

More generally, if $u_1, \ldots, u_m$ are locally bounded psh functions, we can define

$$dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \Theta = dd^c\big(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_m \wedge \Theta\big)$$

by induction on m. (Chern, Levine and Nirenberg, 1969) noticed the following useful inequality. Define the *mass* of a current Θ on a compact set K to be

$$||\Theta||_K = \int_K \sum_{I,J} |\Theta_{I,J}|$$

whenever K is contained in a coordinate patch and $\Theta = \sum \Theta_{I,J} dz_I \wedge d\overline{z}_J$. Up to seminorm equivalence, this does not depend on the choice of coordinates. If K is not contained in a coordinate patch, we use a partition of unity to define a suitable seminorm $||\Theta||_K$. If $\Theta \geq 0$, Exercise 1.15 shows that the mass is controlled by the trace measure, i.e. $||\Theta||_K \leq C \int_K \Theta \wedge \beta^p$.

(2.2) Chern-Levine-Nirenberg inequality. *For all compact subsets K, L of X with $L \subset K^\circ$, there exists a constant $C_{K,L} \geq 0$ such that*

$$||dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \Theta||_L \leq C_{K,L}\ ||u_1||_{L^\infty(K)} \ldots ||u_m||_{L^\infty(K)}\, ||\Theta||_K$$

Proof. By induction, it is sufficient to prove the result for $m = 1$ and $u_1 = u$. There is a covering of L by a family of open balls $B'_j \subset\subset B_j \subset K$ contained in coordinate patches of X. Let (p, p) be the bidimension of Θ, let $\beta = \frac{i}{2} d'd''|z|^2$, and let $\chi \in \mathcal{D}(B_j)$ be equal to 1 on $\overline{B}'_j$. Then

$$||dd^c u \wedge \Theta||_{L \cap \overline{B}'_j} \leq C \int_{\overline{B}'_j} dd^c u \wedge \Theta \wedge \beta^{p-1} \leq C \int_{B_j} \chi\, dd^c u \wedge \Theta \wedge \beta^{p-1}.$$

As Θ and β are closed, an integration by parts yields

$$||dd^c u \wedge \Theta||_{L \cap \overline{B}'_j} \leq C \int_{B_j} u\, \Theta \wedge dd^c \chi \wedge \beta^{p-1} \leq C' ||u||_{L^\infty(K)} ||\Theta||_K$$

where C' is equal to C multiplied by a bound for the coefficients of the smooth form $dd^c \chi \wedge \beta^{p-1}$. □

Various examples (cf. [Kis84]) show however that products of $(1,1)$-currents $dd^c u_j$ cannot be defined in a reasonable way for arbitrary psh functions u_j. However, functions u_j with $-\infty$ poles can be admitted if the polar sets are sufficiently small.

(2.3) Proposition. *Let u be a psh function on X, and let Θ be a closed positive current of bidimension (p, p). Suppose that u is locally bounded on $X \setminus A$, where A is an analytic subset of X of dimension $< p$ at each point. Then $dd^c u \wedge \Theta$ can be defined in such a way that $dd^c u \wedge \Theta = \lim_{\nu \to +\infty} dd^c u_\nu \wedge \Theta$ in the weak topology of currents, for any decreasing sequence $(u_\nu)_{\nu \geq 0}$ of psh functions converging to u.*

Proof. When u is locally bounded everywhere, we have $\lim u_\nu\, \Theta = u\, \Theta$ by the monotone convergence theorem and the result follows from the continuity of dd^c with respect to the weak topology.

First assume that A is discrete. Since our results are local, we may suppose that X is a ball $B(0, R) \subset \mathbb{C}^n$ and that $A = \{0\}$. For every $s \leq 0$, the function $u^{\geq s} = \max(u, s)$ is locally bounded on X, so the product $\Theta \wedge dd^c u^{\geq s}$ is well defined. For $|s|$ large, the function $u^{\geq s}$ differs from u only in a small neighborhood of the origin, at which u may have a $-\infty$ pole. Let γ be a $(p-1, p-1)$-form with constant coefficients and set $s(r) = \liminf_{|z| \to r-0} u(z)$. By Stokes' formula, we see that the integral

$$(2.4)\qquad I(s) := \int_{B(0,r)} dd^c u^{\geq s} \wedge \Theta \wedge \gamma$$

does not depend on s when $s < s(r)$, for the difference $I(s) - I(s')$ of two such integrals involves the dd^c of a current $(u^{\geq s} - u^{\geq s'}) \wedge \Theta \wedge \gamma$ with compact support in $B(0,r)$. Taking $\gamma = (dd^c|z|^2)^{p-1}$, we see that the current $dd^c u \wedge \Theta$ has finite mass on $B(0,r) \smallsetminus \{0\}$ and we can define $\langle \mathbf{1}_{\{0\}}(dd^c u \wedge \Theta), \gamma\rangle$ to be the limit of the integrals (2.4) as r tends to zero and $s < s(r)$. In this case, the weak convergence statement is easily deduced from the locally bounded case discussed above.

In the case where $0 < \dim A < p$, we use a slicing technique to reduce the situation to the discrete case. Set $q = p-1$. There are linear coordinates $(z_1, \ldots, z_n)$ centered at any point of A, such that 0 is an isolated point of $A \cap \big(\{0\} \times \mathbb{C}^{n-q}\big)$. Then there are small balls $B' = B(0,r')$ in $\mathbb{C}^q$, $B'' = B(0,r'')$ in $\mathbb{C}^{n-q}$ such that $A \cap (\overline{B}' \times \partial B'') = \emptyset$, and the projection map

$$\pi : \mathbb{C}^n \to \mathbb{C}^q, \qquad z = (z_1, \ldots, z_n) \mapsto z' = (z_1, \ldots, z_q)$$

defines a finite proper mapping $A \cap (B' \times B'') \to B'$. These properties are preserved if we slightly change the direction of projection. Take sufficiently many projections π_m associated to coordinate systems $(z_1^m, \ldots, z_n^m)$, $1 \leq m \leq N$, in such a way that the family of (q,q)-forms

$$i\, dz_1^m \wedge d\overline{z}_1^m \wedge \ldots \wedge i\, dz_q^m \wedge d\overline{z}_q^m$$

defines a basis of the space of (q,q)-forms. Expressing any compactly supported smooth (q,q)-form in such a basis, we see that we need only define

$$(2.5)\qquad \int_{B'\times B''} dd^c u \wedge \Theta \wedge f(z',z'')\, \mathrm{i}\, dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}\, dz_q \wedge d\overline{z}_q =$$
$$\int_{B'} \Big\{ \int_{B''} f(z',\bullet)\, dd^c u(z',\bullet) \wedge \Theta(z',\bullet) \Big\} \mathrm{i}\, dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}\, dz_q \wedge d\overline{z}_q$$

where f is a test function with compact support in $B' \times B''$, and $\Theta(z',\bullet)$ denotes the slice of Θ on the fiber $\{z'\} \times B''$ of the projection $\pi : \mathbb{C}^n \to \mathbb{C}^q$. Each integral $\int_{B''}$ in the right hand side of (2.5) makes sense since the slices $(\{z'\} \times B'') \cap A$ are discrete. Moreover, the double integral $\int_{B'} \int_{B''}$ is convergent. Indeed, observe that u is bounded on any compact cylinder

$$K_{\delta,\varepsilon} = \overline{B}\big((1-\delta)r'\big) \times \Big(\overline{B}(r'') \smallsetminus \overline{B}\big((1-\varepsilon)r''\big)\Big)$$

disjoint from A. Take $\varepsilon \ll \delta \ll 1$ so small that

$$\operatorname{Supp} f \subset \overline{B}\big((1-\delta)r'\big) \times \overline{B}\big((1-\varepsilon)r''\big).$$

For all $z' \in \overline{B}((1-\delta)r')$, the proof of the Chern-Levine-Nirenberg inequality 2.2 with a cut-off function $\chi(z'')$ equal to 1 on $B((1-\varepsilon)r'')$ and with support in $B((1-\varepsilon/2)r'')$ shows that

$$\int_{B((1-\varepsilon)r'')} dd^c u(z',\bullet) \wedge \Theta(z',\bullet)$$
$$\leq C_\varepsilon ||u||_{L^\infty(K_{\delta,\varepsilon})} \int_{z'' \in B((1-\varepsilon/2)r'')} \Theta(z',z'') \wedge dd^c|z''|^2.$$

This implies that the double integral is convergent. Now replace u everywhere by u_ν and observe that $\lim_{\nu\to+\infty}\int_{B''}$ is the expected integral for every z' such that $\Theta(z',\bullet)$ exists (apply the discrete case already proven). Moreover, the Chern-Levine-Nirenberg inequality yields uniform bounds for all functions u_ν, hence Lebesgue's dominated convergence theorem can be applied to $\int_{B'}$. We conclude from this that the sequence of integrals (2.5) converges when $u_\nu \downarrow u$, as expected. □

(2.6) Remark. In the above proof, the fact that A is an analytic set does not play an essential role. The main point is just that the slices $(\{z'\}\times B'')\cap A$ consist of isolated points for generic choices of coordinates (z', z''). In fact, the proof even works if the slices are totally discontinuous, in particular if they are of zero Hausdorff measure $\mathcal{H}_1$. It follows that Proposition 2.3 still holds whenever A is a closed set such that $\mathcal{H}_{2p-1}(A)=0$. □

2.B. Lelong Numbers

The concept of Lelong number is an analytic analogue of the algebraic notion of multiplicity. It is a very useful technique to extend results in the intersection theory of algebraic cycles to currents. Lelong numbers have been introduced for the first time by Lelong in [Lel57]. See also [Lel69], [Siu74], [Dem82a, 85a, 87] for further developments.

Let us first recall a few definitions. Let Θ be a closed positive current of bidimension (p,p) on a coordinate open set $\Omega\subset\mathbb{C}^n$ of a complex manifold X. The Lelong number of Θ at a point $x\in\Omega$ is defined to be the limit

$$\nu(\Theta,x)=\lim_{r\to 0+}\nu(\Theta,x,r),\qquad \text{where}\ \ \nu(\Theta,x,r)=\frac{\sigma_\Theta(B(x,r))}{\pi^p r^{2p}/p!}$$

measures the ratio of the area of Θ in the ball $B(x,r)$ to the area of the ball of radius r in $\mathbb{C}^p$. As $\sigma_\Theta=\Theta\wedge\frac{1}{p!}(\pi dd^c|z|^2)^p$ by 1.15, we also get

$$\nu(\Theta,x,r)=\frac{1}{r^{2p}}\int_{B(x,r)}\Theta(z)\wedge(dd^c|z|^2)^p. \tag{2.7}$$

The main results concerning Lelong numbers are summarized in the following theorems, due respectively to Lelong, Thie and Siu.

(2.8) Theorem ([Lel57]).

a) *For every positive current Θ, the ratio $\nu(\Theta,x,r)$ is a nonnegative increasing function of r, in particular the limit $\nu(\Theta,x)$ as $r\to 0+$ always exists.*

b) *If $\Theta=dd^c u$ is the bidegree $(1,1)$-current associated with a psh function u, then*

$$\nu(\Theta,x)=\sup\big\{\gamma\ge 0\,;\ u(z)\le\gamma\log|z-x|+O(1)\ \ \text{at } x\big\}.$$

In particular, if $u=\log|f|$ with $f\in H^0(X,\mathcal{O}_X)$ and $\Theta=dd^c u=[Z_f]$, we have

$$\nu([Z_f],x)=\mathrm{ord}_x(f)=\max\{m\in\mathbb{N}\,;\ D^\alpha f(x)=0,\ |\alpha|<m\}.$$

(2.9) Theorem ([Thi67]). *In the case where Θ is a current of integration $[A]$ over an analytic subvariety A, the Lelong number $\nu([A],x)$ coincides with the multiplicity of A at x (defined e.g. as the sheet number in the ramified covering obtained by taking a generic linear projection of the germ (A,x) onto a p-dimensional linear subspace through x in any coordinate patch Ω).*

(2.10) Theorem ([Siu74]). *Let Θ be a closed positive current of bidimension (p,p) on the complex manifold X.*

a) *The Lelong number $\nu(\Theta,x)$ is invariant by holomorphic changes of local coordinates.*

b) *For every $c>0$, the set $E_c(\Theta)=\big\{x\in X\,;\,\nu(\Theta,x)\ge c\big\}$ is a closed analytic subset of X of dimension $\le p$.*

The most important result is 2.10 b), which is a deep application of Hörmander L^2 estimates (see Section 5). The earlier proofs of all other results were rather intricate in spite of their rather simple nature. We reproduce below a sketch of elementary arguments based on the use of a more general and more flexible notion of Lelong number introduced in [Dem87]. Let φ be a continuous psh function with an isolated $-\infty$ pole at x, e.g. a function of the form $\varphi(z)=\log\sum_{1\le j\le N}|g_j(z)|^{\gamma_j}$, $\gamma_j>0$, where $(g_1,\ldots,g_N)$ is an ideal of germs of holomorphic functions in $\mathcal{O}_x$ with $g^{-1}(0)=\{x\}$. The *generalized Lelong number* $\nu(\Theta,\varphi)$ of Θ with respect to the weight φ is simply defined to be the mass of the measure $\Theta\wedge(dd^c\varphi)^p$ carried by the point x (the measure $\Theta\wedge(dd^c\varphi)^p$ is always well defined thanks to Prop. 2.3). This number can also be seen as the limit $\nu(\Theta,\varphi)=\lim_{t\to-\infty}\nu(\Theta,\varphi,t)$, where

$$\nu(\Theta,\varphi,t)=\int_{\varphi(z)<t}\Theta\wedge(dd^c\varphi)^p. \tag{2.11}$$

The relation with our earlier definition of Lelong numbers (as well as part a) of Theorem 2.8) comes from the identity

$$\nu(\Theta,x,r)=\nu(\Theta,\varphi,\log r),\qquad \varphi(z)=\log|z-x|, \tag{2.12}$$

in particular $\nu(\Theta,x)=\nu(\Theta,\log|\bullet-x|)$. This equality is in turn a consequence of the following general formula, applied to $\chi(t)=e^{2t}$ and $t=\log r$:

$$\int_{\varphi(z)<t}\Theta\wedge(dd^c\chi\circ\varphi)^p=\chi'(t-0)^p\int_{\varphi(z)<t}\Theta\wedge(dd^c\varphi)^p, \tag{2.13}$$

where χ is an arbitrary convex increasing function. To prove the formula, we use a regularization and thus suppose that Θ, φ and χ are smooth, and that t is a non critical value of φ. Then Stokes' formula shows that the integrals on the left and right hand side of (2.13) are equal respectively to

$$\int_{\varphi(z)=t}\Theta\wedge\big(dd^c\chi\circ\varphi\big)^{p-1}\wedge d^c(\chi\circ\varphi),\qquad \int_{\varphi(z)=t}\Theta\wedge\big(dd^c\varphi\big)^{p-1}\wedge d^c\varphi,$$

and the differential form of bidegree $(p-1,p)$ appearing in the integrand of the first integral is equal to $(\chi'\circ\varphi)^p\,(dd^c\varphi)^{p-1}\wedge d^c\varphi$. The expected formula follows. Part b) of Theorem 2.8 is a consequence of the Jensen-Lelong formula, whose proof is left as an exercise to the reader.

(2.14) Jensen-Lelong formula. *Let u be any psh function on X. Then u is integrable with respect to the measure $\mu_r = (dd^c\varphi)^{n-1} \wedge d^c\varphi$ supported by the pseudo-sphere $\{\varphi(z) = r\}$ and*

$$\mu_r(u) = \int_{\{\varphi<r\}} u(dd^c\varphi)^n + \int_{-\infty}^{r} \nu(dd^c u, \varphi, t)\, dt. \qquad \square$$

In our case, we set $\varphi(z) = \log|z - x|$. Then $(dd^c\varphi)^n = \delta_x$ and μ_r is just the unitary invariant mean value measure on the sphere $S(x, e^r)$. For $r < r_0$, Formula 2.14 implies

$$\mu_r(u) - \mu_{r_0}(u) = \int_{r_0}^{r} \nu(dd^c u, x, t) \sim (r - r_0)\nu(dd^c u, x) \qquad \text{as } r \to -\infty.$$

From this, using the Harnack inequality for subharmonic functions, we get

$$\liminf_{z\to x} \frac{u(z)}{\log|z - x|} = \lim_{r\to-\infty} \frac{\mu_r(u)}{r} = \nu(dd^c u, x).$$

These equalities imply statement 2.8 b).

Next, we show that the Lelong numbers $\nu(T, \varphi)$ only depend on the asymptotic behaviour of φ near the polar set $\varphi^{-1}(-\infty)$. In a precise way:

(2.15) Comparison theorem. *Let Θ be a closed positive current on X, and let $\varphi, \psi : X \to [-\infty, +\infty[$ be continuous psh functions with isolated poles at some point $x \in X$. Assume that*

$$\ell := \limsup_{z\to x} \frac{\psi(z)}{\varphi(z)} < +\infty.$$

Then $\nu(\Theta, \psi) \le \ell^p \nu(\Theta, \varphi)$, and the equality holds if $\ell = \lim \psi/\varphi$.

Proof. (2.12) shows that $\nu(\Theta, \lambda\varphi) = \lambda^p \nu(\Theta, \varphi)$ for every positive constant λ. It is thus sufficient to verify the inequality $\nu(\Theta, \psi) \le \nu(\Theta, \varphi)$ under the hypothesis $\limsup \psi/\varphi < 1$. For any $c > 0$, consider the psh function

$$u_c = \max(\psi - c, \varphi).$$

Fix $r \ll 0$. For $c > 0$ large enough, we have $u_c = \varphi$ on a neighborhood of $\varphi^{-1}(r)$ and Stokes' formula gives

$$\nu(\Theta, \varphi, r) = \nu(\Theta, u_c, r) \ge \nu(\Theta, u_c).$$

The hypothesis $\limsup \psi/\varphi < 1$ implies on the other hand that there exists $t_0 < 0$ such that $u_c = \psi - c$ on $\{u_c < t_0\}$. We thus get

$$\nu(\Theta, u_c) = \nu(\Theta, \psi - c) = \nu(\Theta, \psi),$$

hence $\nu(\Theta, \psi) \le \nu(\Theta, \varphi)$. The equality case is obtained by reversing the roles of φ and ψ and observing that $\lim \varphi/\psi = 1/l$. □

Part a) of Theorem 2.10 follows immediately from 2.15 by considering the weights $\varphi(z) = \log|\tau(z) - \tau(x)|$, $\psi(z) = \log|\tau'(z) - \tau'(x)|$ associated to coordinates systems $\tau(z) = (z_1, \ldots, z_n)$, $\tau'(z) = (z'_1, \ldots, z'_n)$ in a neighborhood of x. Another application is a direct simple proof of Thie's Theorem 2.9 when $\Theta = [A]$ is the current of integration over an analytic set $A \subset X$ of pure dimension p. For this, we have to observe that Theorem 2.15 still holds provided that x is an isolated point in $\mathrm{Supp}(\Theta) \cap \varphi^{-1}(-\infty)$ and $\mathrm{Supp}(\Theta) \cap \psi^{-1}(-\infty)$ (even though x is not isolated in $\varphi^{-1}(-\infty)$ or $\psi^{-1}(-\infty)$), under the weaker assumption that $\limsup_{\mathrm{Supp}(\Theta) \ni z \to x} \psi(z)/\varphi(z) = \ell$. The reason for this is that all integrals involve currents supported on $\mathrm{Supp}(\Theta)$. Now, by a generic choice of local coordinates $z' = (z_1, \ldots, z_p)$ and $z'' = (z_{p+1}, \ldots, z_n)$ on (X, x), the germ (A, x) is contained in a cone $|z''| \le C|z'|$. If $B' \subset \mathbb{C}^p$ is a ball of center 0 and radius r' small, and $B'' \subset \mathbb{C}^{n-p}$ is the ball of center 0 and radius $r'' = Cr'$, the projection

$$\mathrm{pr} : A \cap (B' \times B'') \longrightarrow B'$$

is a ramified covering with finite sheet number m. When $z \in A$ tends to $x = 0$, the functions

$$\varphi(z) = \log|z| = \log(|z'|^2 + |z''|^2)^{1/2}, \qquad \psi(z) = \log|z'|.$$

satisfy $\lim_{z \to x} \psi(z)/\varphi(z) = 1$. Hence Theorem 2.15 implies

$$\nu([A], x) = \nu([A], \varphi) = \nu([A], \psi).$$

Now, Formula 2.13 with $\chi(t) = e^{2t}$ yields

$$\begin{aligned} \nu([A], \psi, \log t) &= t^{-2p} \int_{\{\psi < \log t\}} [A] \wedge \Big(\frac{1}{2} dd^c e^{2\psi}\Big)^p \\ &= t^{-2p} \int_{A \cap \{|z'| < t\}} \Big(\frac{1}{2} \,\mathrm{pr}^\star\, dd^c |z'|^2\Big)^p \\ &= m\, t^{-2p} \int_{\mathbb{C}^p \cap \{|z'| < t\}} \Big(\frac{1}{2} dd^c |z'|^2\Big)^p = m, \end{aligned}$$

hence $\nu([A], \psi) = m$. Here, we have used the fact that pr is an étale covering with m sheets over the complement of the ramification locus $S \subset B'$, and the fact that S is of zero Lebesgue measure in B'.

(2.16) Proposition. *Under the assumptions of Proposition 2.3, we have*

$$\nu(dd^c u \wedge \Theta, x) \ge \nu(u, x)\, \nu(\Theta, x)$$

at every point $x \in X$.

Proof. Assume that $X = B(0, r)$ and $x = 0$. By definition

$$\nu(dd^c u \wedge \Theta, x) = \lim_{r \to 0} \int_{|z| \le r} dd^c u \wedge \Theta \wedge (dd^c \log|z|)^{p-1}.$$

Set $\gamma = \nu(u,x)$ and

$$u_\nu(z) = \max\big(u(z), (\gamma-\varepsilon)\log|z| - \nu\big)$$

with $0<\varepsilon<\gamma$ (if $\gamma=0$, there is nothing to prove). Then u_ν decreases to u and

$$\int_{|z|\le r} dd^c u \wedge \Theta \wedge (dd^c\log|z|)^{p-1} \ge \limsup_{\nu\to+\infty} \int_{|z|\le r} dd^c u_\nu \wedge \Theta \wedge (dd^c\log|z|)^{p-1}$$

by the weak convergence of $dd^c u_\nu \wedge \Theta$; here $(dd^c\log|z|)^{p-1}$ is not smooth on $\overline{B}(0,r)$, but the integrals remain unchanged if we replace $\log|z|$ by $\chi(\log|z|/r)$ with a smooth convex function χ such that $\chi(t)=t$ for $t\ge -1$ and $\chi(t)=0$ for $t\le -2$. Now, we have $u(z)\le \gamma\log|z|+C$ near 0, so $u_\nu(z)$ coincides with $(\gamma-\varepsilon)\log|z|-\nu$ on a small ball $B(0,r_\nu)\subset B(0,r)$ and we infer

$$\begin{aligned}\int_{|z|\le r} dd^c u_\nu \wedge \Theta \wedge (dd^c\log|z|)^{p-1} &\ge (\gamma-\varepsilon)\int_{|z|\le r_\nu} \Theta\wedge(dd^c\log|z|)^p\\ &\ge (\gamma-\varepsilon)\nu(\Theta,x).\end{aligned}$$

As $r\in\,]0,R[$ and $\varepsilon\in\,]0,\gamma[$ were arbitrary, the desired inequality follows. □

We will later need an important decomposition formula of [Siu74]. We start with the following lemma.

(2.17) Lemma. *If Θ is a closed positive current of bidimension (p,p) and Z is an irreducible analytic set in X, we set*

$$m_Z = \inf\{x\in Z\,;\ \nu(\Theta,x)\}.$$

a) *There is a countable family of proper analytic subsets (Z'_j) of Z such that $\nu(\Theta,x)=m_Z$ for all $x\in Z\smallsetminus\bigcup Z'_j$. We say that m_Z is the generic Lelong number of Θ along Z.*

b) *If $\dim Z = p$, then $\Theta\ge m_Z[Z]$ and $\mathbf{1}_Z\Theta = m_Z[Z]$.*

Proof. a) By definition of m_Z and $E_c(\Theta)$, we have $\nu(\Theta,x)\ge m_Z$ for every $x\in Z$ and

$$\nu(\Theta,x)=m_Z \quad \text{on } Z\smallsetminus \bigcup_{c\in\mathbb{Q},\,c>m_Z} Z\cap E_c(\Theta).$$

However, for $c>m_Z$, the intersection $Z\cap E_c(\Theta)$ is a proper analytic subset of A.
b) Left as an exercise to the reader. It is enough to prove that $\Theta\ge m_Z[Z_{\rm reg}]$ at regular points of Z, so one may assume that Z is a p-dimensional linear subspace in $\mathbb{C}^n$. Show that the measure $(\Theta - m_Z[Z])\wedge(dd^c|z|^2)^p$ has nonnegative mass on every ball $|z-a|<r$ with center $a\in Z$. Conclude by using arbitrary affine changes of coordinates that $\Theta - m_Z[Z]\ge 0$. □

(2.18) Decomposition formula ([Siu74])**.** *Let Θ be a closed positive current of bidimension (p,p). Then Θ can be written as a convergent series of closed positive currents*

$$\Theta = \sum_{k=1}^{+\infty} \lambda_k\,[Z_k] + R,$$

where $[Z_k]$ *is a current of integration over an irreducible analytic set of dimension* p*, and* R *is a residual current with the property that* $\dim E_c(R) < p$ *for every* $c > 0$*. This decomposition is locally and globally unique: the sets* Z_k *are precisely the* p*-dimensional components occurring in the sublevel sets* $E_c(\Theta)$*, and* $\lambda_k = \min_{x\in Z_k} \nu(\Theta, x)$ *is the generic Lelong number of* Θ *along* Z_k*.*

Proof of uniqueness. If Θ has such a decomposition, the p-dimensional components of $E_c(\Theta)$ are $(Z_j)_{\lambda_j\ge c}$, for $\nu(\Theta,x) = \sum \lambda_j \nu([Z_j],x) + \nu(R,x)$ is non zero only on $\bigcup Z_j \cup \bigcup E_c(R)$, and is equal to λ_j generically on Z_j $\big($more precisely, $\nu(\Theta,x) = \lambda_j$ at every regular point of Z_j which does not belong to any intersection $Z_j \cup Z_k$, $k \ne j$ or to $\bigcup E_c(R)\big)$. In particular Z_j and λ_j are unique.

Proof of existence. Let $(Z_j)_{j\ge 1}$ be the countable collection of p-dimensional components occurring in one of the sets $E_c(\Theta)$, $c \in \mathbb{Q}_+^\star$, and let $\lambda_j > 0$ be the generic Lelong number of Θ along Z_j. Then Lemma 2.17 shows by induction on N that $R_N = \Theta - \sum_{1\le j\le N} \lambda_j [Z_j]$ is positive. As R_N is a decreasing sequence, there must be a limit $R = \lim_{N\to+\infty} R_N$ in the weak topology. Thus we have the asserted decomposition. By construction, R has zero generic Lelong number along Z_j, so $\dim E_c(R) < p$ for every $c > 0$. □

It is very important to note that some components of lower dimension can actually occur in $E_c(R)$, but they cannot be subtracted because R has bidimension (p,p). A typical case is the case of a bidimension $(n-1,n-1)$ current $\Theta = dd^c u$ with $u = \log(|f_j|^{\gamma_1} + \ldots |f_N|^{\gamma_N})$ and $f_j \in H^0(X, \mathcal{O}_X)$. In general $\bigcup E_c(\Theta) = \bigcap f_j^{-1}(0)$ has dimension $< n-1$.

Corollary 2.19. *Let* $\Theta_j = dd^c u_j$*,* $1 \le j \le p$*, be closed positive* $(1,1)$*-currents on a complex manifold* X*. Suppose that there are analytic sets* $A_2 \supset \ldots \supset A_p$ *in* X *with* $\operatorname{codim} A_j \ge j$ *at every point such that each* u_j*,* $j \ge 2$*, is locally bounded on* $X \smallsetminus A_j$*. Let* $\{A_{p,k}\}_{k\ge1}$ *be the irreducible components of* A_p *of codimension* p *exactly and let* $\nu_{j,k} = \min_{x\in A_{p,k}} \nu(\Theta_j, x)$ *be the generic Lelong number of* Θ_j *along* $A_{p,k}$*. Then* $\Theta_1 \wedge \ldots \wedge \Theta_p$ *is well-defined and*

$$\Theta_1 \wedge \ldots \wedge \Theta_p \ge \sum_{k=1}^{+\infty} \nu_{1,k} \ldots \nu_{p,k}\,[A_{p,k}].$$

Proof. By induction on p, Prop. 2.3 shows that $\Theta_1 \wedge \ldots \wedge \Theta_p$ is well defined. Moreover, Prop. 2.16 implies

$$\nu(\Theta_1 \wedge \ldots \wedge \Theta_p, x) \ge \nu(\Theta_1, x) \ldots \nu(\Theta_p, x) \ge \nu_{1,k} \ldots \nu_{p,k}$$

at every point $x \in A_{p,k}$. The desired inequality is then a consequence of Siu's decomposition theorem. □

3. Hermitian Vector Bundles, Connections and Curvature

The goal of this section is to recall the most basic definitions of hemitian differential geometry related to the concepts of connection, curvature and first Chern class of a line bundle.

Let F be a complex vector bundle of rank r over a smooth differentiable manifold M. A *connection* D on F is a linear differential operator of order 1

$$D : C^\infty(M, \Lambda^q T_M^\star \otimes F) \to C^\infty(M, \Lambda^{q+1} T_M^\star \otimes F)$$

such that

$$D(f \wedge u) = df \wedge u + (-1)^{\deg f} f \wedge Du \tag{3.1}$$

for all forms $f \in C^\infty(M, \Lambda^p T_M^\star)$, $u \in C^\infty(X, \Lambda^q T_M^\star \otimes F)$. On an open set $\Omega \subset M$ where F admits a trivialization $\theta : F_{|\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}^r$, a connection D can be written

$$Du \simeq_\theta du + \Gamma \wedge u$$

where $\Gamma \in C^\infty(\Omega, \Lambda^1 T_M^\star \otimes \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ is an arbitrary matrix of 1-forms and d acts componentwise. It is then easy to check that

$$D^2 u \simeq_\theta (d\Gamma + \Gamma \wedge \Gamma) \wedge u \quad \text{on } \Omega.$$

Since D^2 is a globally defined operator, there is a global 2-form

$$\Theta(D) \in C^\infty(M, \Lambda^2 T_M^\star \otimes \mathrm{Hom}(F, F)) \tag{3.2}$$

such that $D^2 u = \Theta(D) \wedge u$ for every form u with values in F.

Assume now that F is endowed with a C^∞ hermitian metric along the fibers and that the isomorphism $F_{|\Omega} \simeq \Omega \times \mathbb{C}^r$ is given by a C^∞ frame (e_λ). We then have a canonical sesquilinear pairing

$$\begin{aligned} C^\infty(M, \Lambda^p T_M^\star \otimes F) \times C^\infty(M, \Lambda^q T_M^\star \otimes F) &\longrightarrow C^\infty(M, \Lambda^{p+q} T_M^\star \otimes \mathbb{C}) \\ (u, v) &\longmapsto \{u, v\} \end{aligned} \tag{3.3}$$

given by

$$\{u, v\} = \sum_{\lambda,\mu} u_\lambda \wedge \overline{v}_\mu \langle e_\lambda, e_\mu \rangle, \qquad u = \sum u_\lambda \otimes e_\lambda, \quad v = \sum v_\mu \otimes e_\mu.$$

The connection D is said to be *hermitian* if it satisfies the additional property

$$d\{u, v\} = \{Du, v\} + (-1)^{\deg u} \{u, Dv\}.$$

Assuming that (e_λ) is orthonormal, one easily checks that D is hermitian if and only if $\Gamma^\star = -\Gamma$. In this case $\Theta(D)^\star = -\Theta(D)$, thus

$$\mathrm{i}\Theta(D) \in C^\infty(M, \Lambda^2 T_M^\star \otimes \mathrm{Herm}(F, F)).$$

(3.4) Special case. For a bundle F of rank 1, the connection form Γ of a hermitian connection D can be seen as a 1-form with purely imaginary coefficients $\Gamma = \mathrm{i}A$ (A

real). Then we have $\Theta(D) = d\Gamma = \mathrm{i}dA$. In particular $\mathrm{i}\Theta(F)$ is a closed 2-form. The *First Chern class* of F is defined to be the cohomology class

$$c_1(F)_{\mathbb{R}} = \Big\{\frac{\mathrm{i}}{2\pi}\Theta(D)\Big\} \in H^2_{\mathrm{DR}}(M,\mathbb{R}).$$

The cohomology class is actually independent of the connection, since any other connection D_1 differs by a global 1-form, $D_1u = Du + B\wedge u$, so that $\Theta(D_1) = \Theta(D) + dB$. It is well-known that $c_1(F)_{\mathbb{R}}$ is the image in $H^2(M,\mathbb{R})$ of an integral class $c_1(F)\in H^2(M,\mathbb{Z})$; by using the exponential exact sequence

$$0\to\mathbb{Z}\to\mathcal{E}\to\mathcal{E}^\star\to 0,$$

$c_1(F)$ can be defined in Čech cohomology theory as the image by the coboundary map $H^1(M,\mathcal{E}^\star)\to H^2(M,\mathbb{Z})$ of the cocycle $\{g_{jk}\}\in H^1(M,\mathcal{E}^\star)$ defining F ; see e.g. [GH78] for details. □

We now concentrate ourselves on the complex analytic case. If $M = X$ is a complex manifold X, every connection D on a complex C^∞ vector bundle F can be splitted in a unique way as a sum of a $(1,0)$ and of a $(0,1)$-connection, $D = D' + D''$. In a local trivialization θ given by a C^∞ frame, one can write

$$(3.5') \qquad D'u \simeq_\theta d'u + \Gamma'\wedge u,$$

$$(3.5'') \qquad D''u \simeq_\theta d''u + \Gamma''\wedge u,$$

with $\Gamma = \Gamma' + \Gamma''$. The connection is hermitian if and only if $\Gamma' = -(\Gamma'')^\star$ in any orthonormal frame. Thus there exists a unique hermitian connection D corresponding to a prescribed $(0,1)$ part D''.

Assume now that the bundle F itself has a *holomorphic* structure. The unique hermitian connection for which D'' is the d'' operator defined in § 1 is called the *Chern connection* of F. In a local holomorphic frame (e_λ) of $E_{|\Omega}$, the metric is given by the hermitian matrix $H = (h_{\lambda\mu})$, $h_{\lambda\mu} = \langle e_\lambda, e_\mu\rangle$. We have

$$\{u,v\} = \sum_{\lambda,\mu} h_{\lambda\mu}u_\lambda\wedge\overline{v}_\mu = u^\dagger\wedge H\overline{v},$$

where $u^\dagger$ is the transposed matrix of u, and easy computations yield

$$\begin{aligned} d\{u,v\} &= (du)^\dagger\wedge H\overline{v} + (-1)^{\deg u}u^\dagger\wedge(dH\wedge\overline{v} + H\overline{dv})\\ &= \big(du + \overline{H}^{-1}d'\overline{H}\wedge u\big)^\dagger\wedge H\overline{v} + (-1)^{\deg u}u^\dagger\wedge\overline{(dv + \overline{H}^{-1}d'\overline{H}\wedge v)}\end{aligned}$$

using the fact that $dH = d'H + \overline{d'\overline{H}}$ and $\overline{H}^\dagger = H$. Therefore the Chern connection D coincides with the hermitian connection defined by

$$(3.6) \qquad \begin{cases} Du \simeq_\theta du + \overline{H}^{-1}d'\overline{H}\wedge u,\\ D' \simeq_\theta d' + \overline{H}^{-1}d'\overline{H}\wedge\bullet = \overline{H}^{-1}d'(\overline{H}\bullet), \qquad D'' = d''.\end{cases}$$

It is clear from this relations that $D'^2 = D''^2 = 0$. Consequently D^2 is given by to $D^2 = D'D'' + D''D'$, and the curvature tensor $\Theta(D)$ is of type $(1,1)$. Since $d'd'' + d''d' = 0$, we get

$$\begin{aligned}(D'D''+D''D')u &\simeq_\theta \overline{H}^{-1}d'\overline{H}\wedge d''u + d''(\overline{H}^{-1}d'\overline{H}\wedge u)\\ &= d''(\overline{H}^{-1}d'\overline{H})\wedge u.\end{aligned}$$

(3.7) Proposition. *The Chern curvature tensor $\Theta(F) := \Theta(D)$ is such that*

$$\mathrm{i}\,\Theta(F)\in C^\infty(X,\Lambda^{1,1}T^\star_X\otimes \mathrm{Herm}(F,F)).$$

If $\theta : E_{\restriction\Omega}\to\Omega\times\mathbb{C}^r$ is a holomorphic trivialization and if H is the hermitian matrix representing the metric along the fibers of $F_{\restriction\Omega}$, then

$$\mathrm{i}\,\Theta(F)\simeq_\theta \mathrm{i}\,d''(\overline{H}^{-1}d'\overline{H})\quad\text{on }\Omega.\qquad\square$$

Let $(z_1,\ldots,z_n)$ be holomorphic coordinates on X and let $(e_\lambda)_{1\le\lambda\le r}$ be an orthonormal frame of F. Writing

$$\mathrm{i}\Theta(F)=\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}dz_j\wedge dz_k\otimes e^\star_\lambda\otimes e_\mu,$$

we can identify the curvature tensor to a hermitian form

$$\widetilde{\Theta}(F)(\xi\otimes v)=\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda\overline{v}_\mu \tag{3.8}$$

on $T_X\otimes F$. This leads in a natural way to positivity concepts, following definitions introduced by Kodaira [Kod53], Nakano [Nak55] and Griffiths [Gri66].

(3.9) Definition. *The hermitian vector bundle F is said to be*

a) *positive in the sense of Nakano if $\widetilde{\Theta}(F)(\tau)>0$ for all non zero tensors $\tau=\sum\tau_{j\lambda}\partial/\partial z_j\otimes e_\lambda\in T_X\otimes F$.*

b) *positive in the sense of Griffiths if $\widetilde{\Theta}(F)(\xi\otimes v)>0$ for all non zero decomposable tensors $\xi\otimes v\in T_X\otimes F$;*

Corresponding semipositivity concepts are defined by relaxing the strict inequalities.

(3.10) Special case of rank 1 bundles. Assume that F is a line bundle. The hermitian matrix $H=(h_{11})$ associated to a trivialization $\theta : F_{\restriction\Omega}\simeq\Omega\times\mathbb{C}$ is simply a positive function which we find convenient to denote by $e^{-2\varphi}$, $\varphi\in C^\infty(\Omega,\mathbb{R})$. In this case the curvature form $\Theta(F)$ can be identified to the $(1,1)$-form $2d'd''\varphi$, and

$$\frac{\mathrm{i}}{2\pi}\Theta(F)=\frac{\mathrm{i}}{\pi}d'd''\varphi=dd^c\varphi$$

is a real $(1,1)$-form. Hence F is semipositive (in either Nakano or Griffiths sense) if and only if φ is psh, resp. positive if and only if φ is *strictly psh*. In this setting, the Lelong-Poincaré equation can be generalized as follows: let $\sigma\in H^0(X,F)$ be a non zero holomorphic section. Then

$$dd^c\log\|\sigma\|=[Z_\sigma]-\frac{\mathrm{i}}{2\pi}\Theta(F). \tag{3.11}$$

Formula (3.11) is immediate if we write $\|\sigma\| = |\theta(\sigma)|e^{-\varphi}$ and if we apply (1.20) to the holomorphic function $f = \theta(\sigma)$. As we shall see later, it is very important for applications to consider also singular hermitian metrics.

(3.12) Definition. *A singular (hermitian) metric on a line bundle F is a metric which is given in any trivialization $\theta : F_{\restriction\Omega} \xrightarrow{\simeq} \Omega\times\mathbb{C}$ by*

$$\|\xi\| = |\theta(\xi)|\, e^{-\varphi(x)}, \qquad x\in\Omega,\ \xi\in F_x$$

where $\varphi\in L^1_{\rm loc}(\Omega)$ is an arbitrary function, called the weight of the metric with respect to the trivialization θ.

If $\theta' : F_{\restriction\Omega'} \longrightarrow \Omega'\times\mathbb{C}$ is another trivialization, φ' the associated weight and $g\in\mathcal{O}^\star(\Omega\cap\Omega')$ the transition function, then $\theta'(\xi) = g(x)\,\theta(\xi)$ for $\xi\in F_x$, and so $\varphi' = \varphi + \log|g|$ on $\Omega\cap\Omega'$. The curvature form of F is then given formally by the closed $(1,1)$-current $\frac{\mathrm{i}}{2\pi}\Theta(F) = dd^c\varphi$ on Ω ; our assumption $\varphi\in L^1_{\rm loc}(\Omega)$ guarantees that $\Theta(F)$ exists in the sense of distribution theory. As in the smooth case, $\frac{\mathrm{i}}{2\pi}\Theta(F)$ is globally defined on X and independent of the choice of trivializations, and its De Rham cohomology class is the image of the first Chern class $c_1(F)\in H^2(X,\mathbb{Z})$ in $H^2_{DR}(X,\mathbb{R})$. Before going further, we discuss two basic examples.

(3.13) Example. Let $D = \sum\alpha_j D_j$ be a divisor with coefficients $\alpha_j\in\mathbb{Z}$ and let $F = \mathcal{O}(D)$ be the associated invertible sheaf of meromorphic functions u such that $\mathrm{div}(u) + D \ge 0$; the corresponding line bundle can be equipped with the singular metric defined by $\|u\| = |u|$. If g_j is a generator of the ideal of D_j on an open set $\Omega\subset X$ then $\theta(u) = u\prod g_j^{\alpha_j}$ defines a trivialization of $\mathcal{O}(D)$ over Ω, thus our singular metric is associated to the weight $\varphi = \sum\alpha_j\log|g_j|$. By the Lelong-Poincaré equation, we find

$$\frac{\mathrm{i}}{2\pi}\Theta\big(\mathcal{O}(D)\big) = dd^c\varphi = [D],$$

where $[D] = \sum\alpha_j[D_j]$ denotes the current of integration over D. □

(3.14) Example. Assume that $\sigma_1,\ldots,\sigma_N$ are non zero holomorphic sections of F. Then we can define a natural (possibly singular) hermitian metric on $F^\star$ by

$$\|\xi^\star\|^2 = \sum_{1\le j\le n}\big|\xi^\star.\sigma_j(x)\big|^2 \quad\text{ for }\ \xi^\star\in F^\star_x.$$

The dual metric on F is given by

$$\|\xi\|^2 = \frac{|\theta(\xi)|^2}{|\theta(\sigma_1(x))|^2+\ldots+|\theta(\sigma_N(x))|^2}$$

with respect to any trivialization θ. The associated weight function is thus given by $\varphi(x) = \log\big(\sum_{1\le j\le N}|\theta(\sigma_j(x))|^2\big)^{1/2}$. In this case φ is a psh function, thus $\mathrm{i}\Theta(F)$ is a closed positive current. Let us denote by Σ the linear system defined by $\sigma_1,\ldots,\sigma_N$ and by $B_\Sigma = \bigcap\sigma_j^{-1}(0)$ its base locus. We have a meromorphic map

$$\Phi_\Sigma : X\smallsetminus B_\Sigma\to\mathbb{P}^{N-1}, \qquad x\mapsto(\sigma_1(x):\sigma_2(x):\ldots:\sigma_N(x)).$$

Then $\frac{\mathrm{i}}{2\pi}\Theta(F)$ is equal to the pull-back over $X \smallsetminus B_\Sigma$ of the Fubini-Study metric $\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\log(|z_1|^2 + \ldots + |z_N|^2)$ of $\mathbb{P}^{N-1}$ by Φ_Σ. □

(3.15) Ample and very ample line bundles. *A holomorphic line bundle F over a compact complex manifold X is said to be*

a) *very ample if the map $\Phi_{|F|} : X \to \mathbb{P}^{N-1}$ associated to the complete linear system $|F| = P(H^0(X,F))$ is a regular embedding (by this we mean in particular that the base locus is empty, i.e. $B_{|F|} = \emptyset$).*

b) *ample if some multiple mF, $m > 0$, is very ample.*

Here we use an additive notation for $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^\star)$, hence the symbol mF denotes the line bundle $F^{\otimes m}$. By Example 3.15, every ample line bundle F has a smooth hermitian metric with positive definite curvature form; indeed, if the linear system $|mF|$ gives an embedding in projective space, then we get a smooth hermitian metric on $F^{\otimes m}$, and the m-th root yields a metric on F such that $\frac{\mathrm{i}}{2\pi}\Theta(F) = \frac{1}{m}\Phi^\star_{|mF|}\omega_{\mathrm{FS}}$. Conversely, the Kodaira embedding theorem [Kod54] tells us that every positive line bundle F is ample (see Exercise 5.14 for a straightforward analytic proof of the Kodaira embedding theorem).

4. Bochner Technique and Vanishing Theorems

We first recall briefly a few basic facts of Hodge theory. Assume for the moment that M is a differentiable manifold equipped with a riemannian metric $g = \sum g_{ij} dx_i \otimes dx_j$. Given a q-form u on M with values in F, we consider the global L^2 norm

$$\|u\|^2 = \int_M |u(x)|^2 dV_g(x)$$

where $|u|$ is the pointwise hermitian norm and dV_g is the riemannian volume form. The Laplace-Beltrami operator associated to the connection D is

$$\Delta = DD^\star + D^\star D$$

where

$$D^\star : C^\infty(M, \Lambda^q T^\star_M \otimes F) \to C^\infty(M, \Lambda^{q-1} T^\star_M \otimes F)$$

is the (formal) adjoint of D with respect to the L^2 inner product. Assume that M is *compact*. Since

$$\Delta : C^\infty(M, \Lambda^q T^\star_M \otimes F) \to C^\infty(M, \Lambda^q T^\star_M \otimes F)$$

is a self-adjoint elliptic operator in each degree, standard results of PDE theory show that there is an orthogonal decomposition

$$C^\infty(M, \Lambda^q T^\star_M \otimes F) = \mathcal{H}^q(M,F) \oplus \mathrm{Im}\,\Delta$$

where $\mathcal{H}^q(M,F) = \mathrm{Ker}\,\Delta$ is the space of harmonic forms of degree q; $\mathcal{H}^q(M,F)$ is a finite dimensional space. Assume moreover that the connection D is *integrable*, i.e. that $D^2 = 0$. It is then easy to check that there is an orthogonal direct sum

$$\operatorname{Im} \Delta = \operatorname{Im} D \oplus \operatorname{Im} D^\star,$$

indeed $\langle Du, D^\star v\rangle = \langle D^2 u, v\rangle = 0$ for all u, v. Hence we get an orthogonal decomposition

$$C^\infty(M, \Lambda^q T^\star_M \otimes F) = \mathcal{H}^q(M, F) \oplus \operatorname{Im} D \oplus \operatorname{Im} D^\star,$$

and $\operatorname{Ker} \Delta$ is precisely equal to $\mathcal{H}^q(M, F) \oplus \operatorname{Im} D$. Especially, the q-th cohomology group $\operatorname{Ker} \Delta / \operatorname{Im} \Delta$ is isomorphic to $\mathcal{H}^q(M, F)$. All this can be applied for example in the case of the De Rham groups $H^q_{\mathrm{DR}}(M, \mathbb{C})$, taking F to be the trivial bundle $F = M \times \mathbb{C}$ (notice, however, that a nontrivial bundle F usually does not admit any integrable connection):

(4.1) Hodge Fundamental Theorem. *If M is a compact riemannian manifold, there is an isomorphism*

$$H^q_{\mathrm{DR}}(M, \mathbb{C}) \simeq \mathcal{H}^q(M, \mathbb{C})$$

from De Rham cohomology groups onto spaces of harmonic forms. □

A rather important consequence of the Hodge fundamental theorem is a proof of the *Poincaré duality theorem.* Assume that the Riemannian manifold (M, g) is oriented. Then there is a (conjugate linear) Hodge star operator

$$\star : \Lambda^q T^\star_M \otimes \mathbb{C} \to \Lambda^{m-q} T^\star_M \otimes \mathbb{C}, \qquad m = \dim_{\mathbb{R}} M$$

defined by $u \wedge \star v = \langle u, v\rangle dV_g$ for any two complex valued q-forms u, v. A standard computation shows that $\star$ commutes with Δ, hence $\star u$ is harmonic if and only if u is. This implies that the natural pairing

$$H^q_{\mathrm{DR}}(M, \mathbb{C}) \times H^{m-q}_{\mathrm{DR}}(M, \mathbb{C}), \qquad (\{u\}, \{v\}) \mapsto \int_M u \wedge v \tag{4.2}$$

is a nondegenerate duality, the dual of a class $\{u\}$ represented by a harmonic form being $\{\star u\}$.

Let us now suppose that X is a compact complex manifold equipped with a hermitian metric $\omega = \sum \omega_{jk} dz_j \wedge d\overline{z}_k$. Let F be a holomorphic vector bundle on X equipped with a hermitian metric, and let $D = D' + D''$ be its Chern curvature form. All that we said above for the Laplace-Beltrami operator Δ still applies to the complex Laplace operators

$$\Delta' = D'D'^\star + D'^\star D', \qquad \Delta'' = D''D''^\star + D''^\star D'',$$

with the great advantage that we always have $D'^2 = D''^2 = 0$. Especially, if X is a compact complex manifold, there are isomorphisms

$$H^{p,q}(X, F) \simeq \mathcal{H}^{p,q}(X, F) \tag{4.3}$$

between Dolbeault cohomology groups $H^{p,q}(X, F)$ and spaces $\mathcal{H}^{p,q}(X, F)$ of Δ''-harmonic forms of bidegree (p, q) with values in F. Now, there is a generalized Hodge star operator

$$\star : \Lambda^{p,q} T^\star_X \otimes F \to \Lambda^{n-p,n-q} T^\star_X \otimes F^\star, \qquad n = \dim_{\mathbb{C}} X,$$

such that $u \wedge \star v = \langle u, v\rangle dV_g$, when the for any two F-valued (p,q)-forms, when the wedge product $u \wedge \star v$ is combined with the pairing $F \times F^\star \to \mathbb{C}$. This leads to the *Serre duality theorem* [Ser55]: the bilinear pairing

$$(4.4)\qquad H^{p,q}(X,F) \times H^{n-p,n-q}(X,F^\star), \qquad (\{u\},\{v\}) \mapsto \int_X u \wedge v$$

is a nondegenerate duality. Combining this with the Dolbeault isomorphism, we may restate the result in the form of the duality formula

$$(4.4')\qquad H^q(X, \Omega_X^p \otimes \mathcal{O}(F))^\star \simeq H^{n-q}(X, \Omega_X^{n-p} \otimes \mathcal{O}(F^\star)).$$

We now proceed to explain the basic ideas of the Bochner technique used to prove vanishing theorems. Great simplifications occur in the computations if the hermitian metric on X is supposed to be *Kähler*, i.e. if the associated *fundamental* $(1,1)$*-form*

$$\omega = \mathrm{i} \sum \omega_{jk} dz_j \wedge d\overline{z}_k$$

satisfies $d\omega = 0$. It can be easily shown that ω is Kähler if and only if there are holomorphic coordinates $(z_1, \ldots, z_n)$ centered at any point $x_0 \in X$ such that the matrix of coefficients (ω_{jk}) is tangent to identity at order 2, i.e.

$$\omega_{jk}(z) = \delta_{jk} + O(|z|^2) \quad \text{at } x_0.$$

It follows that all order 1 operators D, D', D'' and their adjoints $D^\star$, $D'^\star$, $D''^\star$ admit at x_0 the same expansion as the analogous operators obtained when all hermitian metrics on X or F are constant. From this, the basic commutation relations of Kähler geometry can be checked. If A, B are differential operators acting on the algebra $C^\infty(X, \Lambda^{\bullet,\bullet}T_X^\star \otimes F)$, their graded commutator (or graded Lie bracket) is defined by

$$[A,B] = AB - (-1)^{ab} BA$$

where a, b are the degrees of A and B respectively. If C is another endomorphism of degree c, the following purely formal *Jacobi identity* holds:

$$(-1)^{ca}\big[A,[B,C]\big] + (-1)^{ab}\big[B,[C,A]\big] + (-1)^{bc}\big[C,[A,B]\big] = 0.$$

(4.5) Basic commutation relations. *Let (X,ω) be a Kähler manifold and let L be the operators defined by $Lu = \omega \wedge u$ and $\Lambda = L^\star$. Then*

$$[D''^\star, L] = \mathrm{i}D', \qquad [D'^\star, L] = -\mathrm{i}D'',$$
$$[\Lambda, D''] = -\mathrm{i}D'^\star, \qquad [\Lambda, D'] = \mathrm{i}D''^\star.$$

Proof (sketch). The first step is to check the identity $[d''^\star, L] = \mathrm{i}d'$ for constant metrics on $X = \mathbb{C}^n$ and $F = X \times \mathbb{C}$, by a brute force calculation. All three other identities follow by taking conjugates or adjoints. The case of variable metrics follows by looking at Taylor expansions up to order 1. □

(4.6) Bochner-Kodaira-Nakano identity. *If (X,ω) is Kähler, the complex Laplace operators Δ' and Δ'' acting on F-valued forms satisfy the identity*

$$\Delta'' = \Delta' + [\mathrm{i}\Theta(F), \Lambda].$$

Proof. The last equality in (4.5) yields $D''^{\star} = -\mathrm{i}[\Lambda, D']$, hence

$$\Delta'' = [D'', \delta''] = -\mathrm{i}[D'', [\Lambda, D']].$$

By the Jacobi identity we get

$$[D'', [\Lambda, D']] = [\Lambda, [D', D'']] + [D', [D'', \Lambda]] = [\Lambda, \Theta(F)] + \mathrm{i}[D', D'^{\star}],$$

taking into account that $[D', D''] = D^2 = \Theta(F)$. The formula follows. □

Assume that X is compact and that $u \in C^\infty(X, \Lambda^{p,q}T^\star X \otimes F)$ is an arbitrary (p,q)-form. An integration by parts yields

$$\langle \Delta' u, u\rangle = \|D'u\|^2 + \|D'^{\star}u\|^2 \geq 0$$

and similarly for Δ'', hence we get the basic a priori inequality

$$\|D''u\|^2 + \|D''^{\star}u\|^2 \geq \int_X \langle [\mathrm{i}\Theta(F), \Lambda]u, u\rangle dV_\omega. \tag{4.7}$$

This inequality is known as the *Bochner-Kodaira-Nakano* inequality (see [Boc48], [Kod53], [Nak55]). When u is Δ''-harmonic, we get

$$\int_X \big(\langle [\mathrm{i}\Theta(F), \Lambda]u, u\rangle + \langle T_\omega u, u\rangle\big) dV \leq 0.$$

If the hermitian operator $[\mathrm{i}\Theta(F), \Lambda]$ acting on $\Lambda^{p,q}T_X^\star \otimes F$ is positive on each fiber, we infer that u must be zero, hence

$$H^{p,q}(X, F) = \mathcal{H}^{p,q}(X, F) = 0$$

by Hodge theory. The main point is thus to compute the curvature form $\Theta(F)$ and find sufficient conditions under which the operator $[\mathrm{i}\Theta(F), \Lambda]$ is positive definite. Elementary (but somewhat tedious) calculations yield the following formulae: if the curvature of F is written as in (3.8) and $u = \sum u_{J,K,\lambda} dz_I \wedge d\overline{z}_J \otimes e_\lambda$, $|J| = p$, $|K| = q$, $1 \leq \lambda \leq r$ is a (p,q)-form with values in F, then

$$\begin{aligned} \langle [\mathrm{i}\Theta(F), \Lambda]u, u\rangle = &\sum_{j,k,\lambda,\mu,J,S} c_{jk\lambda\mu}\, u_{J,jS,\lambda}\, \overline{u_{J,kS,\mu}} \\ &+ \sum_{j,k,\lambda,\mu,R,K} c_{jk\lambda\mu}\, u_{kR,K,\lambda}\, \overline{u_{jR,K,\mu}} \\ &- \sum_{j,\lambda,\mu,J,K} c_{jj\lambda\mu}\, u_{J,K,\lambda}\, \overline{u_{J,K,\mu}}, \end{aligned} \tag{4.8}$$

where the sum is extended to all indices $1 \leq j,k \leq n$, $1 \leq \lambda,\mu \leq r$ and multiindices $|R| = p-1$, $|S| = q-1$ (here the notation $u_{JK\lambda}$ is extended to non necessarily increasing multiindices by making it alternate with respect to permutations). It is usually hard to decide the sign of the curvature term (4.8), except in some special cases.

The easiest case is when $p = n$. Then all terms in the second summation of (4.8) must have $j = k$ and $R = \{1, \ldots, n\} \setminus \{j\}$, therefore the second and third summations are equal. It follows that $[\mathrm{i}\Theta(F), \Lambda]$ is positive on (n, q)-forms under the assumption that F is positive in the sense of Nakano. In this case X is automatically Kähler since

$$\omega = \mathrm{Tr}_F(\mathrm{i}\Theta(F)) = \mathrm{i} \sum_{j,k,\lambda} c_{jk\lambda\lambda} dz_j \wedge d\overline{z}_k = \mathrm{i}\Theta(\det F)$$

is a Kähler metric.

(4.9) Nakano vanishing theorem (1955). *Let X be a compact complex manifold and let F be a Nakano positive vector bundle on X. Then*

$$H^{n,q}(X, F) = H^q(X, K_X \otimes F) = 0 \qquad \textit{for every } q \geq 1. \qquad \square$$

Another tractable case is the case where F is a line bundle ($r = 1$). Indeed, at each point $x \in X$, we may then choose a coordinate system which diagonalizes simultaneously the hermitians forms $\omega(x)$ and $\mathrm{i}\Theta(F)(x)$, in such a way that

$$\omega(x) = \mathrm{i} \sum_{1 \leq j \leq n} dz_j \wedge d\overline{z}_j, \qquad \mathrm{i}\Theta(F)(x) = \mathrm{i} \sum_{1 \leq j \leq n} \gamma_j dz_j \wedge d\overline{z}_j$$

with $\gamma_1 \leq \ldots \leq \gamma_n$. The curvature eigenvalues $\gamma_j = \gamma_j(x)$ are then uniquely defined and depend continuously on x. With our previous notation, we have $\gamma_j = c_{jj11}$ and all other coefficients $c_{jk\lambda\mu}$ are zero. For any (p, q)-form $u = \sum u_{JK} dz_J \wedge d\overline{z}_K \otimes e_1$, this gives

$$\begin{aligned} \langle [\mathrm{i}\Theta(F), \Lambda]u, u \rangle &= \sum_{|J|=p,\, |K|=q} \Big(\sum_{j \in J} \gamma_j + \sum_{j \in K} \gamma_j - \sum_{1 \leq j \leq n} \gamma_j \Big) |u_{JK}|^2 \\ &\geq (\gamma_1 + \ldots + \gamma_q - \gamma_{n-p+1} - \ldots - \gamma_n)|u|^2. \end{aligned} \tag{4.10}$$

Assume that $\mathrm{i}\Theta(F)$ is positive. It is then natural to make the special choice $\omega = \mathrm{i}\Theta(F)$ for the Kähler metric. Then $\gamma_j = 1$ for $j = 1, 2, \ldots, n$ and we obtain $\langle [\mathrm{i}\Theta(F), \Lambda]u, u \rangle = (p + q - n)|u|^2$. As a consequence:

(4.11) Akizuki-Kodaira-Nakano vanishing theorem ([AN54]). *If F is a positive line bundle on a compact complex manifold X, then*

$$H^{p,q}(X, F) = H^q(X, \Omega_X^p \otimes F) = 0 \qquad \textit{for } \; p + q \geq n + 1. \qquad \square$$

More generally, if F is a Griffiths positive (or ample) vector bundle of rank $r \geq 1$, Le Potier [LP75] proved that $H^{p,q}(X, F) = 0$ for $p + q \geq n + r$. The proof is not a direct consequence of the Bochner technique. A rather easy proof has been found by M. Schneider [Sch74], using the Leray spectral sequence associated to the projectivized bundle projection $\mathbb{P}(F) \to X$.

(4.12) Exercise. It is important for various applications to obtain vanishing theorems which are also valid in the case of semipositive line bundles. The easiest case is the following result of Girbau [Gir76]: let (X, ω) be compact Kähler; assume that F is a line bundle and that $\mathrm{i}\Theta(F) \geq 0$ has at least $n-k$ positive eigenvalues at each point, for some integer $k \geq 0$; show that $H^{p,q}(X, F) = 0$ for $p+q \geq n+k+1$.
Hint: use the Kähler metric $\omega_\varepsilon = \mathrm{i}\Theta(F) + \varepsilon\omega$ with $\varepsilon > 0$ small.

A stronger and more natural "algebraic version" of this result has been obtained by Sommese [Som78]: define F to be k-ample if some multiple mF is such that the canonical map

$$\Phi_{|mF|} : X \smallsetminus B_{|mF|} \to \mathbb{P}^{N-1}$$

has at most k-dimensional fibers and $\dim B_{|mF|} \leq k$. If X is projective and F is k-ample, show that $H^{p,q}(X, F) = 0$ for $p+q \geq n+k+1$.
Hint: prove the dual result $H^{p,q}(X, F^{-1}) = 0$ for $p+q \leq n-k-1$ by induction on k. First show that F 0-ample $\Rightarrow$ F positive; then use hyperplane sections $Y \subset X$ to prove the induction step, thanks to the exact sequences

$$0 \longrightarrow \Omega_X^p \otimes F^{-1} \otimes \mathcal{O}(-Y) \longrightarrow \Omega_X^p \otimes F^{-1} \longrightarrow \big(\Omega_X^p \otimes F^{-1}\big)_{\restriction Y} \longrightarrow 0,$$
$$0 \longrightarrow \Omega_Y^{p-1} \otimes F_{\restriction Y}^{-1} \longrightarrow \big(\Omega_X^p \otimes F^{-1}\big)_{\restriction Y} \longrightarrow \Omega_Y^p \otimes F_{\restriction Y}^{-1} \longrightarrow 0. \qquad \square$$

5. L^2 Estimates and Existence Theorems

The starting point is the following L^2 existence theorem, which is essentially due to Hörmander [Hör65, 66], and Andreotti-Vesentini [AV65]. We will only outline the main ideas, referring e.g. to [Dem82b] for a detailed exposition of the technical situation considered here.

(5.1) Theorem. *Let (X, ω) be a Kähler manifold. Here X is not necessarily compact, but we assume that the geodesic distance δ_ω is complete on X. Let F be a hermitian vector bundle of rank r over X, and assume that the curvature operator $A = A_{F,\omega}^{p,q} = [\mathrm{i}\Theta(F), \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{p,q}T_X^\star \otimes F$, $q \geq 1$. Then for any form $g \in L^2(X, \Lambda^{p,q}T_X^\star \otimes F)$ satisfying $D''g = 0$ and $\int_X \langle A^{-1}g, g\rangle\, dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T_X^\star \otimes F)$ such that $D''f = g$ and*

$$\int_X |f|^2\, dV_\omega \leq \int_X \langle A^{-1}g, g\rangle\, dV_\omega.$$

Proof. The assumption that δ_ω is complete implies the existence of cut-off functions ψ_ν with arbitrarily large compact support such that $|d\psi_\nu| \leq 1$ (take ψ_ν to be a function of the distance $x \mapsto \delta_\omega(x_0, x)$, which is an almost everywhere differentiable 1-Lipschitz function, and regularize if necessary). From this, it follows that very form $u \in L^2(X, \Lambda^{p,q}T_X^\star \otimes F)$ such that $D''u \in L^2$ and $D''^\star u \in L^2$ in the sense of distribution theory is a limit of a sequence of smooth forms u_ν with compact support, in such a way that $u_\nu \to u$, $D''u_\nu \to D''u$ and $D''^\star u_\nu \to D''^\star u$ in L^2 (just take u_ν to be a regularization of $\psi_\nu u$). As a consequence, the basic a priori inequality (4.7) extends to arbitrary forms u such that $u,\ D''u, D''^{\star}u \in L^2$. Now, consider the Hilbert space orthogonal decomposition

$$L^2(X, \Lambda^{p,q}T^\star_X \otimes F) = \operatorname{Ker} D'' \oplus (\operatorname{Ker} D'')^\perp,$$

observing that $\operatorname{Ker} D''$ is weakly (hence strongly) closed. Let $v = v_1 + v_2$ be the decomposition of a smooth form $v \in \mathcal{D}^{p,q}(X,F)$ with compact support according to this decomposition (v_1, v_2 do not have compact support in general!). Since $(\operatorname{Ker} D'')^\perp \subset \operatorname{Ker} D''^{\star}$ by duality and $g, v_1 \in \operatorname{Ker} D''$ by hypothesis, we get $D''^{\star} v_2 = 0$ and

$$|\langle g, v\rangle|^2 = |\langle g, v_1\rangle|^2 \le \int_X \langle A^{-1}g, g\rangle\, dV_\omega \int_X \langle Av_1, v_1\rangle\, dV_\omega$$

thanks to the Cauchy-Schwarz inequality. The a priori inequality (4.7) applied to $u = v_1$ yields

$$\int_X \langle Av_1, v_1\rangle\, dV_\omega \le \|D''v_1\|^2 + \|D''^{\star}v_1\|^2 = \|D''^{\star}v_1\|^2 = \|D''^{\star}v\|^2.$$

Combining both inequalities, we find

$$|\langle g, v\rangle|^2 \le \Big(\int_X \langle A^{-1}g, g\rangle\, dV_\omega\Big) \|D''^{\star}v\|^2$$

for every smooth (p,q)-form v with compact support. This shows that we have a well defined linear form

$$w = D''^{\star}v \longmapsto \langle v, g\rangle, \qquad L^2(X, \Lambda^{p,q-1}T^\star_X \otimes F) \supset D''^{\star}(\mathcal{D}^{p,q}(F)) \longrightarrow \mathbb{C}$$

on the range of $D''^{\star}$. This linear form is continuous in L^2 norm and has norm $\le C$ with

$$C = \Big(\int_X \langle A^{-1}g, g\rangle\, dV_\omega\Big)^{1/2}.$$

By the Hahn-Banach theorem, there is an element $f \in L^2(X, \Lambda^{p,q-1}T^\star_X \otimes F)$ with $||f|| \le C$, such that $\langle v, g\rangle = \langle D''^{\star}v, f\rangle$ for every v, hence $D''f = g$ in the sense of distributions. The inequality $||f|| \le C$ is equivalent to the last estimate in the theorem. □

The above L^2 existence theorem can be applied in the fairly general context of *weakly pseudoconvex* manifolds. By this, we mean a complex manifold X such that there exists a smooth psh exhaustion function ψ on X (ψ is said to be an exhaustion if for every $c > 0$ the sublevel set $X_c = \psi^{-1}(c)$ is relatively compact, i.e. $\psi(z)$ tends to $+\infty$ when z is taken outside larger and larger compact subsets of X). In particular, every compact complex manifold X is weakly pseudoconvex (take $\psi = 0$), as well as every Stein manifold, e.g. affine algebraic submanifolds of $\mathbb{C}^N$ (take $\psi(z) = |z|^2$), open balls $X = B(z_0, r)$ $\big($take $\psi(z) = 1/(r - |z - z_0|^2)\big)$, convex open subsets, etc. Now, a basic observation is that every weakly pseudoconvex Kähler manifold (X, ω) carries a *complete* Kähler metric: let $\psi \ge 0$ be a psh exhaustion function and set

$$\omega_\varepsilon = \omega + \varepsilon\, \mathrm{i}d'd''\psi^2 = \omega + 2\varepsilon(2\mathrm{i}\psi d'd''\psi + \mathrm{i}d'\psi \wedge d''\psi).$$

Then $|d\psi|_{\omega_\varepsilon} \le 1/\varepsilon$ and $|\psi(x) - \psi(y)| \le \varepsilon^{-1}\delta_{\omega_\varepsilon}(x, y)$. It follows easily from this estimate that the geodesic balls are relatively compact, hence $\delta_{\omega_\varepsilon}$ is complete for every $\varepsilon > 0$. Therefore, the L^2 existence theorem can be applied to each Kähler metric ω_ε, and by passing to the limit it can even be applied to the non necessarily complete metric ω. An important special case is the following

(5.2) Theorem. *Let (X, ω) be a Kähler manifold, $\dim X = n$. Assume that X is weakly pseudoconvex. Let F be a hermitian line bundle and let*

$$\gamma_1(x) \le \ldots \le \gamma_n(x)$$

be the curvature eigenvalues (i.e. the eigenvalues of $\mathrm{i}\Theta(F)$ with respect to the metric ω) at every point. Assume that the curvature is positive, i.e. $\gamma_1 > 0$ everywhere. Then for any form $g \in L^2(X, \Lambda^{n,q}T_X^\star \otimes F)$ satisfying $D''g = 0$ and $\int_X \langle(\gamma_1 + \ldots + \gamma_q)^{-1}|g|^2\, dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T_X^\star \otimes F)$ such that $D''f = g$ and

$$\int_X |f|^2\, dV_\omega \le \int_X (\gamma_1 + \ldots + \gamma_q)^{-1}|g|^2\, dV_\omega.$$

Proof. Indeed, for $p = n$, Formula 4.10 shows that

$$\langle Au, u\rangle \ge (\gamma_1 + \ldots + \gamma_q)|u|^2,$$

hence $\langle A^{-1}u, u\rangle \ge (\gamma_1 + \ldots + \gamma_q)^{-1}|u|^2$. □

An important observation is that the above theorem still applies when the hermitian metric on F is a singular metric with positive curvature in the sense of currents. In fact, by standard regularization techniques (convolution of psh functions by smoothing kernels), the metric can be made smooth and the solutions obtained by (5.1) or (5.2) for the smooth metrics have limits satisfying the desired estimates. Especially, we get the following

(5.3) Corollary. *Let (X, ω) be a Kähler manifold, $\dim X = n$. Assume that X is weakly pseudoconvex. Let F be a holomorphic line bundle equipped with a singular metric whose local weights are denoted $\varphi \in L^1_{\mathrm{loc}}$. Suppose that*

$$\mathrm{i}\Theta(F) = 2\mathrm{i}d'd''\varphi \ge \varepsilon\omega$$

for some $\varepsilon > 0$. Then for any form $g \in L^2(X, \Lambda^{n,q}T_X^\star \otimes F)$ satisfying $D''g = 0$, there exists $f \in L^2(X, \Lambda^{p,q-1}T_X^\star \otimes F)$ such that $D''f = g$ and

$$\int_X |f|^2 e^{-2\varphi}\, dV_\omega \le \frac{1}{q\varepsilon}\int_X |g|^2 e^{-2\varphi}\, dV_\omega. \qquad \square$$

Here we denoted somewhat incorrectly the metric by $|f|^2 e^{-2\varphi}$, as if the weight φ was globally defined on X (of course, this is so only if F is globally trivial). We will use this notation anyway, because it clearly describes the dependence of the L^2 norm on the psh weights.

We now introduce the concept of *multiplier ideal sheaf*, following A. Nadel [Nad89]. The main idea actually goes back to the fundamental works of Bombieri [Bom70] and H. Skoda [Sko72a].

(5.4) Definition. *Let φ be a psh function on an open subset $\Omega \subset X$; to φ is associated the ideal subsheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$ of germs of holomorphic functions $f \in \mathcal{O}_{\Omega,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x.*

The zero variety $V(\mathcal{I}(\varphi))$ is thus the set of points in a neighborhood of which $e^{-2\varphi}$ is non integrable. Of course, such points occur only if φ has logarithmic poles. This is made precise as follows.

(5.5) Definition. *A psh function φ is said to have a logarithmic pole of coefficient γ at a point $x \in X$ if the Lelong number*

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{\log |z - x|}$$

is non zero and if $\nu(\varphi, x) = \gamma$.

(5.6) Lemma (Skoda [Sko72a])**.** *Let φ be a psh function on an open set Ω and let $x \in \Omega$.*

a) *If $\nu(\varphi, x) < 1$, then $e^{-2\varphi}$ is integrable in a neighborhood of x, in particular $\mathcal{I}(\varphi)_x = \mathcal{O}_{\Omega,x}$.*

b) *If $\nu(\varphi, x) \geq n + s$ for some integer $s \geq 0$, then $e^{-2\varphi} \geq C|z - x|^{-2n-2s}$ in a neighborhood of x and $\mathcal{I}(\varphi)_x \subset \mathfrak{m}_{\Omega,x}^{s+1}$, where $\mathfrak{m}_{\Omega,x}$ is the maximal ideal of $\mathcal{O}_{\Omega,x}$.*

c) *The zero variety $V(\mathcal{I}(\varphi))$ of $\mathcal{I}(\varphi)$ satisfies*

$$E_n(\varphi) \subset V(\mathcal{I}(\varphi)) \subset E_1(\varphi)$$

where $E_c(\varphi) = \{x \in X \,;\, \nu(\varphi, x) \geq c\}$ is the c-sublevel set of Lelong numbers of φ.

Proof. a) Set $\Theta = dd^c\varphi$ and $\gamma = \nu(\Theta, x) = \nu(\varphi, x)$. Let χ be a cut-off function will support in a small ball $B(x, r)$, equal to 1 in $B(x, r/2)$. As $(dd^c \log |z|)^n = \delta_0$, we get

$$\begin{aligned}\varphi(z) &= \int_{B(x,r)} \chi(\zeta)\varphi(\zeta)(dd^c \log |\zeta - z|)^n \\ &= \int_{B(x,r)} dd^c(\chi(\zeta)\varphi(\zeta)) \wedge \log |\zeta - z|(dd^c \log |\zeta - z|)^{n-1}\end{aligned}$$

for $z \in B(x, r/2)$. Expanding $dd^c(\chi\varphi)$ and observing that $d\chi = dd^c\chi = 0$ on $B(x, r/2)$, we find

$$\varphi(z) = \int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge \log |\zeta - z|(dd^c \log |\zeta - z|)^{n-1} + \text{smooth terms}$$

on $B(x,r/2)$. Fix r so small that

$$\int_{B(x,r)} \chi(\zeta)\Theta(\zeta)\wedge(dd^c\log|\zeta-x|)^{n-1} \le \nu(\Theta,x,r) < 1.$$

By continuity, there exists $\delta,\varepsilon>0$ such that

$$I(z) := \int_{B(x,r)} \chi(\zeta)\Theta(\zeta)\wedge(dd^c\log|\zeta-z|)^{n-1} \le 1-\delta$$

for all $z\in B(x,\varepsilon)$. Applying Jensen's convexity inequality to the probability measure

$$d\mu_z(\zeta) = I(z)^{-1}\chi(\zeta)\Theta(\zeta)\wedge(dd^c\log|\zeta-z|)^{n-1},$$

we find

$$-\varphi(z) = \int_{B(x,r)} I(z)\log|\zeta-z|^{-1}\,d\mu_z(\zeta) + O(1) \quad\Longrightarrow$$
$$e^{-2\varphi(z)} \le C\int_{B(x,r)} |\zeta-z|^{-2I(z)}\,d\mu_z(\zeta).$$

As

$$d\mu_z(\zeta) \le C_1|\zeta-z|^{-(2n-2)}\Theta(\zeta)\wedge(dd^c|\zeta|^2)^{n-1} = C_2|\zeta-z|^{-(2n-2)}d\sigma_\Theta(\zeta),$$

we get

$$e^{-2\varphi(z)} \le C_3\int_{B(x,r)} |\zeta-z|^{-2(1-\delta)-(2n-2)}d\sigma_\Theta(\zeta),$$

and the Fubini theorem implies that $e^{-2\varphi(z)}$ is integrable on a neighborhood of x.

b) If $\nu(\varphi,x)=\gamma$, the convexity properties of psh functions, namely, the convexity of $\log r\mapsto \sup_{|z-x|=r}\varphi(z)$ implies that

$$\varphi(z) \le \gamma\log|z-x|/r_0 + M,$$

where M is the supremum on $B(x,r_0)$. Hence there exists a constant $C>0$ such that $e^{-2\varphi(z)} \ge C|z-x|^{-2\gamma}$ in a neighborhood of x. The desired result follows from the identity

$$\int_{B(0,r_0)} \frac{\big|\sum a_\alpha z^\alpha\big|^2}{|z|^{2\gamma}}dV(z) = \mathrm{Const}\int_0^{r_0}\Big(\sum|a_\alpha|^2r^{2|\alpha|}\Big)r^{2n-1-2\gamma}\,dr,$$

which is an easy consequence of Parseval's formula. In fact, if γ has integral part $[\gamma]=n+s$, the integral converges if and only if $a_\alpha=0$ for $|\alpha|\le s$.

c) is just a simple formal consequence of a) and b). □

(5.7) Proposition ([Nad89]). *For any psh function φ on $\Omega\subset X$, the sheaf $\mathcal{I}(\varphi)$ is a coherent sheaf of ideals over Ω.*

Proof. Since the result is local, we may assume that Ω is the unit ball in $\mathbb{C}^n$. Let E be the set of all holomorphic functions f on Ω such that $\int_\Omega |f|^2e^{-2\varphi}\,d\lambda < +\infty$. By the strong noetherian property of coherent sheaves, the set E generates a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_\Omega$. It is clear that $\mathcal{J} \subset \mathcal{I}(\varphi)$; in order to prove the equality, we need only check that $\mathcal{J}_x + \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1} = \mathcal{I}(\varphi)_x$ for every integer s, in view of the Krull lemma. Let $f \in \mathcal{I}(\varphi)_x$ be defined in a neighborhood V of x and let θ be a cut-off function with support in V such that $\theta = 1$ in a neighborhood of x. We solve the equation $d''u = g := d''(\theta f)$ by means of Hörmander's L^2 estimates 5.3, where F is the trivial line bundle $\Omega \times \mathbb{C}$ equipped with the strictly psh weight

$$\widetilde{\varphi}(z) = \varphi(z) + (n+s)\log|z-x| + |z|^2.$$

We get a solution u such that $\int_\Omega |u|^2 e^{-2\varphi}|z-x|^{-2(n+s)}d\lambda < \infty$, thus $F = \theta f - u$ is holomorphic, $F \in E$ and $f_x - F_x = u_x \in \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1}$. This proves our contention. □

The multiplier ideal sheaves satisfy the following basic fonctoriality property with respect to direct images of sheaves by modifications.

(5.8) Proposition. *Let $\mu : X' \to X$ be a modification of non singular complex manifolds (i.e. a proper generically 1:1 holomorphic map), and let φ be a psh function on X. Then*

$$\mu_\star\big(\mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi \circ \mu)\big) = \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi).$$

Proof. Let $n = \dim X = \dim X'$ and let $S \subset X$ be an analytic set such that $\mu : X' \smallsetminus S' \to X \smallsetminus S$ is a biholomorphism. By definition of multiplier ideal sheaves, $\mathcal{O}(K_X) \otimes \mathcal{I}(\varphi)$ is just the sheaf of holomorphic n-forms f on open sets $U \subset X$ such that $\mathrm{i}^{n^2} f \wedge \overline{f}\, e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)$. Since φ is locally bounded from above, we may even consider forms f which are a priori defined only on $U \smallsetminus S$, because f will be in $L^2_{\mathrm{loc}}(U)$ and therefore will automatically extend through S. The change of variable formula yields

$$\int_U \mathrm{i}^{n^2} f \wedge \overline{f}\, e^{-2\varphi} = \int_{\mu^{-1}(U)} \mathrm{i}^{n^2} \mu^\star f \wedge \overline{\mu^\star f}\, e^{-2\varphi\circ\mu},$$

hence $f \in \Gamma(U, \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi))$ iff $\mu^\star f \in \Gamma(\mu^{-1}(U), \mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi \circ \mu))$. Proposition 5.8 is proved. □

(5.9) Remark. If φ has analytic singularities (according to Definition 1.10), the computation of $\mathcal{I}(\varphi)$ can be reduced to a purely algebraic problem.

The first observation is that $\mathcal{I}(\varphi)$ can be computed easily if φ has the form $\varphi = \sum \alpha_j \log|g_j|$ where $D_j = g_j^{-1}(0)$ are nonsingular irreducible divisors with normal crossings. Then $\mathcal{I}(\varphi)$ is the sheaf of functions h on open sets $U \subset X$ such that

$$\int_U |h|^2 \prod |g_j|^{-2\alpha_j} dV < +\infty.$$

Since locally the g_j can be taken to be coordinate functions from a local coordinate system $(z_1, \ldots, z_n)$, the condition is that h is divisible by $\prod g_j^{m_j}$ where $m_j - \alpha_j > -1$ for each j, i.e. $m_j \geq \lfloor \alpha_j \rfloor$ (integer part). Hence

$$\mathcal{I}(\varphi) = \mathcal{O}(-\lfloor D \rfloor) = \mathcal{O}(-\sum \lfloor \alpha_j \rfloor D_j)$$

where $\lfloor D \rfloor$ denotes the integral part of the $\mathbb{Q}$-divisor $D = \sum \alpha_j D_j$.

Now, consider the general case of analytic singularities and suppose that $\varphi \sim \frac{\alpha}{2} \log\big(|f_1|^2 + \cdots + |f_N|^2\big)$ near the poles. By the remarks after Definition 1.10, we may assume that the (f_j) are generators of the integrally closed ideal sheaf $\mathcal{J} = \mathcal{J}(\varphi/\alpha)$, defined as the sheaf of holomorphic functions h such that $|h| \le C \exp(\varphi/\alpha)$. In this case, the computation is made as follows (see also L. Bonavero's work [Bon93], where similar ideas are used in connection with "singular" holomorphic Morse inequalities).

First, one computes a smooth modification $\mu : \widetilde{X} \to X$ of X such that $\mu^\star \mathcal{J}$ is an invertible sheaf $\mathcal{O}(-D)$ associated with a normal crossing divisor $D = \sum \lambda_j D_j$, where (D_j) are the components of the exceptional divisor of $\widetilde{X}$ (take the blow-up X' of X with respect to the ideal $\mathcal{J}$ so that the pull-back of $\mathcal{J}$ to X' becomes an invertible sheaf $\mathcal{O}(-D')$, then blow up again by Hironaka [Hir64] to make X' smooth and D' have normal crossings). Now, we have $K_{\widetilde{X}} = \mu^\star K_X + R$ where $R = \sum \rho_j D_j$ is the zero divisor of the Jacobian function J_μ of the blow-up map. By the direct image formula 5.8, we get

$$\mathcal{I}(\varphi) = \mu_\star\big(\mathcal{O}(K_{\widetilde{X}} - \mu^\star K_X) \otimes \mathcal{I}(\varphi \circ \mu)\big) = \mu_\star\big(\mathcal{O}(R) \otimes \mathcal{I}(\varphi \circ \mu)\big).$$

Now, $(f_j \circ \mu)$ are generators of the ideal $\mathcal{O}(-D)$, hence

$$\varphi \circ \mu \sim \alpha \sum \lambda_j \log |g_j|$$

where g_j are local generators of $\mathcal{O}(-D_j)$. We are thus reduced to computing multiplier ideal sheaves in the case where the poles are given by a $\mathbb{Q}$-divisor with normal crossings $\sum \alpha \lambda_j D_j$. We obtain $\mathcal{I}(\varphi \circ \mu) = \mathcal{O}(-\sum \lfloor \alpha \lambda_j \rfloor D_j)$, hence

$$\mathcal{I}(\varphi) = \mu_\star \mathcal{O}_{\widetilde{X}}\big(\sum (\rho_j - \lfloor \alpha \lambda_j \rfloor) D_j\big). \qquad \square$$

(5.10) Exercise. Compute the multiplier ideal sheaf $\mathcal{I}(\varphi)$ associated with $\varphi = \log(|z_1|^{\alpha_1} + \ldots + |z_p|^{\alpha_p})$ for arbitrary real numbers $\alpha_j > 0$.
Hint: using Parseval's formula and polar coordinates $z_j = r_j e^{\mathrm{i}\theta_j}$, show that the problem is equivalent to determining for which p-tuples $(\beta_1, \ldots, \beta_p) \in \mathbb{N}^p$ the integral

$$\int_{[0,1]^p} \frac{r_1^{2\beta_1} \ldots r_p^{2\beta_p}\, r_1 dr_1 \ldots r_p dr_p}{r_1^{2\alpha_1} + \ldots + r_p^{2\alpha_p}} = \int_{[0,1]^p} \frac{t_1^{(\beta_1+1)/\alpha_1} \ldots t_p^{(\beta_p+1)/\alpha_p}}{t_1 + \ldots + t_p} \frac{dt_1}{t_1} \cdots \frac{dt_p}{t_p}$$

is convergent. Conclude from this that $\mathcal{I}(\varphi)$ is generated by the monomials $z_1^{\beta_1} \ldots z_p^{\beta_p}$ such that $\sum (\beta_p + 1)/\alpha_p > 1$. (This exercise shows that the analytic definition of $\mathcal{I}(\varphi)$ is sometimes also quite convenient for computations). $\square$

Let F be a line bundle over X with a singular metric h of curvature current $\Theta_h(F)$. If φ is the weight representing the metric in an open set $\Omega \subset X$, the ideal sheaf $\mathcal{I}(\varphi)$ is independent of the choice of the trivialization and so it is the restriction to Ω of a global coherent sheaf $\mathcal{I}(h)$ on X. We will sometimes still write $\mathcal{I}(h) = \mathcal{I}(\varphi)$ by abuse of notation. In this context, we have the following fundamental vanishing theorem, which is probably one of the most central results of analytic and algebraic geometry (as we will see later, it contains the Kawamata-Viehweg vanishing theorem as a special case).

(5.11) Nadel vanishing theorem ([Nad89], [Dem93b]). *Let (X,ω) be a Kähler weakly pseudoconvex manifold, and let F be a holomorphic line bundle over X equipped with a singular hermitian metric h of weight φ. Assume that $\mathrm{i}\Theta_h(F)\ge\varepsilon\omega$ for some continuous positive function ε on X. Then*

$$H^q\big(X,\mathcal{O}(K_X+F)\otimes\mathcal{I}(h)\big)=0\qquad \textit{for all } q\ge 1.$$

Proof. Let $\mathcal{L}^q$ be the sheaf of germs of (n,q)-forms u with values in F and with measurable coefficients, such that both $|u|^2e^{-2\varphi}$ and $|d''u|^2e^{-2\varphi}$ are locally integrable. The d'' operator defines a complex of sheaves $(\mathcal{L}^\bullet,d'')$ which is a resolution of the sheaf $\mathcal{O}(K_X+F)\otimes\mathcal{I}(\varphi)$: indeed, the kernel of d'' in degree 0 consists of all germs of holomorphic n-forms with values in F which satisfy the integrability condition; hence the coefficient function lies in $\mathcal{I}(\varphi)$; the exactness in degree $q\ge1$ follows from Corollary 5.3 applied on arbitrary small balls. Each sheaf $\mathcal{L}^q$ is a $\mathcal{C}^\infty$-module, so $\mathcal{L}^\bullet$ is a resolution by acyclic sheaves. Let ψ be a smooth psh exhaustion function on X. Let us apply Corollary 5.3 globally on X, with the original metric of F multiplied by the factor $e^{-\chi\circ\psi}$, where χ is a convex increasing function of arbitrary fast growth at infinity. This factor can be used to ensure the convergence of integrals at infinity. By Corollary 5.3, we conclude that $H^q\big(\Gamma(X,\mathcal{L}^\bullet)\big)=0$ for $q\ge1$. The theorem follows. □

(5.12) Corollary. *Let (X,ω), F and φ be as in Theorem 5.11 and let $x_1,\ldots,x_N$ be isolated points in the zero variety $V(\mathcal{I}(\varphi))$. Then there is a surjective map*

$$H^0(X,K_X+F)\longrightarrow\!\!\!\!\!\rightarrow\bigoplus_{1\le j\le N}\mathcal{O}(K_X+L)_{x_j}\otimes\big(\mathcal{O}_X/\mathcal{I}(\varphi)\big)_{x_j}.$$

Proof. Consider the long exact sequence of cohomology associated to the short exact sequence $0\to\mathcal{I}(\varphi)\to\mathcal{O}_X\to\mathcal{O}_X/\mathcal{I}(\varphi)\to0$ twisted by $\mathcal{O}(K_X+F)$, and apply Theorem 5.11 to obtain the vanishing of the first H^1 group. The asserted surjectivity property follows. □

(5.13) Corollary. *Let (X,ω), F and φ be as in Theorem 5.11 and suppose that the weight function φ is such that $\nu(\varphi,x)\ge n+s$ at some point $x\in X$ which is an isolated point of $E_1(\varphi)$. Then $H^0(X,K_X+F)$ generates all s-jets at x.*

Proof. The assumption is that $\nu(\varphi,y)<1$ for y near x, $y\ne x$. By Skoda's lemma 5.6 b), we conclude that $e^{-2\varphi}$ is integrable at all such points y, hence $\mathcal{I}(\varphi)_y=\mathcal{O}_{X,y}$, whilst $\mathcal{I}(\varphi)_x\subset\mathfrak{m}_{X,x}^{s+1}$ by 5.6 a). Corollary 5.13 is thus a special case of 5.12. □

The philosophy of these results (which can be seen as generalizations of the Hörmander-Bombieri-Skoda theorem [Bom70], [Sko72a, 75]) is that the problem of constructing holomorphic sections of $K_X + F$ can be solved by constructing suitable hermitian metrics on F such that the weight φ has isolated poles at given points x_j.

(5.14) Exercise. Assume that X is compact and that L is a positive line bundle on X. Let $\{x_1, \ldots, x_N\}$ be a finite set. Show that there are constants $a, b \geq 0$ depending only on L and N such that $H^0(X, mL)$ generates jets of any order s at all points x_j for $m \geq as + b$.
Hint: Apply Corollary 5.12 to $F = -K_X + mL$, with a singular metric on L of the form $h = h_0 e^{-\varepsilon\psi}$, where h_0 is smooth of positive curvature, $\varepsilon > 0$ small and $\psi(z) \sim \log |z - x_j|$ in a neighborhood of x_j.
Derive the Kodaira embedding theorem from the above result:

(5.15) Theorem (Kodaira)**.** *If L is a line bundle on a compact complex manifold, then L is ample if and only if L is positive.* □

(5.16) Exercise (solution of the Levi problem)**.** Show that the following two properties are equivalent.

a) X is strongly pseudoconvex, i.e. X admits a strongly psh exhaustion function.

b) X is Stein, i.e. the global holomorphic functions $H^0(X, \mathcal{O}_X)$ separate points and yield local coordinates at any point, and X is holomorphically convex (this means that for any discrete sequence z_ν there is a function $f \in H^0(X, \mathcal{O}_X)$ such that $|f(z_\nu)| \to \infty$). □

(5.17) Remark. As long as forms of bidegree (n, q) are considered, the L^2 estimates can be extended to complex spaces with arbitrary singularities. In fact, if X is a complex space and φ is a psh weight function on X, we may still define a sheaf $K_X(\varphi)$ on X, such that the sections on an open set U are the holomorphic n-forms f on the regular part $U \cap X_{\text{reg}}$, satisfying the integrability condition $\mathrm{i}^{n^2} f \wedge \overline{f}\, e^{-2\varphi} \in L^1_{\text{loc}}(U)$. In this setting, the fonctoriality property 5.8 becomes

$$\mu_\star\big(K_{X'}(\varphi \circ \mu)\big) = K_X(\varphi)$$

for arbitrary complex spaces X, X' such that $\mu : X' \to X$ is a modification. If X is nonsingular we have $K_X(\varphi) = \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi)$, however, if X is singular, the symbols K_X and $\mathcal{I}(\varphi)$ must not be dissociated. The statement of the Nadel vanishing theorem becomes $H^q(X, \mathcal{O}(F) \otimes K_X(\varphi)) = 0$ for $q \geq 1$, under the same assumptions (X Kähler and weakly pseudoconvex, curvature $\geq \varepsilon\omega$). The proof can be obtained by restricting everything to X_{reg}. Although in general X_{reg} is not weakly pseudoconvex (e.g. in case $\operatorname{codim} X_{\text{sing}} \geq 2$), X_{reg} is always Kähler complete (the complement of a proper analytic subset in a Kähler weakly pseudoconvex space is complete Kähler, see e.g. [Dem82a]). As a consequence, Nadel's vanishing theorem is essentially insensitive to the presence of singularities. □

6. Numerically Effective Line Bundles

Many problems of algebraic geometry (e.g. problems of classification of algebraic surfaces or higher dimensional varieties) lead in a natural way to the study of line bundles satisfying semipositivity conditions. It turns out that semipositivity in the sense of curvature (at least, as far as smooth metrics are considered) is not a very satisfactory notion. A more flexible notion perfectly suitable for algebraic purposes is the notion of *numerical effectivity.* The goal of this section is to give a few fundamental algebraic definitions and to discuss their differential geometric counterparts. We first suppose that X is a projective algebraic manifold, $\dim X = n$.

(6.1) Definition. *A holomorphic line bundle L over a projective manifold X is said to be numerically effective, nef for short, if $L \cdot C = \int_C c_1(L) \geq 0$ for every curve $C \subset X$.*

If L is nef, it can be shown that $L^p \cdot Y = \int_Y c_1(L)^p \geq 0$ for any p-dimensional subvariety $Y \subset X$ (see e.g. [Har70]). In relation with this, let us recall the Nakai-Moishezon ampleness criterion: a line bundle L is ample if and only if $L^p \cdot Y > 0$ for every p-dimensional subvariety Y. From this, we easily infer

(6.2) Proposition. *Let L be a line bundle on a projective algebraic manifold X, on which an ample line bundle A and a hermitian metric ω are given. The following properties are equivalent:*

a) *L is nef;*

b) *for any integer $k \geq 1$, the line bundle $kL + A$ is ample;*

c) *for every $\varepsilon > 0$, there is a smooth metric h_ε on L such that $\mathrm{i}\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$.*

Proof. a) ⇒ b). If L is nef and A is ample then clearly $kL + A$ satisfies the Nakai-Moishezon criterion, hence $kL + A$ is ample.

b) ⇒ c). Condition c) is independent of the choice of the hermitian metric, so we may select a metric h_A on A with positive curvature and set $\omega = \mathrm{i}\Theta(A)$. If $kL + A$ is ample, this bundle has a metric h_{kL+A} of positive curvature. Then the metric $h_L = (h_{kL+A} \otimes h_A^{-1})^{1/k}$ has curvature

$$\mathrm{i}\Theta(L) = \frac{1}{k}\big(\mathrm{i}\Theta(kL + A) - \mathrm{i}\Theta(A)\big) \geq -\frac{1}{k}\mathrm{i}\Theta(A)\,;$$

in this way the negative part can be made smaller than $\varepsilon\,\omega$ by taking k large enough.

c) ⇒ a). Under hypothesis c), we get $L \cdot C = \int_C \frac{\mathrm{i}}{2\pi}\Theta_{h_\varepsilon}(L) \geq -\frac{\varepsilon}{2\pi}\int_C \omega$ for every curve C and every $\varepsilon > 0$, hence $L \cdot C \geq 0$ and L is nef. □

Let now X be an arbitrary compact complex manifold. Since there need not exist any curve in X, Property 6.2 c) is simply taken as a definition of nefness ([DPS94]):

(6.3) Definition. *A line bundle L on a compact complex manifold X is said to be nef if for every $\varepsilon > 0$, there is a smooth hermitian metric h_ε on L such that $\mathrm{i}\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$.*

In general, it is not possible to extract a smooth limit h_0 such that $\mathrm{i}\Theta_{h_0}(L) \geq 0$. The following simple example is given in [DPS94] (Example 1.7). Let E be a non trivial extension $0 \to \mathcal{O} \to E \to \mathcal{O} \to 0$ over an elliptic curve C and let $X = P(E)$ be the corresponding ruled surface over C. Then $L = \mathcal{O}_{P(E)}(1)$ is nef but does not admit any smooth metric of nonnegative curvature. This example answers negatively a question raised by [Fuj83].

Let us now introduce the important concept of *Kodaira-Iitaka dimension* of a line bundle.

(6.4) Definition. *If L is a line bundle, the Kodaira-Iitaka dimension $\kappa(L)$ is the supremum of the rank of the canonical maps*

$$\Phi_m : X \smallsetminus B_m \longrightarrow P(V_m^\star), \qquad x \longmapsto H_x = \{\sigma \in V_m\,;\, \sigma(x) = 0\}, \qquad m \geq 1$$

with $V_m = H^0(X, mL)$ and $B_m = \bigcap_{\sigma \in V_m} \sigma^{-1}(0) =$ base locus of V_m. In case $V_m = \{0\}$ for all $m \geq 1$, we set $\kappa(L) = -\infty$.
A line bundle is said to be big if $\kappa(L) = \dim X$.

The following lemma is well-known (the proof is a rather elementary consequence of the Schwarz lemma).

(6.5) Serre-Siegel lemma ([Ser54], [Sie55])**.** *Let L be any line bundle on a compact complex manifold. Then we have*

$$h^0(X, mL) \leq O(m^{\kappa(L)}) \qquad \text{for } m \geq 1,$$

and $\kappa(L)$ is the smallest constant for which this estimate holds. □

We now discuss the various concepts of positive cones in the space of numerical classes of line bundles, and establish a simple dictionary relating these concepts to corresponding concepts in the context of differential geometry.

Let us recall that an integral cohomology class in $H^2(X, \mathbb{Z})$ is the first Chern class of a holomorphic (or algebraic) line bundle if and only if it lies in the *Neron-Severi* group

$$\mathrm{NS}(X) = \mathrm{Ker}\,\big(H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)\big)$$

(this fact is just an elementary consequence of the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\star \to 0$). If X is compact Kähler, as we will suppose from now on in this section, this is the same as saying that the class is of type $(1,1)$ with respect to Hodge decomposition.

Let $\mathrm{NS}_\mathbb{R}(X)$ be the real vector space $\mathrm{NS}(X) \otimes \mathbb{R} \subset H^2(X, \mathbb{R})$. We define four convex cones

$$N_{\mathrm{amp}}(X) \subset N_{\mathrm{eff}}(X) \subset \mathrm{NS}_\mathbb{R}(X),$$
$$N_{\mathrm{nef}}(X) \subset N_{\mathrm{psef}}(X) \subset \mathrm{NS}_\mathbb{R}(X)$$

which are, respectively, the *convex cones* generated by Chern classes $c_1(L)$ of ample and effective line bundles, resp. the *closure of the convex cones* generated by numerically effective and pseudo-effective line bundles; we say that L is effective if mL has a section for some $m > 0$, i.e. if $\mathcal{O}(mL) \simeq \mathcal{O}(D)$ for some effective divisor D; and we say that L pseudo-effective if $c_1(L)$ is the cohomology class of some closed positive current T, i.e. if L can be equipped with a singular hermitian metric h with $T = \frac{\mathrm{i}}{2\pi}\Theta_h(L) \geq 0$ as a current. For each of the ample, effective, nef and pseudo-effective cones, the first Chern class $c_1(L)$ of a line bundle L lies in the cone if and only if L has the corresponding property (for N_{psef} use the fact that the space of positive currents of mass 1 is weakly compact; the case of all other cones is obvious).

(6.6) Proposition. *Let (X, ω) be a compact Kähler manifold. The numerical cones satisfy the following properties.*

a) $N_{\mathrm{amp}} = N^\circ_{\mathrm{amp}} \subset N^\circ_{\mathrm{nef}}$, $\quad N_{\mathrm{nef}} \subset N_{\mathrm{psef}}$.

b) *If moreover X is projective algebraic, we have $N_{\mathrm{amp}} = N^\circ_{\mathrm{nef}}$ (therefore $\overline{N}_{\mathrm{amp}} = N_{\mathrm{nef}}$), and $\overline{N}_{\mathrm{eff}} = N_{\mathrm{psef}}$.*

If L is a line bundle on X and h denotes a hermitian metric on L, the following properties are equivalent:

c) $c_1(L) \in N_{\mathrm{amp}} \Leftrightarrow \exists\varepsilon > 0$, $\exists h$ *smooth such that* $\mathrm{i}\Theta_h(L) \geq \varepsilon\omega$.

d) $c_1(L) \in N_{\mathrm{nef}} \Leftrightarrow \forall\varepsilon > 0$, $\exists h$ *smooth such that* $\mathrm{i}\Theta_h(L) \geq -\varepsilon\omega$.

e) $c_1(L) \in N_{\mathrm{psef}} \Leftrightarrow \exists h$ *possibly singular such that* $\mathrm{i}\Theta_h(L) \geq 0$.

f) *If moreover X is projective algebraic, then*
$c_1(L) \in N^\circ_{\mathrm{eff}} \Leftrightarrow \kappa(L) = \dim X$
$\Leftrightarrow \exists\varepsilon > 0$, $\exists h$ *possibly singular such that* $\mathrm{i}\Theta_h(L) \geq \varepsilon\omega$.

Proof. c) and d) are already known and e) is a definition.

a) The ample cone N_{amp} is always open by definition and contained in N_{nef}, so the first inclusion is obvious (N_{amp} is of course empty if X is not projective algebraic). Let us now prove that $N_{\mathrm{nef}} \subset N_{\mathrm{psef}}$. Let L be a line bundle with $c_1(L) \in N_{\mathrm{nef}}$. Then for every $\varepsilon > 0$, there is a current $T_\varepsilon = \frac{\mathrm{i}}{2\pi}\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$. Then $T_\varepsilon + \varepsilon\omega$ is a closed positive current and the family is uniformly bounded in mass for $\varepsilon \in]0, 1]$, since

$$\int_X (T_\varepsilon + \varepsilon\omega) \wedge \omega^{n-1} = \int_X c_1(L) \wedge \omega^{n-1} + \varepsilon \int_X \omega^n.$$

By weak compactness, some subsequence converges to a weak limit $T \geq 0$ and $T \in c_1(L)$ (the cohomology class $\{T\}$ of a current is easily shown to depend continuously on T with respect to the weak topology; use e.g. Poincaré duality to check this).

b) If X is projective, the equality $N_{\mathrm{amp}} = N^\circ_{\mathrm{nef}}$ is a simple consequence of 6.2 b) and of the fact that ampleness (or positivity) is an open property. It remains to show that $N_{\mathrm{psef}} \subset \overline{N}_{\mathrm{eff}}$. Let L be a line bundle with $c_1(L) \in N_{\mathrm{psef}}$ and let h_L be a singular hermitian on L such that $T = \frac{\mathrm{i}}{2\pi}\Theta(L) \geq 0$. Fix a point $x_0 \in X$ such that the Lelong number of T at x_0 is zero, and take a sufficiently positive line bundle A (replacing A by a multiple if necessary), such that $A - K_X$ has a singular metric

h_{A-K_X} of curvature $\ge \varepsilon\omega$ and such that h_{A-K_X} is smooth on $X \smallsetminus \{x_0\}$ and has an isolated logarithmic pole of Lelong number $\ge n$ at x_0. Then apply Corollary 5.13 to $F = mL + A - K_X$ equipped with the metric $h_L^{\otimes m} \otimes h_{A-K_X}$. Since the weight φ of this metric has a Lelong number $\ge n$ at x_0 and a Lelong number equal to the Lelong number of $T = \frac{\mathrm{i}}{2\pi}\Theta(L)$ at nearby points, $\limsup_{x\to x_0} \nu(T,x) = \nu(T,x_0) = 0$, Corollary 5.13 implies that $H^0(X, K_X + F) = H^0(X, mL + A)$ has a section which does not vanish at x_0. Hence there is an effective divisor D_m such that $\mathcal{O}(mL+A) = \mathcal{O}(D_m)$ and $c_1(L) = \frac{1}{m}\{D_m\} - \frac{1}{m}c_1(A) = \lim \frac{1}{m}\{D_m\}$ is in $\overline{N}_{\mathrm{eff}}$. □

f) Fix a nonsingular ample divisor A. If $c_1(L) \in N^\circ_{\mathrm{eff}}$, there is an integer $m > 0$ such that $c_1(L) - \frac{1}{m}c_1(A)$ is still effective, hence for m, p large we have $mpL - pA = D + F$ with an effective divisor D and a numerically trivial line bundle F. This implies $\mathcal{O}(kmpL) = \mathcal{O}(kpA + kD + kF) \supset \mathcal{O}(kpA + kF)$, hence $h^0(X, kmpL) \ge h^0(X, kpA + kF) \sim (kp)^n A^n/n!$ by the Riemann-Roch formula. Therefore $\kappa(L) = n$.

If $\kappa(L) = n$, then $h^0(X, kL) \ge ck^n$ for $k \ge k_0$ and $c > 0$. The exact cohomology sequence

$$0 \longrightarrow H^0(X, kL - A) \longrightarrow H^0(X, kL) \longrightarrow H^0(A, kL_{\restriction A})$$

where $h^0(A, kL_{\restriction A}) = O(k^{n-1})$ shows that $kL - A$ has non zero sections for k large. If D is the divisor of such a section, then $kL \simeq \mathcal{O}(A + D)$. Select a smooth metric on A such that $\frac{\mathrm{i}}{2\pi}\Theta(A) \ge \varepsilon_0\omega$ for some $\varepsilon_0 > 0$, and take the singular metric on $\mathcal{O}(D)$ with weight function $\varphi_D = \sum \alpha_j \log |g_j|$ described in Example 3.13. Then the metric with weight $\varphi_L = \frac{1}{k}(\varphi_A + \varphi_D)$ on L yields

$$\frac{\mathrm{i}}{2\pi}\Theta(L) = \frac{1}{k}\Big(\frac{\mathrm{i}}{2\pi}\Theta(A) + [D]\Big) \ge (\varepsilon_0/k)\,\omega,$$

as desired.

Finally, the curvature condition $\mathrm{i}\Theta_h(L) \ge \varepsilon\omega$ in the sense of currents yields by definition $c_1(L) \in N^\circ_{\mathrm{psef}}$. Moreover, b) implies $N^\circ_{\mathrm{psef}} = N^\circ_{\mathrm{eff}}$. □

Before going further, we need a lemma.

(6.7) Lemma. *Let X be a compact Kähler n-dimensional manifold, let L be a nef line bundle on X, and let E be an arbitrary holomorphic vector bundle. Then $h^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL)) = o(k^n)$ as $k \to +\infty$, for every $q \ge 1$. If X is projective algebraic, the following more precise bound holds:*

$$h^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL)) = O(k^{n-q}), \qquad \forall q \ge 0.$$

Proof. The Kähler case will be proved in Section 12, as a consequence of the holomorphic Morse inequalities. In the projective algebraic case, we proceed by induction on $n = \dim X$. If $n = 1$ the result is clear, as well as if $q = 0$. Now let A be a nonsingular ample divisor such that $E \otimes \mathcal{O}(A - K_X)$ is Nakano positive. Then the Nakano vanishing theorem applied to the vector bundle $F = E \otimes \mathcal{O}(kL + A - K_X)$ shows that $H^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL + A)) = 0$ for all $q \ge 1$. The exact sequence

$$0 \to \mathcal{O}(kL) \to \mathcal{O}(kL + A) \to \mathcal{O}(kL + A)_{\restriction A} \to 0$$

twisted by E implies

$$H^q(X,\mathcal{O}(E)\otimes\mathcal{O}(kL))\simeq H^{q-1}(A,\mathcal{O}(E_{\restriction A}\otimes\mathcal{O}(kL+A)_{\restriction A}),$$

and we easily conclude by induction since $\dim A=n-1$. Observe that the argument does not work any more if X is not algebraic. It seems to be unknown whether the $O(k^{n-q})$ bound still holds in that case. □

(6.8) Corollary. *If L is nef, then L is big (i.e. $\kappa(L)=n$) if and only if $L^n>0$. Moreover, if L is nef and big, then for every $\delta>0$, L has a singular metric $h=e^{-2\varphi}$ such that $\max_{x\in X}\nu(\varphi,x)\le\delta$ and $\mathrm{i}\Theta_h(L)\ge\varepsilon\,\omega$ for some $\varepsilon>0$. The metric h can be chosen to be smooth on the complement of a fixed divisor D, with logarithmic poles along D.*

Proof. By Lemma 6.7 and the Riemann-Roch formula, we have $h^0(X,kL)=\chi(X,kL)+o(k^n)=k^nL^n/n!+o(k^n)$, whence the first statement. If L is big, the proof made in 6.5 f) shows that there is a singular metric h_1 on L such that

$$\frac{\mathrm{i}}{2\pi}\Theta_{h_1}(L)=\frac{1}{k}\Big(\frac{\mathrm{i}}{2\pi}\Theta(A)+[D]\Big)$$

with a positive line bundle A and an effective divisor D. Now, for every $\varepsilon>0$, there is a smooth metric h_ε on L such that $\frac{\mathrm{i}}{2\pi}\Theta_{h_\varepsilon}(L)\ge-\varepsilon\omega$, where $\omega=\frac{\mathrm{i}}{2\pi}\Theta(A)$. The convex combination of metrics $h'_\varepsilon=h_1^{k\varepsilon}h_\varepsilon^{1-k\varepsilon}$ is a singular metric with poles along D which satisfies

$$\frac{\mathrm{i}}{2\pi}\Theta_{h'_\varepsilon}(L)\ge\varepsilon(\omega+[D])-(1-k\varepsilon)\varepsilon\omega\ge k\varepsilon^2\omega.$$

Its Lelong numbers are $\varepsilon\nu(D,x)$ and they can be made smaller than δ by choosing $\varepsilon>0$ small. □

We still need a few elementary facts about the numerical dimension of nef line bundles.

(6.9) Definition. *Let L be a nef line bundle on a compact Kähler manifold X. One defines the numerical dimension of L to be*

$$\nu(L)=\max\big\{k=0,\ldots,n\,;\ c_1(L)^k\ne0\ \mathit{in}\ H^{2k}(X,\mathbb{R})\big\}.$$

By Corollary 6.8, we have $\kappa(L)=n$ if and only if $\nu(L)=n$. In general, we merely have an inequality.

(6.10) Proposition. *If L is a nef line bundle on a compact Kähler manifold, then $\kappa(L)\le\nu(L)$.*

Proof. By induction on $n=\dim X$. If $\nu(L)=n$ or $\kappa(L)=n$ the result is true, so we may assume $r:=\kappa(L)\le n-1$ and $k:=\nu(L)\le n-1$. Fix $m>0$ so that $\Phi=\Phi_{|mL|}$ has generic rank r. Select a nonsingular ample divisor A in X such that the restriction of $\Phi_{|mL|}$ to A still has rank r (for this, just take A passing through a point $x \notin B_{|mL|}$ at which $\mathrm{rank}(d\Phi_x) = r < n$, in such a way that the tangent linear map $d\Phi_{x\restriction T_{A,x}}$ still has rank r). Then $\kappa(L_{\restriction A}) \geq r = \kappa(L)$ (we just have an equality because there might exist sections in $H^0(A, mL_{\restriction A})$ xhich do not extend to X). On the other hand, we claim that $\nu(L_{\restriction A}) = k = \nu(L)$. The inequality $\nu(L_{\restriction A}) \geq \nu(L)$ is clear. Conversely, if we set $\omega = \frac{\mathrm{i}}{2\pi}\Theta(A) > 0$, the cohomology class $c_1(L)^k$ can be represented by a closed positive current of bidegree (k, k)

$$T = \lim_{\varepsilon \to 0} \Big(\frac{\mathrm{i}}{2\pi}\Theta_{h_\varepsilon}(L) + \varepsilon\omega\Big)^k$$

after passing to some subsequence (there is a uniform bound for the mass thanks to the Kähler assumption, taking wedge products with ω^{n-k}). The current T must be non zero since $c_1(L)^k \neq 0$ by definition of $k = \nu(L)$. Then $\{[A]\} = \{\omega\}$ as cohomology classes, and

$$\int_A c_1(L_{\restriction A})^k \wedge \omega^{n-1-k} = \int_X c_1(L)^k \wedge [A] \wedge \omega^{n-1-k} = \int_X T \wedge \omega^{n-k} > 0.$$

This implies $\nu(L_{\restriction A}) \geq k$, as desired. The induction hypothesis with X replaced by A yields

$$\kappa(L) \leq \kappa(L_{\restriction A}) \leq \nu(L_{\restriction A}) \leq \nu(L). \qquad \square$$

(6.11) Remark. It may happen that $\kappa(L) < \nu(L)$: take e.g.

$$L \to X = X_1 \times X_2$$

equal to the total tensor product of an ample line bundle L_1 on a projective manifold X_1 and of a unitary flat line bundle L_2 on an elliptic curve X_2 given by a representation $\pi_1(X_2) \to U(1)$ such that no multiple kL_2 with $k \neq 0$ is trivial. Then $H^0(X, kL) = H^0(X_1, kL_1) \otimes H^0(X_2, kL_2) = 0$ for $k > 0$, and thus $\kappa(L) = -\infty$. However $c_1(L) = \mathrm{pr}_1^\star c_1(L_1)$ has numerical dimension equal to $\dim X_1$. The same example shows that the Kodaira dimension may increase by restriction to a subvariety (if $Y = X_1 \times \{\mathrm{point}\}$, then $\kappa(L_{\restriction Y}) = \dim Y$). □

We now derive an algebraic version of the Nadel vanishing theorem in the context of nef line bundles. This algebraic vanishing theorem has been obtained independently by Kawamata [Kaw82] and Viehweg [Vie82], who both reduced it to the Kodaira-Nakano vanishing theorem by cyclic covering constructions. Since then, a number of other proofs have been given, one based on connections with logarithmic singularities [EV86], another on Hodge theory for twisted coefficient systems [Kol85], a third one on the Bochner technique [Dem89] (see also [EV92] for a general survey, and [Eno93] for an extension to the compact Kähler case). Since the result is best expressed in terms of multiplier ideal sheaves (avoiding then any unnecessary desingularization in the statement), we feel that the direct approach via Nadel's vanishing theorem is probably the most natural one.

If $D = \sum \alpha_j D_j \geq 0$ is an effective $\mathbb{Q}$-divisor, we define the *multiplier ideal sheaf* $\mathcal{I}(D)$ to be equal to $\mathcal{I}(\varphi)$ where $\varphi = \sum \alpha_j |g_j|$ is the corresponding psh function defined by generators g_j of $\mathcal{O}(-D_j)$; as we saw in Remark 5.9, the computation of

$\mathcal{I}(D)$ can be made algebraically by using desingularizations $\mu : \widetilde{X} \to X$ such that $\mu^\star D$ becomes a divisor with normal crossings on $\widetilde{X}$.

(6.12) Kawamata-Viehweg vanishing theorem. *Let X be a projective algebraic manifold and let F be a line bundle over X such that some positive multiple mF can be written $mF = L + D$ where L is a nef line bundle and D an effective divisor. Then*

$$H^q\big(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(m^{-1}D)\big) = 0 \quad \textit{for} \;\; q > n - \nu(L).$$

(6.13) Special case. *If F is a nef line bundle, then*

$$H^q\big(X, \mathcal{O}(K_X + F)\big) = 0 \quad \textit{for} \;\; q > n - \nu(F).$$

Proof of Theorem 6.12. First suppose that $\nu(L) = n$, i.e. that L is big. By the proof of 6.5 f), there is a singular hermitian metric on L such that the corresponding weight $\varphi_{L,0}$ has algebraic singularities and

$$\mathrm{i}\Theta_0(L) = 2\mathrm{i}d'd''\varphi_L \geq \varepsilon_0\omega$$

for some $\varepsilon_0 > 0$. On the other hand, since L is nef, there are metrics given by weights $\varphi_{L,\varepsilon}$ such that $\frac{\mathrm{i}}{2\pi}\Theta_\varepsilon(L) \geq \varepsilon\omega$ for every $\varepsilon > 0$, ω being a Kähler metric. Let $\varphi_D = \sum \alpha_j \log|g_j|$ be the weight of the singular metric on $\mathcal{O}(D)$ described in Example 3.13. We define a singular metric on F by

$$\varphi_F = \frac{1}{m}\big((1-\delta)\varphi_{L,\varepsilon} + \delta\varphi_{L,0} + \varphi_D\big)$$

with $\varepsilon \ll \delta \ll 1$, δ rational. Then φ_F has algebraic singularities, and by taking δ small enough we find $\mathcal{I}(\varphi_F) = \mathcal{I}(\frac{1}{m}\varphi_D) = \mathcal{I}(\frac{1}{m}D)$. In fact, $\mathcal{I}(\varphi_F)$ can be computed by taking integer parts of $\mathbb{Q}$-divisors (as explained in Remark 5.9), and adding $\delta\varphi_{L,0}$ does not change the integer part of the rational numbers involved when δ is small. Now

$$\begin{aligned} dd^c\varphi_F &= \frac{1}{m}\big((1-\delta)dd^c\varphi_{L,\varepsilon} + \delta dd^c\varphi_{L,0} + dd^c\varphi_D\big) \\ &\geq \frac{1}{m}\big(-(1-\delta)\varepsilon\omega + \delta\varepsilon_0\omega + [D] \geq \frac{\delta\varepsilon}{m}\omega, \end{aligned}$$

if we choose $\varepsilon \leq \delta\varepsilon_0$. Nadel's theorem thus implies the desired vanishing result for all $q \geq 1$.

Now, if $\nu(L) < n$, we use hyperplane sections and argue by induction on $n = \dim X$. Since the sheaf $\mathcal{O}(K_X) \otimes \mathcal{I}(m^{-1}D)$ behaves fonctorially with respect to modifications (and since the L^2 cohomology complex is "the same" upstairs and downstairs), we may assume after blowing-up that D is a divisor with normal crossings. By Remark 5.9, the multiplier ideal sheaf $\mathcal{I}(m^{-1}D) = \mathcal{O}(-\lfloor m^{-1}D\rfloor)$ is locally free. By Serre duality, the expected vanishing is equivalent to

$$H^q(X, \mathcal{O}(-F) \otimes \mathcal{O}(\lfloor m^{-1}D\rfloor)) = 0 \qquad \text{for } q < \nu(L).$$

Then select a nonsingular ample divisor A such that A meets all components D_j transversally. Select A positive enough so that $\mathcal{O}(A + F - \lfloor m^{-1}D\rfloor)$ is ample. Then

$H^q(X, \mathcal{O}(-A-F)\otimes\mathcal{O}(\lfloor m^{-1}D\rfloor)) = 0$ for $q < n$ by Kodaira vanishing, and the exact sequence $0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to (i_A)_\star\mathcal{O}_A \to 0$ twisted by $\mathcal{O}(-F) \otimes \mathcal{O}(\lfloor m^{-1}D\rfloor)$ yields an isomorphism

$$H^q(X, \mathcal{O}(-F) \otimes \mathcal{O}(\lfloor m^{-1}D\rfloor)) \simeq H^q(A, \mathcal{O}(-F_{\restriction A}) \otimes \mathcal{O}(\lfloor m^{-1}D_{\restriction A}\rfloor).$$

The proof of 5.8 showed that $\nu(L_{\restriction A}) = \nu(L)$, hence the induction hypothesis implies that the cohomology group on A in the right hand side is zero for $q < \nu(L)$. □

(6.14) Remark. Enoki [Eno92] proved that the special case 6.13 of the Kawamata-Viehweg vansihing theorem still holds when X is a compact Kähler manifold. The idea is to replace the induction on dimension (hyperplane section argument) by a clever use of the Aubin-Calabi-Yau theorem. We conjecture that the general case 6.12 is also valid for compact Kähler manifolds. □

7. Seshadri Constants and the Fujita Conjecture

In questions related to the search of effective bounds for the existence of sections in multiples of ample line bundles, one is lead to study the "local positivity" of such bundles. In fact, there is a simple way of giving a precise measurement of positivity. We more or less follow the original exposition given in [Dem90], and discuss some interesting results obtained in [EL92] and [EKL94].

(7.1) Definition. *Let L be a nef line bundle over a projective algebraic manifold X. The Seshadri constant of L at a point $x \in X$ is the nonnegative real number*

$$\varepsilon(L,x) = \sup\{\varepsilon \geq 0\,;\ \mu^\star L - \varepsilon E \text{ is nef}\}, \tag{7.1$'$}$$

where $\pi : \widetilde{X} \to X$ is the blow-up of X at x and E is the exceptional divisor. An equivalent definition is

$$\varepsilon(L,x) = \inf_{C \ni x} \frac{L \cdot C}{\nu(C,x)}, \tag{7.1$''$}$$

where the infimum is taken over all irreducible curves C passing through x and $\nu(C,x)$ is the multiplicity of C at x.

To get the equivalence between the two definitions, we just observe that for any irreducible curve $\widetilde{C} \subset \widetilde{X}$ not contained in the exceptional divisor and $C = \pi(\widetilde{C})$, then (exercise to the reader!)

$$(\pi^\star L - \varepsilon E) \cdot \widetilde{C} = L \cdot C - \varepsilon\nu(C,x).$$

The infimum

$$\varepsilon(L) = \inf_{x\in X} \varepsilon(L,x) = \inf_C \frac{L \cdot C}{\nu(C)} \quad \text{where } \nu(C) = \max_{x\in C} \nu(C,x) \tag{7.2}$$

will be called the *global Seshadri constant* of L. It is well known that L is ample if and only if $\varepsilon(L) > 0$ (Seshadri's criterion for ampleness, see [Har70] Chapter 1). It is useful to think of the Seshadri constant $\varepsilon(L,x)$ as measuring how positive L is along curves passing through x. The following exercise presents some illustrations of this intuitive idea.

(7.3) Exercise.

a) If L is an ample line bundle such that mL is *very ample*, then $\varepsilon(L) \geq \frac{1}{m}$. (This is the elementary half of Seshadri's criterion for ampleness).

b) For any two nef line bundles L_1, L_2, show that

$$\varepsilon(L_1 + L_2, x) \geq \varepsilon(L_1, x) + \varepsilon(L_2, x) \qquad \text{for all } x \in X. \qquad \square$$

If L is a nef line bundle, we are also interested in singular metrics with isolated logarithmic poles (e.g. in view of applying Corollary 5.13): we say that a logarithmic pole x of the weight φ is *isolated* if φ is finite and continuous on $V \smallsetminus \{x\}$ for some neighborhood V of x and we define

$$\gamma(L,x) = \sup \left\{ \begin{array}{l} \gamma \in \mathbb{R}_+ \text{ such that } L \text{ has a singular metric} \\ \text{with } \mathrm{i}\Theta(L) \geq 0 \text{ and with an isolated} \\ \text{logarithmic pole of coefficient } \gamma \text{ at } x \end{array} \right\}. \tag{7.4}$$

If there are no such metrics, we set $\gamma(L,x) = 0$.

The numbers $\varepsilon(L,x)$ and $\gamma(L,x)$ will be seen to carry a lot of useful information about the global sections of L and its multiples kL. To make this precise, we first introduce some further definitions. Let $s(L,x)$ be the largest integer $s \in \mathbb{N}$ such that the global sections in $H^0(X,L)$ generate all s-jets $J^s_x L = \mathcal{O}_x(L)/\mathfrak{m}_x^{s+1}\mathcal{O}_x(L)$. If L_x is not generated, i.e. if all sections of L vanish at x, we set $s(L,x) = -\infty$. We also introduce the limit value

$$\sigma(L,x) = \limsup_{k \to +\infty} \frac{1}{k} s(kL,x) = \sup_{k \in \mathbb{N}^\star} \frac{1}{k} s(kL,x) \tag{7.5}$$

if $s(kL,x) \neq -\infty$ for some k, and $\sigma(L,x) = 0$ otherwise. The limsup is actually equal to the sup thanks to the superadditivity property

$$s(L_1 + L_2, x) \geq s(L_1,x) + s(L_2,x).$$

The limsup is in fact a limit as soon as kL spans at x for $k \geq k_0$, e.g. when L is ample.

(7.6) Theorem. *Let L be a line bundle over X.*

a) *If L is nef then $\varepsilon(L,x) \geq \gamma(L,x) \geq \sigma(L,x)$ for every $x \in X$.*

b) *If L is ample then $\varepsilon(L,x) = \gamma(L,x) = \sigma(L,x)$ for every $x \in X$.*

c) *If L is big and nef, the equality holds for all x outside some divisor D in X.*

Proof. Fix a point $x \in X$ and a coordinate system $(z_1, \ldots, z_n)$ centered at x. If $s = s(kL,x)$, then $H^0(X,kL)$ generates all s-jets at x and we can find holomorphic sections $f_1, \ldots, f_N$ whose s-jets are all monomials z^α, $|\alpha| = s$. We define a global singular metric on L by

$$(7.7) \qquad |\xi| = \Big(\sum_{1\le j\le N} |f_j(z)\cdot\xi^{-k}|^2\Big)^{-1/2k}, \ \xi \in L_z$$

associated to the weight function $\varphi(z) = \frac{1}{2k}\log\sum\big|\theta(f_j(z))\big|^2$ in any trivialization $L_{\restriction\Omega} \simeq \Omega\times\mathbb{C}$. Then φ has an isolated logarithmic pole of coefficient s/k at x, thus

$$\gamma(L,x) \ge \frac{1}{k}s(kL,x)$$

and in the limit we get $\gamma(L,x)\ge\sigma(L,x)$.

Now, suppose that L has a singular metric with an isolated log pole of coefficient $\ge\gamma$ at x. Set $\frac{\mathrm{i}}{2\pi}\Theta(L) = dd^c\varphi$ on a neighborhood Ω of x and let C be an irreducible curve passing through x. Then all weight functions associated to the metric of L must be locally integrable along C (since φ has an isolated pole at x). We infer

$$L\cdot C = \int_C \frac{\mathrm{i}}{2\pi}\Theta(L) \ge \int_{C\cap\Omega} dd^c\varphi \ge \gamma\,\nu(C,x)$$

because the last integral is larger than the Lelong number of the current $[C]$ with respect to the weight φ and we may apply our comparison theorem 2.15 with the ordinary Lelong number associated to the weight $\log|z-x|$. Therefore

$$\varepsilon(L,x) = \inf\frac{L\cdot C}{\nu(C,x)} \ge \sup\gamma = \gamma(L,x).$$

Finally, we show that $\sigma(L,x)\ge\varepsilon(L,x)$ when L is ample. This is done essentially by same arguments as in the proof of Seshadri's criterion, as explained in [Har70]. Consider the blow-up $\pi:\widetilde X\to X$ at point x, the exceptional divisor $E=\pi^{-1}(x)$ and the line bundles $F_{p,q} = \mathcal{O}(p\,\pi^\star L - q\,E)$ over $\widetilde X$, where $p,q>0$. Recall that $\mathcal{O}(-E)_{\restriction E}$ is the canonical line bundle $\mathcal{O}_E(1)$ over $E\simeq\mathbb{P}^{n-1}$, in particular we have $E^n = \mathcal{O}_E(-1)^{n-1} = (-1)^{n-1}$. For any irreducible curve $\widetilde C\subset\widetilde X$, either $\widetilde C\subset E$ and

$$F_{p,q}\cdot\widetilde C = \mathcal{O}(-q\,E)\cdot\widetilde C = q\,\mathcal{O}_E(1)\cdot\widetilde C = q\deg\widetilde C$$

or $\pi(\widetilde C) = C$ is a curve and

$$F_{p,q}\cdot C = p\,L\cdot C - q\,\nu(C,x) \ge \big(p - q/\varepsilon(L,x)\big)L\cdot C.$$

Thus $F_{p,q}$ is nef provided that $p\ge q/\varepsilon(L,x)$. Since $F_{p,q}$ is ample when p/q is large, a simple interpolation argument shows that $F_{p,q}$ is ample for $p>q/\varepsilon(L,x)$. In that case, the Kodaira-Serre vanishing theorem gives

$$H^1(\widetilde X, k\,F_{p,q}) = H^1\big(\widetilde X, \mathcal{O}(kp\,\pi^\star L - kq\,E)\big) = 0$$

for k large. Hence we get a surjective map

$$H^0(\widetilde X, kp\,\pi^\star L) \longrightarrow\!\!\!\!\to H^0\Big(\widetilde X, \mathcal{O}(kp\,\pi^\star L)\otimes(\mathcal{O}/\mathcal{O}(-kq\,E))\Big) \simeq J_x^{kq-1}(kp\,L),$$

that is, $H^0(X, kp\,L)$ generates all $(kq-1)$ jets at x. Therefore $p > q/\varepsilon(L,x)$ implies $s(kp\,L,x) \ge kq-1$ for k large, so $\sigma(L,x) \ge q/p$. At the limit we get $\sigma(L,x) \ge \varepsilon(L,x)$.

Assume now that L is nef and big and that $\varepsilon(L,x) > 0$. By the proof of 6.6 f), there exist an integer $k_0 \ge 1$ and effective divisors A, D such that $k_0 L \simeq A + D$ where A is ample. Then $a\,\pi^\star A - E$ is ample for a large. Hence there are integers $a, b > 0$ such that $a\,\pi^\star A - b\,E - K_{\widetilde{X}}$ is ample. When $F_{p,q}$ is nef, the sum with any positive multiple $k\,F_{p,q}$ is still ample and the Akizuki-Kodaira-Nakano vanishing theorem gives

$$H^1(\widetilde{X}, k\,F_{p,q} + a\,\pi^\star A - b\,E) = H^1\big(\widetilde{X}, (kp + k_0 a)\,\pi^\star L - a\,\pi^\star D - (kq+b)E\big) = 0$$

when we substitute $A = k_0 L - D$. As above, this implies that we have a surjective map

$$H^0\big(X, (kp+k_0 a)\,L - a\,D\big) \longrightarrow\!\!\!\!\to J_x^{kq+b-1}\big((kp+k_0 a)\,L - a\,D\big)$$

when $p \ge q/\varepsilon(L,x)$. Since $\mathcal{O}(-aD) \subset \mathcal{O}$, we infer $s\big((kp+k_0 a)L, x\big) \ge kq + b - 1$ at every point $x \in X \smallsetminus D$ and at the limit $\sigma(L,x) \ge \varepsilon(L,x)$. □

(7.8) Remark. Suppose that the line bundle L is ample. The same arguments show that if $\pi : \widetilde{X} \to X$ is the blow-up at two points x, y and if $E_x + E_y$ is the exceptional divisor, then $F_{p,q} = p\,\pi^\star L - q\,E_x - E_y$ is ample for $p > q/\varepsilon(L,x) + 1/\varepsilon(L,y)$. In that case, $H^0(X, kp\,L)$ generates $J_x^{kq-1}(kp\,L) \oplus J_y^{k-1}(kp\,L)$ for k large. Take $p > q/\varepsilon(L,x) + 1/\varepsilon(L)$ and let y run over $X \smallsetminus \{x\}$. For k large, we obtain sections $f_j \in H^0(X, kp\,L)$ whose jets at x are all monomials z^α, $|\alpha| = kq-1$, and with no other common zeros. Moreover, Formula (7.7) produces a metric on L which is smooth and has positive definite curvature on $X \smallsetminus \{x\}$, and which has a log pole of coefficient $(kq-1)/kp$ at x. Therefore the supremum $\gamma(L,x) = \sup\{\gamma\}$ is always achieved by metrics which are smooth and have positive definite curvature on $X \smallsetminus \{x\}$. □

(7.9) Remark. If Y is a p-dimensional algebraic subset of X passing through x, then

$$L^p \cdot Y \ge \varepsilon(L,x)^p \nu(Y,x)$$

where $L^p \cdot Y = \int_Y c_1(L)^p$ and $\nu(Y,x)$ is the multiplicity of Y at x. In fact, if L is ample, we can take a metric on L which is smooth on $X \smallsetminus \{x\}$ and defined on a neighborhood Ω of x by a weight function φ with a log pole of coefficient γ at x. By the comparison theorem for Lelong numbers, we get

$$L^p \cdot Y \ge \int_{Y\cap\Omega} (dd^c\varphi)^p \ge \gamma^p \nu(Y,x)$$

and γ can be chosen arbitrarily close to $\varepsilon(L,x)$. If L is nef, we apply the inequality to $k\,L + M$ with M ample and take the limit as $k \to +\infty$. □

The Seshadri constants $\varepsilon(L,x)$ and $\varepsilon(L) = \inf \varepsilon(L,x)$ are especially interesting because they provide effective results concerning the existence of sections of the so called *adjoint line bundle* $K_X + L$. The following proposition illustrates this observation.

(7.10) Proposition. *Let L be a big nef line bundle over X.*

a) *If $\varepsilon(L,x) > n+s$, then $H^0(X, K_X+L)$ generates all s-jets at x.*

b) *If $\varepsilon(L) > 2n$, then K_X+L is very ample.*

Proof. a) By the proof of Theorem 7.6, the line bundle $\pi^\star L - q\,E$ is nef for $q \le \varepsilon(L,x)$. Moreover, its n-th self intersection is equal to $L^n + (-q)^n E^n = L^n - q^n$ and as $L^n \ge \varepsilon(L,x)^n$ by remark 3.5, we see that $\pi^\star L - q\,E$ is big for $q < \varepsilon(L,x)$. The Kawamata-Viehweg vanishing theorem 5.2 then gives

$$H^1(\widetilde X, K_{\widetilde X} + \pi^\star L - q\,E) = H^1\big(\widetilde X, \pi^\star K_X + \pi^\star L - (q-n+1)E\big) = 0,$$

since $K_{\widetilde X} = \pi^\star K_X + (n-1)E$. Thus we get a surjective map

$$\begin{array}{ccc} H^0\big(\widetilde X, \pi^\star K_X + \pi^\star L\big) & \twoheadrightarrow & H^0\big(\widetilde X, \pi^\star\mathcal{O}(K_X+L)\otimes\mathcal{O}/\mathcal{O}(-(q-n+1)E)\big) \\ \| & & \| \\ H^0(X, K_X+L) & \twoheadrightarrow & J_x^{q-n}(K_X+L) \end{array}$$

provided that $\varepsilon(L,x) > q$. The first statement is proved. To show that K_X+L is very ample, we blow up at two points x, y. The line bundle $\pi^\star L - n\,E_x - n\,E_y$ is ample for $1/\varepsilon(L,x) + 1/\varepsilon(L,y) < 1/n$, a sufficient condition for this is $\varepsilon(L) > 2n$. Then we see that

$$H^0(X, K_X+L) \longrightarrow (K_X+L)_x \oplus (K_X+L)_y$$

is also surjective. □

(7.11) Exercise. Derive Proposition 7.10 directly from Corollary 5.13, assuming that L is ample and using the equality $\varepsilon(L,x) = \gamma(L,x)$. □

In relation with these questions, Fujita [Fuj87, 88] made the following interesting conjecture about adjoint line bundles.

(7.12) Conjecture (Fujita). *If L is an ample line bundle, then K_X+mL is generated by global sections for $m \ge n+1$ and very ample for $m \ge n+2$.*

Using Mori theory, Fujita proved that K_X+mL is nef for $m \ge n+1$ and ample for $m \ge n+2$, but these results are of course much weaker than the conjecture; they can be derived rather easily by a direct application of the Kawamata-Viehweg vanishing theorem (see Section 8). Observe that if the conjecture holds, it would be actually optimal, as shown by the case of projective space: if $X = \mathbb{P}^n$, then $K_X = \mathcal{O}(-n-1)$, hence the bounds $m = n+1$ for global generation and $m = n+2$ for very ampleness are sharp when $L = \mathcal{O}(1)$. The case of curves ($n = 1$) is easily settled in the affirmative:

(7.13) Exercise. If X is a curve and L is a line bundle of positive degree on X, show by Riemann-Roch that $\varepsilon(L,x) = \sigma(L,x) = \deg L$ at every point. Show in this case that Fujita's conjecture follows from Proposition 7.10. □

In the case of surfaces ($n=2$), Fujita's conjecture has been proved by I. Reider in [Rei88], as a special case of a stronger numerical criterion for very ampleness (a deep generalization of Bombieri's work [Bom73] on pluricanonical embeddings of surfaces of general type). Reider's method is based on the Serre construction for rank 2 bundles and on the Bogomolov instability criterion. Since then, various other proofs have been obtained (Sakai [Sak88], Ein-Lazarsfeld [EL93]). We will give in the next section an algebraic proof of Reider's result based on vanishing theorems, following closely [EL93]. In higher dimensions, the global generation part of the statement has been proved by Ein-Lazarsfeld [EL93] for $n=3$. However, up to now, there is no strong indication that the conjecture should be true in higher dimensions.

In a naive attempt to prove Fujita's conjecture using part a) of Proposition 7.10, it is natural to ask whether one has $\varepsilon(L,x)\ge 1$ when L is an ample line bundle. Unfortunately, simple examples due to R. Miranda show that $\varepsilon(L,x)$ may be arbitrarily small as soon as $\dim X\ge 2$.

(7.14) Proposition (R. Miranda). *Given $\varepsilon>0$, there exists a rational surface X, a point $x\in X$ and an ample line bundle L on X such that $\varepsilon(L,x)\le\varepsilon$.*

Proof. Let $C\subset\mathbb{P}^2$ be an irreducible curve of large degree d with a point x of multiplicity m. Let C' be another irreducible curve of the same degree meeting C transversally. Blow-up the points of $C\cap C'$ to obtain a rational surface X, admitting a map $\varphi: X\to\mathbb{P}^1$ (the map φ is simply given by $\varphi(z)=P'(z)/P(z)$ where $P=0$, $P'=0$ are equations of C and C'). The fibers of φ are the curves in the pencil spanned by C and C'. If C' is chosen general enough, then all these fibers are irreducible. As the fibers are disjoint, we have $C^2=C\cdot C'=0$ in X. Now, let E be one of the components of the exceptional divisor of X. Fix an integer $a\ge 2$. It follows from the Nakai-Moishezon criterion that the divisor $L=aC+E$ is ample: in fact $L^2=2aC\cdot E+E^2=2a-1$, $L\cdot E=a-1$, $L\cdot\Gamma=L\cdot C=1$ if Γ is a fiber of φ; all other irreducible curves $\Gamma\subset X$ must satisfy $\varphi(\Gamma)=\mathbb{P}^1$, hence they have non empty intersection with C and $L\cdot\Gamma\ge a$. However $\nu(C,x)=m$ by construction, hence $\varepsilon(L,x)\le 1/m$. □

Although Seshadri constants fail to be bounded below in a uniform way, the following statement (which would imply the Fujita conjecture at generic points) is expected to be true.

(7.15) Conjecture. *If L is a big nef line bundle, then $\varepsilon(L,x)\ge 1$ for x generic.*

By a generic point, we mean a point in the complement of a proper algebraic subvariety, or possibly, in a countable union of proper algebraic subvarieties. Quite recently, major progress has been made on this question by Ein-Lazarsfeld [EL92] and Ein-Küchle-Lazarsfeld [EKL94].

(7.16) Theorem ([EL92]). *Let L be an ample line bundle on a smooth projective surface X. Then $\varepsilon(L,x)\ge 1$ for all except perhaps countably many points $x\in X$, and moreover if $L^2>1$ then the set of exceptional points is finite. If $L^2\ge 5$ and $L\cdot C\ge 2$ for all curves $C\subset X$, then $\varepsilon(L,x)\ge 2$ for all but finitely many $x\in X$.* □

(7.17) Theorem ([EKL94]). *Let L be a big nef line bundle on a projective n-dimensional manifold X. Then $\varepsilon(L,x) \ge 1/n$ for all x outside a countable union of proper algebraic subsets.* □

8. Algebraic Approach to the Fujita Conjecture

This section is devoted to a proof of various results related to the Fujita conjecture. The main ideas occuring here are inspired by a recent work of Y.T. Siu [Siu94a]. His method, which is algebraic in nature and quite elementary, consists in a combination of the Riemann-Roch formula together with Nadel's vanishing theorem (in fact, only the algebraic case is needed, thus the original Kawamata-Viehweg vanishing theorem would be sufficient). Slightly later, Angehrn and Siu [AS94], [Siu94b] introduced other closely related methods, producing better bounds for the global generation question; since their method is rather delicate, we can only refer the ready to the above references. In the sequel, X denotes a projective algebraic n-dimensional manifold. The first observation is the following well-known consequence of the Riemann-Roch formula.

(8.1) Special case of Riemann-Roch. *Let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent ideal sheaf on X such that the subscheme $Y = V(\mathcal{J})$ has dimension d (with possibly some lower dimensional components). Let $[Y] = \sum \lambda_j [Y_j]$ be the effective algebraic cycle of dimension d associated to the d dimensional components of Y (taking into account multiplicities λ_j given by the ideal $\mathcal{J}$). Then for any line bundle F, the Euler characteristic*

$$\chi(Y, \mathcal{O}(F+mL)_{\restriction Y}) = \chi(X, \mathcal{O}(F+mL) \otimes \mathcal{O}_X/\mathcal{J})$$

is a polynomial $P(m)$ of degree d and leading coefficient $L^d \cdot [Y]/d!$

The second fact is an elementary lemma about numerical polynomials (polynomials with rational coefficients, mapping $\mathbb{Z}$ into $\mathbb{Z}$).

(8.2) Lemma. *Let $P(m)$ be a numerical polynomial of degree $d > 0$ and leading coefficient $a_d/d!$, $a_d \in \mathbb{Z}$, $a_d > 0$. Suppose that $P(m) \ge 0$ for $m \ge m_0$. Then*

a) *For every integer $N \ge 0$, there exists $m \in [m_0, m_0 + Nd]$ such that $P(m) \ge N$.*

b) *For every $k \in \mathbb{N}$, there exists $m \in [m_0, m_0 + kd]$ such that $P(m) \ge a_d k^d / 2^{d-1}$.*

c) *For every integer $N \ge 2d^2$, there exists $m \in [m_0, m_0 + N]$ such that $P(m) \ge N$.*

Proof. a) Each of the N equations $P(m) = 0$, $P(m) = 1$, ..., $P(m) = N-1$ has at most d roots, so there must be an integer $m \in [m_0, m_0 + dN]$ which is not a root of these.

b) By Newton's formula for iterated differences $\Delta P(m) = P(m+1) - P(m)$, we get

$$\Delta^d P(m) = \sum_{1 \le j \le d} (-1)^j \binom{d}{j} P(m+d-j) = a_d, \qquad \forall m \in \mathbb{Z}.$$

Hence if $j \in \{0, 2, 4, \ldots, 2\lfloor d/2 \rfloor\} \subset [0, d]$ is the even integer achieving the maximum of $P(m_0 + d - j)$ over this finite set, we find

$$2^{d-1} P(m_0 + d - j) = \left(\binom{d}{0} + \binom{d}{2} + \ldots \right) P(m_0 + d - j) \geq a_d,$$

whence the existence of an integer $m \in [m_0, m_0 + d]$ with $P(m) \geq a_d/2^{d-1}$. The case $k = 1$ is thus proved. In general, we apply the above case to the polynomial $Q(m) = P(km - (k-1)m_0)$, which has leading coefficient $a_d k^d/d!$

c) If $d = 1$, part a) already yields the result. If $d = 2$, a look at the parabola shows that

$$\max_{m \in [m_0, m_0+N]} P(m) \geq \begin{cases} a_2 N^2/8 & \text{if } N \text{ is even,} \\ a_2(N^2 - 1)/8 & \text{if } N \text{ is odd;} \end{cases}$$

thus $\max_{m \in [m_0, m_0+N]} P(m) \geq N$ whenever $N \geq 8$. If $d \geq 3$, we apply b) with k equal to the smallest integer such that $k^d/2^{d-1} \geq N$, i.e. $k = \lceil 2(N/2)^{1/d} \rceil$, where $\lceil x \rceil \in \mathbb{Z}$ denotes the round-up of $x \in \mathbb{R}$. Then $kd \leq (2(N/2)^{1/d} + 1)d \leq N$ whenever $N \geq 2d^2$, as a short computation shows. □

We now apply Nadel's vanishing theorem pretty much in the same way as Siu [Siu94a], but with substantial simplifications in the technique and improvements in the bounds. Our method yields simultaneously a simple proof of the following basic result.

(8.3) Theorem. *If L is an ample line bundle over a projective n-fold X, then the adjoint line bundle $K_X + (n+1)L$ is nef.*

By using Mori theory and the base point free theorem ([Mor82], [Kaw84]), one can even show that $K_X + (n+1)L$ is semiample, i.e., there exists a positive integer m such that $m(K_X + (n+1)L)$ is generated by sections (see [Kaw85] and [Fuj87]). The proof rests on the observation that $n+1$ is the maximal length of extremal rays of smooth projective n-folds. Our proof of (8.3) is different and will be given simultaneously with the proof of Th. (8.4) below.

(8.4) Theorem. *Let L be an ample line bundle and let G be a nef line bundle on a projective n-fold X. Then the following properties hold.*

a) $2K_X + mL + G$ *generates simultaneous jets of order $s_1, \ldots, s_p \in \mathbb{N}$ at arbitrary points $x_1, \ldots, x_p \in X$, i.e., there is a surjective map*

$$H^0(X, 2K_X + mL + G) \longrightarrow\!\!\!\!\!\rightarrow \bigoplus_{1 \leq j \leq p} \mathcal{O}(2K_X + mL + G) \otimes \mathcal{O}_{X,x_j}/\mathfrak{m}_{X,x_j}^{s_j+1},$$

provided that $m \geq 2 + \displaystyle\sum_{1 \leq j \leq p} \binom{3n + 2s_j - 1}{n}$.

In particular $2K_X + mL + G$ is very ample for $m \geq 2 + \binom{3n+1}{n}$.

b) $2K_X+(n+1)L+G$ *generates simultaneous jets of order* $s_1,\ldots,s_p$ *at arbitrary points* $x_1,\ldots,x_p\in X$ *provided that the intersection numbers* $L^d\cdot Y$ *of* L *over all d-dimensional algebraic subsets* Y *of* X *satisfy*

$$L^d\cdot Y>\frac{2^{d-1}}{\lfloor n/d\rfloor^d}\sum_{1\le j\le p}\binom{3n+2s_j-1}{n}.$$

Proof. The proofs of (8.3) and (8.4 a, b) go along the same lines, so we deal with them simultaneously (in the case of (8.3), we simply agree that $\{x_1,\ldots,x_p\}=\emptyset$). The idea is to find an integer (or rational number) m_0 and a singular hermitian metric h_0 on K_X+m_0L with strictly positive curvature current $\mathrm{i}\Theta_{h_0}\ge\varepsilon\omega$, such that $V(\mathcal{I}(h_0))$ is 0-dimensional and the weight φ_0 of h_0 satisfies $\nu(\varphi_0,x_j)\ge n+s_j$ for all j. As L and G are nef, $(m-m_0)L+G$ has for all $m\ge m_0$ a metric h' whose curvature $\mathrm{i}\Theta_{h'}$ has arbitrary small negative part (see [Dem90]), e.g., $\mathrm{i}\Theta_{h'}\ge-\frac{\varepsilon}{2}\omega$. Then $\mathrm{i}\Theta_{h_0}+\mathrm{i}\Theta_{h'}\ge\frac{\varepsilon}{2}\omega$ is again positive definite. An application of Cor (1.5) to $F=K_X+mL+G=(K_X+m_0L)+((m-m_0)L+G)$ equipped with the metric $h_0\otimes h'$ implies the existence of the desired sections in $K_X+F=2K_X+mL+G$ for $m\ge m_0$.

Let us fix an embedding $\Phi_{|\mu L|}:X\to\mathbb{P}^N$, $\mu\gg0$, given by sections $\lambda_0,\ldots,\lambda_N\in H^0(X,\mu L)$, and let h_L be the associated metric on L of positive definite curvature form $\omega=\frac{\mathrm{i}}{2\pi}\Theta(L)$. In order to obtain the desired metric h_0 on K_X+m_0L, we fix $a\in\mathbb{N}^\star$ and use a double induction process to construct singular metrics $(h_{k,\nu})_{\nu\ge1}$ on aK_X+b_kL for a non increasing sequence of positive integers $b_1\ge b_2\ge\ldots\ge b_k\ge\ldots$. Such a sequence much be stationary and m_0 will just be the stationary limit $m_0=\lim b_k/a$. The metrics $h_{k,\nu}$ are taken to satisfy the following properties:

α) $h_{k,\nu}$ is an algebraic metric of the form

$$\|\xi\|^2_{h_{k,\nu}}=\frac{|\tau_k(\xi)|^2}{\big(\sum_{1\le i\le\nu,\,0\le j\le N}\big|\tau_k^{(a+1)\mu}(\sigma_i^{a\mu}\cdot\lambda_j^{(a+1)b_k-am_i})\big|^2\big)^{1/(a+1)\mu}},$$

defined by sections $\sigma_i\in H^0(X,(a+1)K_X+m_iL)$, $m_i<\frac{a+1}{a}b_k$, $1\le i\le\nu$, where $\xi\mapsto\tau_k(\xi)$ is an arbitrary local trivialization of aK_X+b_kL ; note that $\sigma_i^{a\mu}\cdot\lambda_j^{(a+1)b_k-am_i}$ is a section of

$$a\mu((a+1)K_X+m_iL)+((a+1)b_k-am_i)\mu L=(a+1)\mu(aK_X+b_kL).$$

β) $\mathrm{ord}_{x_j}(\sigma_i)\ge(a+1)(n+s_j)$ for all i,j ;

γ) $\mathcal{I}(h_{k,\nu+1})\supset\mathcal{I}(h_{k,\nu})$ and $\mathcal{I}(h_{k,\nu+1})\ne\mathcal{I}(h_{k,\nu})$ whenever the zero variety $V(\mathcal{I}(h_{k,\nu}))$ has positive dimension.

The weight $\varphi_{k,\nu}=\frac{1}{2(a+1)\mu}\log\sum\big|\tau_k^{(a+1)\mu}(\sigma_i^{a\mu}\cdot\lambda_j^{(a+1)b_k-am_i})\big|^2$ of $h_{k,\nu}$ is plurisubharmonic and the condition $m_i<\frac{a+1}{a}b_k$ implies $(a+1)b_k-am_i\ge1$, thus the difference $\varphi_{k,\nu}-\frac{1}{2(a+1)\mu}\log\sum|\tau(\lambda_j)|^2$ is also plurisubharmonic. Hence $\frac{\mathrm{i}}{2\pi}\Theta_{h_{k,\nu}}(aK_X+b_kL)=\frac{\mathrm{i}}{\pi}d'd''\varphi_{k,\nu}\ge\frac{1}{(a+1)}\omega$. Moreover, condition β) clearly implies $\nu(\varphi_{k,\nu},x_j)\ge a(n+s_j)$. Finally, condition γ) combined with the strong Noetherian property of coherent sheaves ensures that the sequence $(h_{k,\nu})_{\nu\ge1}$ will finally produce a zero dimensional subscheme $V(\mathcal{I}(h_{k,\nu}))$. We agree that the sequence $(h_{k,\nu})_{\nu\ge 1}$ stops at this point, and we denote by $h_k = h_{k,\nu}$ the final metric, such that $\dim V(\mathcal{I}(h_k)) = 0$.

For $k = 1$, it is clear that the desired metrics $(h_{1,\nu})_{\nu\ge 1}$ exist if b_1 is taken large enough (so large, say, that $(a+1)K_X + (b_1-1)L$ generates jets of order $(a+1)(n+\max s_j)$ at every point; then the sections $\sigma_1,\ldots,\sigma_\nu$ can be chosen with $m_1 = \ldots = m_\nu = b_1 - 1$). Suppose that the metrics $(h_{k,\nu})_{\nu\ge 1}$ and h_k have been constructed and let us proceed with the construction of $(h_{k+1,\nu})_{\nu\ge 1}$. We do this again by induction on ν, assuming that $h_{k+1,\nu}$ is already constructed and that $\dim V(\mathcal{I}(h_{k+1,\nu})) > 0$. We start in fact the induction with $\nu = 0$, and agree in this case that $\mathcal{I}(h_{k+1,0}) = 0$ (this would correspond to an infinite metric of weight identically equal to $-\infty$). By Nadel's vanishing theorem applied to

$$F_m = aK_X + mL = (aK_X + b_kL) + (m-b_k)L$$

with the metric $h_k \otimes (h_L)^{\otimes m-b_k}$, we get

$$H^q(X, \mathcal{O}((a+1)K_X + mL)\otimes\mathcal{I}(h_k)) = 0 \qquad \text{for } q\ge 1,\ m\ge b_k.$$

As $V(\mathcal{I}(h_k))$ is 0-dimensional, the sheaf $\mathcal{O}_X/\mathcal{I}(h_k)$ is a skyscraper sheaf, and the exact sequence $0\to\mathcal{I}(h_k)\to\mathcal{O}_X\to\mathcal{O}_X/\mathcal{I}(h_k)\to 0$ twisted with the invertible sheaf $\mathcal{O}((a+1)K_X+mL)$ shows that

$$H^q(X, \mathcal{O}((a+1)K_X + mL)) = 0 \qquad \text{for } q\ge 1,\ m\ge b_k.$$

Similarly, we find

$$H^q(X, \mathcal{O}((a+1)K_X + mL)\otimes\mathcal{I}(h_{k+1,\nu})) = 0 \qquad \text{for } q\ge 1,\ m\ge b_{k+1}$$

(also true for $\nu = 0$, since $\mathcal{I}(h_{k+1,0}) = 0$), and when $m\ge\max(b_k,b_{k+1}) = b_k$, the exact sequence $0\to\mathcal{I}(h_{k+1,\nu})\to\mathcal{O}_X\to\mathcal{O}_X/\mathcal{I}(h_{k+1},\nu)\to 0$ implies

$$H^q(X, \mathcal{O}((a+1)K_X + mL)\otimes\mathcal{O}_X/\mathcal{I}(h_{k+1,\nu})) = 0 \qquad \text{for } q\ge 1,\ m\ge b_k.$$

In particular, since the H^1 group vanishes, every section u' of $(a+1)K_X+mL$ on the subscheme $V(\mathcal{I}(h_{k+1,\nu}))$ has an extension u to X. Fix a basis $u'_1,\ldots,u'_N$ of the sections on $V(\mathcal{I}(h_{k+1,\nu}))$ and take arbitrary extensions $u_1,\ldots,u_N$ to X. Look at the linear map assigning the collection of jets of order $(a+1)(n+s_j)-1$ at all points x_j

$$u = \sum_{1\le j\le N} a_j u_j \longmapsto \bigoplus J_{x_j}^{(a+1)(n+s_j)-1}(u).$$

Since the rank of the bundle of s-jets is $\binom{n+s}{n}$, the target space has dimension

$$\delta = \sum_{1\le j\le p}\binom{n+(a+1)(n+s_j)-1}{n}.$$

In order to get a section $\sigma_{\nu+1} = u$ satisfying condition β) with non trivial restriction $\sigma'_{\nu+1}$ to $V(\mathcal{I}(h_{k+1,\nu}))$, we need at least $N = \delta+1$ independent sections $u'_1,\ldots,u'_N$. This condition is achieved by applying Lemma (8.2) to the numerical polynomial

$$\begin{aligned} P(m) &= \chi(X, \mathcal{O}((a+1)K_X + mL)\otimes\mathcal{O}_X/\mathcal{I}(h_{k+1,\nu}))\\ &= h^0(X, \mathcal{O}((a+1)K_X + mL)\otimes\mathcal{O}_X/\mathcal{I}(h_{k+1,\nu}))\ge 0, \qquad m\ge b_k.\end{aligned}$$

The polynomial P has degree $d = \dim V(\mathcal{I}(h_{k+1,\nu})) > 0$. We get the existence of an integer $m \in [b_k, b_k + \eta]$ such that $N = P(m) \geq \delta + 1$ with some explicit integer $\eta \in \mathbb{N}$ (for instance $\eta = n(\delta + 1)$ always works by (8.2 a), but we will also use the other possibilities to find an optimal choice in each case). Then we find a section $\sigma_{\nu+1} \in H^0(X, (a+1)K_X + mL)$ with non trivial restriction $\sigma'_{\nu+1}$ to $V(\mathcal{I}(h_{k+1,\nu}))$, vanishing at order $\geq (a+1)(n+s_j)$ at each point x_j. We just set $m_{\nu+1} = m$, and the condition $m_{\nu+1} < \frac{a+1}{a} b_{k+1}$ is satisfied if $b_k + \eta < \frac{a+1}{a} b_{k+1}$. This shows that we can take inductively

$$b_{k+1} = \left\lfloor \frac{a}{a+1}(b_k + \eta) \right\rfloor + 1.$$

By definition, $h_{k+1,\nu+1} \leq h_{k+1,\nu}$, hence $\mathcal{I}(h_{k+1,\nu+1}) \supset \mathcal{I}(h_{k+1,\nu})$. We necessarily have $\mathcal{I}(h_{k+1,\nu+1}) \neq \mathcal{I}(h_{k+1,\nu})$, for $\mathcal{I}(h_{k+1,\nu+1})$ contains the ideal sheaf associated with the zero divisor of $\sigma_{\nu+1}$, whilst $\sigma_{\nu+1}$ does not vanish identically on $V(\mathcal{I}(h_{k+1,\nu}))$. Now, an easy computation shows that the iteration of $b_{k+1} = \lfloor \frac{a}{a+1}(b_k + \eta) \rfloor + 1$ stops at $b_k = a(\eta + 1) + 1$ for any large initial value b_1. In this way, we obtain a metric h_∞ of positive definite curvature on $aK_X + (a(\eta+1)+1)L$, with $\dim V(\mathcal{I}(h_\infty)) = 0$ and $\nu(\varphi_\infty, x_j) \geq a(n + s_j)$ at each point x_j.

Proof of (8.3). In this case, the set $\{x_j\}$ is taken to be empty, thus $\delta = 0$. By (8.2 a), the condition $P(m) \geq 1$ is achieved for some $m \in [b_k, b_k + n]$ and we can take $\eta = n$. As μL is very ample, there exists on μL a metric with an isolated logarithmic pole of Lelong number 1 at any given point x_0 (e.g., the algebraic metric defined with all sections of μL vanishing at x_0). Hence

$$F'_a = aK_X + (a(n+1)+1)L + n\mu L$$

has a metric h'_a such that $V(\mathcal{I}(h'_a))$ is zero dimensional and contains $\{x_0\}$. By Cor (1.5), we conclude that

$$K_X + F'_a = (a+1)K_X + (a(n+1) + 1 + n\mu)L$$

is generated by sections, in particular $K_X + \frac{a(n+1)+1+n\mu}{a+1} L$ is nef. As a tends to $+\infty$, we infer that $K_X + (n+1)L$ is nef. □

Proof of (8.4 a). Here, the choice $a = 1$ is sufficient for our purposes. Then

$$\delta = \sum_{1 \leq j \leq p} \binom{3n + 2s_j - 1}{n}.$$

If $\{x_j\} \neq \emptyset$, we have $\delta + 1 \geq \binom{3n-1}{n} + 1 \geq 2n^2$ for $n \geq 2$. Lemma (8.2 c) shows that $P(m) \geq \delta + 1$ for some $m \in [b_k, b_k + \eta]$ with $\eta = \delta + 1$. We can start in fact the induction procedure $k \mapsto k+1$ with $b_1 = \eta + 1 = \delta + 2$, because the only property needed for the induction step is the vanishing property

$$H^0(X, 2K_X + mL) = 0 \qquad \text{for } q \geq 1,\ m \geq b_1,$$

which is true by the Kodaira vanishing theorem and the ampleness of $K_X + b_1 L$ (here we use Fujita's result (8.3), observing that $b_1 > n + 1$). Then the recursion formula $b_{k+1} = \lfloor \frac{1}{2}(b_k + \eta) \rfloor + 1$ yields $b_k = \eta + 1 = \delta + 2$ for all k, and (8.4 a) follows. □

Proof of (8.4 b). Quite similar to (8.4 a), except that we take $\eta = n$, $a = 1$ and $b_k = n+1$ for all k. By Lemma (8.2 b), we have $P(m) \ge a_d k^d / 2^{d-1}$ for some integer $m \in [m_0, m_0 + kd]$, where $a_d > 0$ is the coefficient of highest degree in P. By Lemma (8.1) we have $a_d \ge \inf_{\dim Y = d} L^d \cdot Y$. We take $k = \lfloor n/d \rfloor$. The condition $P(m) \ge \delta + 1$ can thus be realized for some $m \in [m_0, m_0 + kd] \subset [m_0, m_0 + n]$ as soon as

$$\inf_{\dim Y = d} L^d \cdot Y \ \lfloor n/d \rfloor^d / 2^{d-1} > \delta,$$

which is equivalent to the condition given in (8.4 b). □

(8.5) Corollary. *Let X be a smooth projective n-fold, let L be an ample line bundle and G a nef line bundle over X. Then $m(K_X + (n+2)L) + G$ is very ample for $m \ge \binom{3n+1}{n} - 2n$.*

Proof. Apply Th. (8.4 a) with $G' = a(K_X + (n+1)L) + G$, so that

$$2K_X + mL + G' = (a+2)(K_X + (n+2)L) + (m - 2n - 4 - a)L + G,$$

and take $m = a + 2n + 4 \ge 2 + \binom{3n+1}{n}$. □

The main drawback of the above technique is that multiples of L at least equal to $(n+1)L$ are required to avoid zeroes of the Hilbert polynomial. In particular, it is not possible to obtain directly a very ampleness criterion for $2K_X + L$ in the statement of (8.4 b). Nevertheless, using different ideas from Angehrn-Siu [AS94], [Siu94b] has obtained such a criterion. We derive here a slightly weaker version, thanks to the following elementary Lemma.

(8.6) Lemma. *Assume that for some integer $\mu \in \mathbb{N}^\star$ the line bundle μF generates simultaneously all jets of order $\mu(n+s_j)+1$ at any point x_j in a subset $\{x_1, \ldots, x_p\}$ of X. Then $K_X + F$ generates simultaneously all jets of order s_j at x_j.*

Proof. Take the algebraic metric on F defined by a basis of sections $\sigma_1, \ldots, \sigma_N$ of μF which vanish at order $\mu(n + s_j) + 1$ at all points x_j. Since we are still free to choose the homogeneous term of degree $\mu(n + s_j) + 1$ in the Taylor expansion at x_j, we find that $x_1, \ldots, x_p$ are isolated zeroes of $\bigcap \sigma_j^{-1}(0)$. If φ is the weight of the metric of F near x_j, we thus have $\varphi(z) \sim (n + s_j + \frac{1}{\mu}) \log |z - x_j|$ in suitable coordinates. We replace φ in a neighborhood of x_j by

$$\varphi'(z) = \max\big(\varphi(z)\,,\, |z|^2 - C + (n + s_j) \log |z - x_j|\big)$$

and leave φ elsewhere unchanged (this is possible by taking $C > 0$ very large). Then $\varphi'(z) = |z|^2 - C + (n + s_j) \log |z - x_j|$ near x_j, in particular φ' is strictly plurisubharmonic near x_j. In this way, we get a metric h' on F with semipositive curvature everywhere on X, and with positive definite curvature on a neighborhood of $\{x_1, \ldots, x_p\}$. The conclusion then follows directly from Hörmander's L^2 estimates (5.1) and (5.2). □

(8.7) Theorem. *Let X be a smooth projective n-fold, and let L be an ample line bundle over X. Then $2K_X + L$ generates simultaneous jets of order $s_1, \ldots, s_p$ at arbitrary points $x_1, \ldots, x_p \in X$ provided that the intersection numbers $L^d \cdot Y$ of L over all d-dimensional algebraic subsets Y of X satisfy*

$$L^d \cdot Y > \frac{2^{d-1}}{\lfloor n/d \rfloor^d} \sum_{1 \le j \le p} \binom{(n+1)(4n+2s_j+1)-2}{n}, \qquad 1 \le d \le n.$$

Proof. By Lemma (8.6) applied with $F = K_X + L$ and $\mu = n+1$, the desired jet generation of $2K_X + L$ occurs if $(n+1)(K_X+L)$ generates jets of order $(n+1)(n+s_j)+1$ at x_j. By Lemma (8.5) again with $F = aK_X + (n+1)L$ and $\mu = 1$, we see by backward induction on a that we need the simultaneous generation of jets of order $(n+1)(n+s_j)+1+(n+1-a)(n+1)$ at x_j. In particular, for $2K_X + (n+1)L$ we need the generation of jets of order $(n+1)(2n+s_j-1)+1$. Theorem (8.4 b) yields the desired condition. □

We now list a few immediate consequences of Theorem 8.4, in connection with some classical questions of algebraic geometry.

(8.8) Corollary. *Let X be a projective n-fold of general type with K_X ample. Then mK_X is very ample for $m \ge m_0 = \binom{3n+1}{n} + 4$.*

(8.9) Corollary. *Let X be a Fano n-fold, that is, a n-fold such that $-K_X$ is ample. Then $-mK_X$ is very ample for $m \ge m_0 = \binom{3n+1}{n}$.*

Proof. Corollaries 8.8, 8.9 follow easily from Theorem 8.4 a) applied to $L = \pm K_X$. Hence we get pluricanonical embeddings $\Phi : X \to \mathbb{P}^N$ such that $\Phi^\star \mathcal{O}(1) = \pm m_0 K_X$. The image $Y = \Phi(X)$ has degree

$$\deg(Y) = \int_Y c_1\big(\mathcal{O}(1)\big)^n = \int_X c_1\big(\pm m_0 K_X\big)^n = m_0^n |K_X^n|.$$

It can be easily reproved from this that there are only finitely many deformation types of Fano n-folds, as well as of n-folds of general type with K_X ample, corresponding to a given discriminant $|K_X^n|$ (from a theoretical viewpoint, this result is a consequence of Matsusaka's big theorem [Mat72] and [KoM72], but the bounds which can be obtained from it are probably extremely huge). In the Fano case, a fundamental result obtained indepently by Kollár-Miyaoka-Mori [KoMM92] and Campana [Cam92] shows that the discriminant K_X^n is in fact bounded by a constant C_n depending only on n. Therefore, one can find an explicit bound C'_n for the degree of the embedding Φ, and it follows that there are only finitely many families of Fano manifolds in each dimension. □

In the case of surfaces, much more is known. We will content ourselves with a brief account of recent results. If X is a surface, the failure of an adjoint bundle $K_X + L$ to be globally generated or very ample is described in a very precise way by the following result of I. Reider [Rei88].

(8.10) Reider's Theorem. *Let X be a smooth projective surface and let L be a nef line bundle on X.*

a) *Assume that $L^2 \geq 5$ and let $x \in X$ be a given point. Then $K_X + L$ has a section which does not vanish at x, unless there is an effective divisor $D \subset X$ passing through x such that either*

$$\begin{aligned} &L \cdot D = 0 \quad \textit{and} \quad D^2 = -1\,; \qquad \textit{or} \\ &L \cdot D = 1 \quad \textit{and} \quad D^2 = 0. \end{aligned}$$

b) *Assume that $L^2 \geq 10$. Then any two points $x, y \in X$ (possibly infinitely near) are separated by sections of $K_X + L$, unless there is an effective divisor $D \subset X$ passing through x and y such that either*

$$\begin{aligned} &L \cdot D = 0 \quad \textit{and} \quad D^2 = -1 \ \textit{or} \ -2\,; \qquad \textit{or} \\ &L \cdot D = 1 \quad \textit{and} \quad D^2 = 0 \quad \textit{or} \ -1\,; \qquad \textit{or} \\ &L \cdot D = 2 \quad \textit{and} \quad D^2 = 0. \end{aligned}$$

□

(8.11) Corollary. *Let L be an ample line bundle on a smooth projective surface X. Then $K_X + 3L$ is globally generated and $K_X + 4L$ is very ample. If $L^2 \geq 2$ then $K_X + 2L$ is globally generated and $K_X + 3L$ is very ample.* □

The case of higher order jets can be treated similarly. The most general result in this direction has been obtained by Beltrametti and Sommese [BeS93].

(8.12) Theorem ([BeS93])**.** *Let X be a smooth projective surface and let L be a nef line bundle on X. Let p be a positive integer such that $L^2 > 4p$. Then for every 0-dimensional subscheme $Z \subset X$ of length $h^0(Z, \mathcal{O}_Z) \leq p$ the restriction*

$$\rho_Z : H^0(X, \mathcal{O}_X(K_X + L)) \longrightarrow H^0(Z, \mathcal{O}_Z(K_X + L))$$

is surjective, unless there is an effective divisor $D \subset X$ intersecting the support $|Z|$ such that

$$L \cdot D - p \leq D^2 < \frac{1}{2} L \cdot D.$$

□

Proof (Sketch). The proof the above theorems rests in an essential way on the construction of rank 2 vector bundles sitting in an exact sequence

$$0 \to \mathcal{O}_X \to E \to L \otimes \mathcal{I}_Z \to 0.$$

Arguing by induction on the length of Z, we may assume that Z is a 0-dimensional subscheme such that ρ_Z is not surjective, but such that $\rho_{Z'}$ is surjective for every proper subscheme $Z' \subset Z$. The existence of E is obtained by a classical construction of Serre (unfortunately, this construction only works in dimension 2). The numerical condition on L^2 in the hypotheses ensures that $c_1(E)^2 - 4\,c_2(E) > 0$, hence E is unstable in the sense of Bogomolov. The existence of the effective divisor D asserted in 8.10 or 8.12 follows. We refer to [Rei88], [BeS93] and [Laz93] for details. The reader will find in [FdB93] a proof of the Bogomolov inequality depending only on the Kawamata-Viehweg vanishing theorem. □

(8.13) Exercise. The goal of the exercise is to prove the following weaker form of Theorems 8.10 and 8.12, by a simple direct method based on Nadel's vanishing theorem:

> *Let L be a nef line bundle on a smooth projective surface X. Fix points $x_1, \ldots, x_N$ and corresponding multiplicities $s_1, \ldots, s_N$, and set $p = \sum (2+s_j)^2$. Then $H^0(X, K_X + L)$ generates simultaneously jets of order s_j at all points x_j provided that $L^2 > p$ and $L \cdot C > p$ for all curves C passing through one of the points x_j.*

a) Using the Riemann-Roch formula, show that the condition $L^2 > p$ implies the existence of a section of a large multiple mL vanishing at order $> m(2+s_j)$ at each of the points.

b) Construct a sequence of singular hermitian metrics on L with positive definite curvature, such that the weights φ_ν have algebraic singularities, $\nu(\varphi_\nu, x_j) \geq 2 + s_j$ at each point, and such that for some integer $m_1 > 0$ the multiplier ideal sheaves satisfy $\mathcal{I}(m_1\varphi_{\nu+1}) \supsetneq \mathcal{I}(m_1\varphi_\nu)$ if $V(\mathcal{I}(\varphi_\nu))$ is not 0-dimensional near some x_j.

Hint: a) starts the procedure. Fix $m_0 > 0$ such that $m_0 L - K_X$ is ample. Use Nadel's vanishing theorem to show that

$$H^q(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{I}(\lambda m \varphi_\nu)) = 0 \qquad \text{for all } q \geq 1,\ m \geq 0,\ \lambda \in [0,1].$$

Let D_ν be the effective $\mathbb{Q}$-divisor describing the 1-dimensional singularities of φ_ν. Then $\mathcal{I}(\lambda m \varphi_\nu) \subset \mathcal{O}(-\lfloor \lambda m D_\nu \rfloor)$ and the quotient has 0-dimensional support, hence

$$H^q(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{O}(-\lfloor \lambda m D_\nu \rfloor)) = 0 \qquad \text{for all } q \geq 1,\ m \geq 0,\ \lambda \in [0,1].$$

By Riemann-Roch again prove that

$$(\star) \quad h^0(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{O}/\mathcal{O}(-\lfloor \lambda m D_\nu \rfloor)) = \frac{m^2}{2}(2\lambda\, L \cdot D_\nu - \lambda^2 D_\nu^2) + O(m).$$

As the left hand side of $(\star)$ is increasing with λ, one must have $D_\nu^2 \leq L \cdot D_\nu$. If $V(\mathcal{I}(\varphi_\nu))$ is not 0-dimensional at x_j, then the coefficient of some component of D_ν passing through x_j is at least 1, hence

$$2\, L \cdot D_\nu - D_\nu^2 \geq L \cdot D_\nu \geq p + 1.$$

Show the existence of an integer $m_1 > 0$ independent of ν such that

$$h^0(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{O}/\mathcal{O}(-\lfloor m D_\nu \rfloor)) > \sum_{1 \leq j \leq N} \binom{(m+m_0)(2+s_j)+2}{2}$$

for $m \geq m_1$, and infer the existence of a suitable section of $(m_1+m_0)L$ which is not in $H^0(X, \mathcal{O}((m_1+m_0)L - \lfloor m_1 D_\nu \rfloor))$. Use this section to construct $\varphi_{\nu+1}$ such that $\mathcal{I}(m_1\varphi_{\nu+1}) \supsetneq \mathcal{I}(m_1\varphi_\nu)$.

9. Regularization of Currents and Self-intersection Inequalities

Let X be a compact complex n-dimensional manifold. It will be convenient to work with currents that are not necessarily positive, but such that their negative part is locally bounded. We say that a bidimension (p,p) current T is *almost positive* if there exists a smooth form v of bidegree $(n-p,n-p)$ such that $T+v\ge 0$. Similarly, a function φ on X is said to be *almost psh* if φ is locally equal to the sum of a psh function and of a smooth function; then the $(1,1)$-current $dd^c\varphi$ is almost positive; conversely, if a locally integrable function φ is such that $dd^c\varphi$ is almost positive, then φ is equal a.e. to an almost psh function. If T is closed and almost positive, the Lelong numbers $\nu(T,x)$ are well defined, since the negative part always contributes to zero.

(9.1) Theorem. *Let T be a closed almost positive $(1,1)$-current and let α be a smooth real $(1,1)$-form in the the same dd^c-cohomology class as T, i.e. $T=\alpha+dd^c\psi$ where ψ is an almost psh function. Let γ be a continuous real $(1,1)$-form such that $T\ge\gamma$. Suppose that $\mathcal{O}_{T_X}(1)$ is equipped with a smooth hermitian metric such that the curvature form satisfies*

$$\frac{\mathrm{i}}{2\pi}\Theta\big(\mathcal{O}_{T_X}(1)\big)+\pi^\star u\ge 0$$

with $\pi:P(T^\star X)\to X$ and with some nonnegative smooth $(1,1)$-form u on X. Fix a hermitian metric ω on X. Then for every $c>0$, there is a sequence of closed almost positive $(1,1)$-currents $T_{c,k}=\alpha+dd^c\psi_{c,k}$ such that $\psi_{c,k}$ is smooth on $X\smallsetminus E_c(T)$ and decreases to ψ as k tends to $+\infty$ (in particular, $T_{c,k}$ is smooth on $X\smallsetminus E_c(T)$ and converges weakly to T on X), and

$$T_{c,k}\ge\gamma-\lambda_{c,k}u-\varepsilon_k\omega$$

where

a) $\lambda_{c,k}(x)$ *is a decreasing sequence of continuous functions on X such that* $\lim_{k\to+\infty}\lambda_{c,k}(x)=\min\big(\nu(T,x),c\big)$ *at every point,*

b) $\lim_{k\to+\infty}\varepsilon_k=0$,

c) $\nu(T_{c,k},x)=\big(\nu(T,x)-c\big)_+$ *at every point $x\in X$.*

Here $\mathcal{O}_{T_X}(1)$ is the canonical line bundle associated with T_X over the hyperplane bundle $P(T^\star X)$. Observe that the theorem gives in particular approximants $T_{c,k}$ which are smooth everywhere on X if c is taken such that $c>\max_{x\in X}\nu(T,x)$. The equality in c) means that the procedure kills all Lelong numbers that are $\le c$ and shifts all others downwards by c. Hence Theorem 9.1 is an analogue over manifolds of Kiselman's procedure [Kis78 ,79] for killing Lelong numbers of a psh function on an open subset of $\mathbb{C}^n$.

Proof (Sketch). We refer to [Dem92] for a detailed proof. We only sketch a special case for which the main idea is simple to explain. The special case we wish to consider is the following: X is projective algebraic, $u = \frac{\mathrm{i}}{2\pi}\Theta(G)$ is the curvature form of a nef $\mathbb{Q}$-divisor G, and T has the form

$$T = \alpha + dd^c\psi, \qquad \psi = \log(|f_1^2| + \ldots + |f_N|^2)^{1/2},$$

where α is smooth and the f_j's are sections of some C^∞ hermitian line bundle L on X. The lower bound for T is

$$T \geq \gamma := \alpha - \frac{\mathrm{i}}{2\pi}\Theta(L),$$

hence T is almost positive. Somehow, the general situation can be reduced locally to this one by an approximation theorem for currents based on Hörmander's L^2 estimates, in the form given by Ohsawa-Takegoshi ([OT87], [Ohs88]); the main point is to show that any closed positive $(1,1)$-current is locally a weak limit of effective $\mathbb{Q}$-divisors which have roughly the same Lelong numbers as the given current, up to small errors converging to 0; the proof is then completed by means of rather tricky gluing techniques for psh functions (see [Dem92]). Now, let A be an ample divisor and let $\omega = \frac{\mathrm{i}}{2\pi}\Theta(A)$ be a positive curvature form for A. After adding εA to G ($\varepsilon \in \mathbb{Q}^+$), which is the same as adding $\varepsilon\omega$ to u, we may assume that u is positive definite and that $\mathcal{O}_{T_X}(1) + \pi^\star G$ is ample ($\mathcal{O}_{T_X}(1)$ is relatively ample, so adding something ample from X is enough to make it ample on $P(T_X^\star)$). The Lelong numbers of T are given by the simple formula

$$\nu(T,x) = \min_{1\leq i\leq N} \mathrm{ord}_x(f_i).$$

The basic idea is to decrease the Lelong numbers of T by adding some "derivatives" of the f_i's in the sum of squares. The derivatives have of course to be computed as global sections on X. For this, we introduce two positive integers m, p which will be selected later more carefully, and we consider the m-jet sections $J^m f_i^{\otimes p}$, viewed as sections of the m-jet vector bundle $J^m L^{\otimes p}$. The bundle $J^m L^{\otimes p}$ has a filtration whose graded terms are $S^\nu T_X^\star \otimes L^{\otimes p}$, $0 \leq \nu \leq m$, hence $(J^m L^{\otimes p})^\star \otimes L^{\otimes p}$ has a filtration with graded terms $S^\nu T_X$, $0 \leq \nu \leq m$. Our hypothesis $T_X \otimes \mathcal{O}(G)$ ample implies that $(J^m L^{\otimes p})^\star \otimes \mathcal{O}(pL + mG)$ is ample. Hence, raising to some symmetric power S^ν where ν is a multiple of a denominator for the $\mathbb{Q}$-divisor G, we obtain that $S^\nu(J^m L^{\otimes p})^\star \otimes \mathcal{O}(\nu pL + \nu mG)$ is generated by global sections $P_1, \ldots, P_{N'}$ for ν large (we view the P_j's as some kind of "differential operators" acting on sections of L). Now, the ν-th symmetric product $S^\nu(J^m f_i^{\otimes p})$ takes values in $S^\nu(J^m L^{\otimes p})$, hence the pairing with the dual bundle yields a section

$$F^{p,\nu}_{m,i,j} := P_j \cdot S^\nu(J^m f_i^{\otimes p}) \in H^0(X, \mathcal{O}(\nu pL + \nu mG)).$$

Since the P_j generate sections and $J^m f_i^{\otimes p}$ vanishes at order $(p\ \mathrm{ord}_x(f_i) - m)_+$ at any point x, we get

$$\min_j \mathrm{ord}_x(F^{p,\nu}_{m,i,j}) = \nu(p\ \mathrm{ord}_x(f_i) - m)_+.$$

We set

$$\begin{aligned} T_{c,k} &= \alpha + dd^c\psi_{c,k}, \\ \psi_{c,k} &= \frac{1}{2\nu p}\log\Big(\big(\sum_i |f_i|^2\big)^{\nu p} + \sum_{1\leq\mu\leq m} k^{-\mu}\sum_{i,j}\big|F^{p,\nu}_{\mu,i,j}\big|^2\Big) \end{aligned}$$

Clearly, $\psi_{c,k}$ is decreasing and converges to ψ, and its Lelong numbers are

$$\nu(\psi_{c,k},x)=\min_i\Big(\mathrm{ord}_x(f_i)-\frac{m}{p}\Big)_+=\Big(\nu(\psi,x)-\frac{m}{p}\Big)_+.$$

Select $c=\frac{m}{p}$ (or take a very close rational approximation if an arbitrary real number $c\in\mathbb{R}^+$ is to be considered). As $F^{p,\nu}_{m,i,j}$ takes values in $\mathcal{O}(\nu pL+\nu mG)$, an easy computation yields

$$T_{c,k}\ge\alpha-\frac{\mathrm{i}}{2\pi}\Theta\Big(L+\frac{m}{p}G\Big)=\gamma-\frac{m}{p}u=\gamma-cu.$$

However, at points where $\nu(T,x)<c$, there is already a term with $\frac{\mu}{p}\simeq\nu(T,x)$ such that the contribution of the terms indexed by μ give a non zero contribution. The terms corresponding to higher indices will then not contribute much to the lower bound of $dd^c\psi_{c,k}$ since in the summation $k^{-\mu+1}$ becomes negligible against $k^{-\mu}$ as $k\to+\infty$. This gives a lower bound $T_{c,k}\ge\gamma-\lambda_{c,k}-\varepsilon_k\omega$ of the expected form. □

(9.2) Corollary. *Let Θ be a closed almost positive current of bidimension (p,p) and let $\alpha_1,\ldots,\alpha_q$ be closed almost positive $(1,1)$-currents such that $\alpha_1\wedge\ldots\wedge\alpha_q\wedge\Theta$ is well defined by application of criteria* 2.3 *or* 2.5*, when α_j is written locally as $\alpha_j=dd^cu_j$. Then*

$$\{\alpha_1\wedge\ldots\wedge\alpha_q\wedge\Theta\}=\{\alpha_1\}\cdots\{\alpha_q\}\cdot\{\Theta\}.$$

Proof. Theorem 9.1 and the monotone continuity theorem for Monge-Ampère operators show that

$$\alpha_1\wedge\ldots\wedge\alpha_q\wedge\Theta=\lim_{k\to+\infty}\alpha_1^k\wedge\ldots\wedge\alpha_q^k\wedge\Theta$$

where $\alpha_j^k\in\{\alpha_j\}$ is smooth. Since the result is by definition true for smooth forms, we conclude by the weak continuity of cohomology class assignment. □

Now, let X be a compact Kähler manifold equipped with a Kähler metric ω. The *degree* of a closed positive current Θ with respect to ω is by definition

(9.3) $$\deg_\omega\Theta=\int_X\Theta\wedge\omega^p,\qquad \mathrm{bidim}\,\Theta=(p,p).$$

In particular, the degree of a p-dimensional analytic set $A\subset X$ is its volume $\int_A\omega^p$ with respect to ω. We are interested in the following problem.

(9.4) Problem. *Let T be a closed positive $(1,1)$-current on X. Is it possible to derive a bound for the codimension p components in the Lelong upperlevel sets $E_c(T)$ in terms of the cohomology class $\{T\}\in H^2_{DR}(X,\mathbb{R})$?*

Let $\Xi\subset X$ be an arbitrary subset. We introduce the sequence

$$0=b_1\le\ldots\le b_n\le b_{n+1}$$

of "*jumping values*" of $E_c(T)$ over Ξ, defined by the property that the dimension of $E_c(T)$ in a neighborhood of Ξ drops by one unit when c gets larger than b_p, namely

$$b_p = \inf\big\{c > 0\,;\ \mathrm{codim}\big(E_c(T), x\big) \ge p,\ \forall x \in \Xi\big\}.$$

Then, when $c \in \,]b_p, b_{p+1}]$, we have $\mathrm{codim}\, E_c(T) = p$ in a neighborhood of Ξ, with at least one component of codimension p meeting Ξ. Let $(Z_{p,k})_{k\ge1}$ be the family of all these p-codimensional components $\big($occurring in any of the sets $E_c(T)$ for $c \in \,]b_p, b_{p+1}]\big)$, and let

$$\nu_{p,k} = \min_{x\in Z_{p,k}} \nu(T,x) \in \,]b_p, b_{p+1}]$$

be the generic Lelong number of T along $Z_{p,k}$. Then we have the following self-intersection inequality.

(9.5) Theorem. *Suppose that X is Kähler and that $\mathcal{O}_{T_X}(1)$ has a hermitian metric such that $\frac{\mathrm{i}}{2\pi}\Theta\big(\mathcal{O}_{T_X}(1)\big) + \pi^\star u \ge 0$, where u is a smooth closed semipositive $(1,1)$-form. Let $\Xi \subset X$ be an arbitrary subset, let T be a closed positive current of bidegree $(1,1)$, and let (b_p), $(Z_{p,k}$ be the corresponding jumping values and p-codimensional components of $E_c(T)$ meeting Ξ. Assume either that $\Xi = X$ or that the cohomology class $\{T\} \in H^{1,1}(X)$ is nef (i.e. in the closure of the Kähler cone). Then, for each $p = 1,\ldots,n$, the De Rham cohomology class $(\{T\} + b_1\{u\})\cdots(\{T\} + b_p\{u\})$ can be represented by a closed positive current Θ_p of bidegree (p,p) such that*

$$\Theta_p \ge \sum_{k\ge1} (\nu_{p,k} - b_1)\ldots(\nu_{p,k} - b_p)\,[Z_{p,k}] + (T_{\mathrm{ac}} + b_1 u)\wedge\ldots\wedge(T_{\mathrm{ac}} + b_p u)$$

where $T_{\mathrm{ac}} \ge 0$ is the absolutely continuous part in the Lebesgue decomposition of T (decomposition of the coefficients of T into absolutely continuous and singular measures), $T = T_{\mathrm{ac}} + T_{\mathrm{sing}}$.

By neglecting the second term in the right hand side and taking the wedge product with ω^{n-p}, we get the following interesting consequence:

(9.6) Corollary. *If ω is a Kähler metric on X and if $\{u\}$ is a nef cohomology class such that $c_1\big(\mathcal{O}_{T_X}(1)\big) + \pi^\star\{u\}$ is nef, the degrees of the components $Z_{p,k}$ with respect to ω satisfy the estimate*

$$\sum_{k=1}^{+\infty} (\nu_{p,k} - b_1)\ldots(\nu_{p,k} - b_p)\int_X [Z_{p,k}]\wedge\omega^{n-p} \le \big(\{T\} + b_1\{u\}\big)\cdots\big(\{T\} + b_p\{u\}\big)\cdot\{\omega\}^{n-p}.$$

As a special case, if D is an effective divisor and $T = [D]$, we get a bound for the degrees of the p-codimensional singular strata of D in terms of a polynomial of degree p in the cohomology class $\{D\}$; the multiplicities $(\nu_{p,k} - b_1)\ldots(\nu_{p,k} - b_p)$ are then positive integers. The case when X is $\mathbb{P}^n$ or a homogeneous manifold is especially simple: then T_X is generated by sections and we can take $u = 0$; the bound is thus simply $\{D\}^p \cdot \{\omega\}^{n-p}$; the same is true more generally as soon as $\mathcal{O}_{T_X}(1)$ is nef. The main idea of the proof is to kill the Lelong numbers of T up to the level b_j; then the singularities of the resulting current T_j occur only in codimension j and it becomes possible to define the wedge product $T_1 \wedge \ldots \wedge T_p$ by means of Proposition 2.3. Here are the details:

Proof of Theorem 9.5. First suppose that $\Xi = X$. We argue by induction on p. For $p = 1$, Siu's decomposition formula shows that

$$T = \sum \nu_{1,k}[Z_{1,k}] + R,$$

and we have $R \geq T_{\mathrm{ac}}$ since the other part has singular measures as coefficients. The result is thus true with $\Theta_1 = T$. Now, suppose that Θ_{p-1} has been constructed. For $c > b_p$, the current $T_{c,k} = \alpha + dd^c\psi_{c,k}$ produced by Theorem 9.1 is such that the codimension of the set of poles $\psi_{c,k}^{-1}(-\infty) = E_c(T)$ is at least p at every point $x \in X$ (recall that $\Xi = X$). Then Proposition 2.5 shows that

$$\Theta_{p,c,k} = \Theta_{p-1} \wedge (T_{c,k} + c\,u + \varepsilon_k\omega)$$

is well defined. If ε_k tends to zero slowly enough, $T_{c,k} + c\,u + \varepsilon_k\omega$ is positive by (9.1 a), so $\Theta_{p,c,k} \geq 0$. Moreover, by Corollary 9.2, the cohomology class of $\Theta_{p,c,k}$ is $\{\Theta_{p-1}\} \cdot (\{T\} + c\{u\} + \varepsilon_k\{\omega\})$, converging to $\{\Theta_{p-1}\} \cdot (\{T\} + c\{u\})$. Since the mass $\int_X \Theta_{p,c,k} \wedge \omega^{n-p}$ remains uniformly bounded, the family $(\Theta_{p,c,k})_{c\in]b_p,b_p+1],k\geq 1}$ is relatively compact in the weak topology. We define

$$\Theta_p = \lim_{c\to b_p+0}\ \lim_{k\to+\infty} \Theta_{p,c,k},$$

possibly after extracting some weakly convergent subsequence. Then $\{\Theta_p\} = \{\Theta_{p-1}\}\cdot(\{T\}+b_p\{u\})$, and so $\{\Theta_p\} = (\{T\}+b_1\{u\})\cdots(\{T\}+b_p\{u\})$. Moreover, it is well-known (and easy to check) that Lelong numbers are upper semi-continuous with respect to weak limits of currents. Therefore

$$\begin{aligned}
\nu(\Theta_p, x) &\geq \limsup_{c\to b_p+0}\ \limsup_{k\to+\infty}\ \nu\big(\Theta_{p-1} \wedge (T_{c,k} + c\,u + \varepsilon_k\omega), x\big) \\
&\geq \nu(\Theta_{p-1}, x) \times \limsup_{c\to b_p+0}\ \limsup_{k\to+\infty}\ \nu(T_{c,k}, x) \\
&\geq \nu(\Theta_{p-1}, x)\big(\nu(T,x) - b_p\big)_+
\end{aligned}$$

by application of Proposition 2.16 and (9.1 a). Hence by induction we get

$$\nu(\Theta_p, x) \geq \big(\nu(T,x) - b_1\big)_+ \ldots \big(\nu(T,x) - b_p\big)_+,$$

in particular, the generic Lelong number of Θ_p along $Z_{p,k}$ is at least equal to the product $(\nu_{p,k} - b_1)\ldots(\nu_{p,k} - b_p)$. This already implies

$$\Theta_p \geq \sum_{k\geq 1} (\nu_{p,k} - b_1)\ldots(\nu_{p,k} - b_p)\,[Z_{p,k}].$$

Since the right hand side is Lebesgue singular, the desired inequality will be proved if we show in addition that

$$\Theta_{p,\mathrm{ac}} \geq (T_{\mathrm{ac}} + b_1 u) \wedge \ldots \wedge (T_{\mathrm{ac}} + b_p u),$$

or inductively, that $\Theta_{p,\mathrm{ac}} \geq \Theta_{p-1,\mathrm{ac}} \wedge (T_{\mathrm{ac}} + b_p u)$. In order to do this, we simply have to make sure that $\lim_{k\to+\infty} T_{c,k,\mathrm{ac}} = T_{\mathrm{ac}}$ almost everywhere and use induction again. But our arguments are not affected if we replace $T_{c,k}$ by $T'_{c,k} = \alpha + dd^c\psi'_{c,k}$ where $\psi'_{c,k} = \max\{\psi, \psi_{c,k} - A_k\}$ and (A_k) is a sequence converging quickly to $+\infty$. Lemma 9.7 below shows that a suitable choice of A_k gives $\lim(dd^c\psi'_{c,k})_{\mathrm{ac}} = (dd^c\psi)_{\mathrm{ac}}$ almost everywhere. This concludes the proof in the case $\Xi = X$.

When $\Xi \neq X$, a slight difficulty appears: in fact, there may remain in $T_{c,k}$ some poles of codimension $\leq p-1$, corresponding to components of $E_c(T)$ which do not meet Ξ (since we completely forgot these components in the definition of the jumping values). It follows that the wedge product $\Theta_{p-1} \wedge (T'_{c,k} + c\,u + \varepsilon_k\omega)$ is no longer well defined. In this case, we proceed as follows. The assumption that T is nef implies that are smooth functions $\widetilde\psi_k$ such that the cohomology class $\{T\}$ has a representative $\alpha + dd^c\widetilde\psi_k \geq -\varepsilon_k\omega$. We replace $T'_{c,k}$ in the above arguments by $T'_{c,k,\nu} = \alpha + dd^c\psi'_{c,k,\nu}$ where

$$\psi'_{c,k,\nu} = \max\{\psi, \psi_{c,k} - A_k, \widetilde\psi_k - \nu\}.$$

Then certainly $T'_{c,k,\nu} + c\,u + \varepsilon_k\omega \geq 0$ and we can define a closed positive current $\Theta_{p-1} \wedge (T'_{c,k,\nu} + c\,u + \varepsilon_k\omega)$ without any difficulty since $\psi'_{c,k,\nu}$ is locally bounded. We first extract a weak limit $\Theta_{p,c,k}$ as $\nu \to +\infty$. By monotone continuity of Monge-Ampère operators, we find

$$\Theta_{p,c,k} = \Theta_{p-1} \wedge (T'_{c,k} + c\,u + \varepsilon_k\omega)$$

in the neighborhood of Ξ where this product is well defined. All other arguments are the same as before.

(9.7) Lemma. *Let $\Omega \subset \mathbb{C}^n$ be an open subset and let φ be an arbitrary psh function on Ω. Set $\varphi_\nu = \max(\varphi, \psi_\nu)$ where ψ_ν is a decreasing sequence of psh functions converging to $-\infty$, each ψ_ν being locally bounded in Ω (or perhaps only in the complement of an analytic subset of codimension $\geq p$). Let Θ be a closed positive current of bidegree $(p-1,p-1)$. If $\Theta \wedge dd^c\varphi_\nu$ converges to a weak limit Θ', then*

$$\Theta'_{\mathrm{ac}} \geq \Theta_{\mathrm{ac}} \wedge (dd^c\varphi)_{\mathrm{ac}}.$$

Proof. Let (ρ_ε) (resp. $(\widetilde\rho_\varepsilon)$) be a family of regularizing kernels on $\mathbb{C}^n$ (resp. on $\mathbb{R}^2$), and let $\max_\varepsilon(x,y) = (\max\star\widetilde\rho_\varepsilon)(x,y)$ be a regularized max function. For $\varepsilon > 0$ small enough, the function

$$\varphi_{\nu,\varepsilon} = \max\nolimits_\varepsilon(\varphi\star\rho_\varepsilon, \psi_\nu\star\rho_\varepsilon)$$

is psh and well defined on any preassigned open set $\Omega' \subset\subset \Omega$. As $\varphi_{\nu,\varepsilon}$ decreases to φ_ν when ε decreases to 0, proposition 10.2 shows that

$$\lim_{\varepsilon\to 0} \Theta \wedge dd^c\varphi_{\nu,\varepsilon} = \Theta \wedge dd^c\varphi_\nu$$

in the weak topology. Let (β_j) be a sequence of test forms which is dense in the space of test forms of bidegree $(n-p,n-p)$ and contains strongly positive forms with arbitrary large compact support in Ω. Select $\varepsilon_\nu > 0$ so small that

$$\langle \Theta \wedge dd^c\varphi_{\nu,\varepsilon_\nu} - \Theta \wedge dd^c\varphi_\nu\,,\, \beta_j\rangle \le \frac{1}{\nu} \quad \text{for} \;\; j \le \nu.$$

Then the sequence $\Theta \wedge dd^c\varphi_{\nu,\varepsilon_\nu}$ is locally uniformly bounded in mass and converges weakly to the same limit Θ' as $\Theta \wedge dd^c\varphi_\nu$. Moreover, at every point $x \in \Omega$ such that $\varphi(x) > -\infty$, we have $\varphi_{\nu,\varepsilon_\nu}(x) \ge \varphi(x) > \psi_\nu \star \rho_{\varepsilon_\nu}(x) + 1$ for ν large, because $\lim_{\nu\to-\infty} \psi_\nu = -\infty$ locally uniformly. Hence $\varphi_{\nu,\varepsilon_\nu} = \varphi \star \rho_{\varepsilon_\nu}$ on a neighborhood of x (which may depend on ν) and $dd^c\varphi_{\nu,\varepsilon_\nu}(x) = (dd^c\varphi) \star \rho_{\varepsilon_\nu}(x)$ for $\nu \ge \nu(x)$. By the Lebesgue density theorem, if μ is a measure of absolutely continuous part $\mu_{\rm ac}$, the sequence $\mu \star \rho_{\varepsilon_\nu}(x)$ converges to $\mu_{\rm ac}(x)$ at almost every point. Therefore $\lim dd^c\varphi_{\nu,\varepsilon_\nu}(x) = (dd^c\varphi)_{\rm ac}(x)$ almost everywhere For any strongly positive test form $\alpha = i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_{n-p} \wedge \overline{\alpha}_{n-p}$ of bidegree $(n-p, n-p)$ on Ω, we get

$$\begin{aligned}\int_\Omega \Theta' \wedge \alpha &= \lim_{\nu\to+\infty} \int_\Omega \Theta \wedge dd^c\varphi_{\nu,\varepsilon_\nu} \wedge \alpha \\ &\ge \liminf_{\nu\to+\infty} \int_\Omega \Theta_{\rm ac} \wedge dd^c\varphi_{\nu,\varepsilon_\nu} \wedge \alpha \ge \int_\Omega \Theta_{\rm ac} \wedge dd^c\varphi_{\rm ac} \wedge \alpha.\end{aligned}$$

Indeed, the first inequality holds because $dd^c\varphi_{\nu,\varepsilon_\nu}$ is smooth, and the last one results from Fatou's lemma. This implies $\Theta'_{\rm ac} \ge \Theta_{\rm ac} \wedge (dd^c\varphi)_{\rm ac}$ and Lemma 9.7 follows. □

10. Use of Monge-Ampère Equations

The goal of the next two sections is to find numerical criteria for an adjoint line bundle $K_X + L$ to be generated by sections (resp. very ample, s-jet ample). The conditions ensuring these conclusions should be ideally expressed in terms of explicit lower bounds for the intersection numbers $L^p \cdot Y$, where Y runs over p-dimensional subvarieties of X (as the form of Reider's theorem suggests in the case of surfaces). Unfortunately, the simple algebraic approach described in the proof of Theorems 8.4 and 8.5 does not seem to be applicable to get criteria for the very ampleness of $K_X + mL$ in the range $m \le n+1$. We will now explain how this can be achieved by an alternative analytic method. The essential idea is to construct directly the psh weight function φ needed in Nadel's vanishing theorem by solving a Monge-Ampère equation (Aubin-Calabi-Yau theorem). Quite recently, Ein and Lazarsfeld [EL94] have developed purely algebraic methods which yield similar results; however, up to now, the bounds obtained with the algebraic method are not as good as with the analytic approach.

Let us first recall a special case of the well-known theorem of Aubin-Yau related to the Calabi conjecture. The special case we need is the following fundamental existence result about solutions of Monge-Ampère equations.

(10.1) Theorem ([Yau78], see also [Aub78]). *Let X be a compact complex n-dimensional manifold with a smooth Kähler metric ω. Then for any smooth volume form $f > 0$ with $\int_X f = \int_X \omega^n$, there exists a unique Kähler metric $\widetilde{\omega} = \omega + dd^c\psi$ in the Kähler class $\{\omega\}$ such that $\widetilde{\omega}^n = (\omega + dd^c\psi)^n = f$.*

There are several equivalent ways of formulating this result. Usually, one starts with a $(1,1)$-form γ representing the first Chern class $c_1(X) = c_1(\Lambda^n T_X^\star)$. Then, under the normalization $\int_X f = \int_X \omega^n$, there is a unique volume form f on X which, viewed as a hermitian form on $\Lambda^n T_X$, yields $\frac{\mathrm{i}}{2\pi}\Theta_f(\Lambda^n T_X) = \gamma$. Then Theorem 10.1 is actually equivalent to finding a Kähler metric $\widetilde{\omega}$ in the Kähler class $\{\omega\}$, such that $\mathrm{Ricci}(\widetilde{\omega}) = \gamma$. We will not use this viewpoint here, and will be essentially concerned instead with the Monge-Ampère equation $(\omega + dd^c\psi)^n = f$.

There are two different ways in which the Monge-Ampère equation will be used. The most essential idea is that the Monge-Ampère equation can be used to produce weights with logarithmic singularities (as needed for the application of Corollary 5.13), when the right hand side f is taken to be a linear combination of Dirac measures (in fact, f has to be smooth so we rather find solutions ψ_ε corresponding to smooth approximations f_ε of the Dirac measures). This will be explained later.

Another useful consequence of the Monge-Ampère equation is a general version of convexity inequality due to Hovanski [Hov79] and Teissier [Tei79, 82], which is a natural generalization of the usual Hodge index theorem for surfaces. This inequality is reproved along similar lines in [BBS89], where it is applied to the study of projective n-folds of log-general type. For the sake of completeness, we include here a different and slightly simpler proof, based on the Aubin-Yau theorem 10.1 instead of the Hodge index theorem. Our proof also has the (relatively minor) advantage of working over arbitrary Kähler manifolds.

(10.2) Proposition. *The following inequalities hold in any dimension n.*

a) *If $\alpha_1, \ldots, \alpha_n$ are semipositive $(1,1)$-forms on $\mathbb{C}^n$, then*

$$\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n \geq (\alpha_1^n)^{1/n}(\alpha_2^n)^{1/n} \ldots (\alpha_n^n)^{1/n}.$$

b) *If $u_1, \ldots, u_n$ are nef cohomology classes of type $(1,1)$ on a Kähler manifold X of dimension n, then*

$$u_1 \cdot u_2 \cdots u_n \geq (u_1^n)^{1/n}(u_2^n)^{1/n} \ldots (u_n^n)^{1/n}.$$

By a nef cohomology class of type $(1,1)$, we mean a class in the closed convex cone of $H^{1,1}(X,\mathbb{R})$ generated by Kähler classes, that is, a class $\{u\}$ admitting representatives u_ε with $u_\varepsilon \geq -\varepsilon\omega$ for every $\varepsilon > 0$. For instance, inequality b) can be applied to $u_j = c_1(L_j)$ when $L_1, \ldots, L_n$ are nef line bundles over a projective manifold.

Proof. Observe that a) is a pointwise inequality between (n,n)-forms whereas b) is an inequality of a global nature for the cup product intersection form. We first show that a) holds when only two of the forms α_j are distinct, namely that

$$\alpha^p \wedge \beta^{n-p} \geq (\alpha^n)^{p/n}(\beta^n)^{(n-p)/n}$$

for all $\alpha, \beta \geq 0$. By a density argument, we may suppose $\alpha, \beta > 0$. Then there is a simultaneous orthogonal basis in which

$$\alpha = i \sum_{1\leq j\leq n} \lambda_j \, dz_j \wedge d\overline{z}_j, \qquad \beta = i \sum_{1\leq j\leq n} dz_j \wedge d\overline{z}_j$$

with $\lambda_j > 0$, and a) is equivalent to

$$p!(n-p)! \sum_{j_1<\ldots<j_p} \lambda_{j_1}\ldots\lambda_{j_p} \ge n!\,(\lambda_1\ldots\lambda_n)^{p/n}.$$

As both sides are homogeneous of degree p in (λ_j), we may assume $\lambda_1\ldots\lambda_n = 1$. Then our inequality follows from the inequality between the arithmetic and geometric means of the numbers $\lambda_{j_1}\ldots\lambda_{j_p}$. Next, we show that statements a) and b) are equivalent in any dimension n.

a) $\Longrightarrow$ b). By density, we may suppose that $u_1,\ldots,u_n$ are Kähler classes. Fix a positive (n,n) form f such that $\int_X f = 1$. Then Theorem 10.1 implies that there is a Kähler metric α_j representing u_j such that $\alpha_j^n = u_j^n f$. Inequality a) combined with an integration over X yields

$$u_1\cdots u_n = \int_X \alpha_1\wedge\ldots\wedge\alpha_n \ge (u_1^n)^{1/n}\ldots(u_n^n)^{1/n}\int_X f.$$

b) $\Longrightarrow$ a). The forms $\alpha_1,\ldots,\alpha_n$ can be considered as constant $(1,1)$-forms on any complex torus $X = \mathbb{C}^n/\Gamma$. Inequality b) applied to the associated cohomology classes $u_j \in H^{1,1}(X,\mathbb{R})$ is then equivalent to a).

Finally we prove a) by induction on n, assuming the result already proved in dimension $n-1$. We may suppose that α_n is positive definite, say $\alpha_n = i\sum dz_j\wedge d\overline{z}_j$ in a suitable basis. Denote by $u_1,\ldots,u_n$ the associated cohomology classes on the abelian variety $X = \mathbb{C}^n/\mathbb{Z}[i]^n$. Then u_n has integral periods, so some multiple of u_n is the first Chern class of a very ample line bundle $\mathcal{O}(D)$ where D is a smooth irreducible divisor in X. Without loss of generality, we may suppose $u_n = c_1(\mathcal{O}(D))$. Thus

$$u_1\cdots u_{n-1}\cdot u_n = u_{1\restriction D}\cdots u_{n-1\restriction D}$$

and by the induction hypothesis we get

$$u_1\cdots u_n \ge (u_{1\restriction D}^{n-1})^{1/(n-1)}\ldots(u_{n-1\restriction D}^{n-1})^{1/(n-1)}.$$

However $u_{j\restriction D}^{n-1} = u_j^{n-1}\cdot u_n \ge (u_j^n)^{(n-1)/n}(u_n^n)^{1/n}$, since a) and b) are equivalent and a) is already proved in the case of two forms. b) follows in dimension n, and therefore a) holds in $\mathbb{C}^n$. □

(10.3) Remark. In case α_j (resp. u_j) are positive definite, the equality holds in 10.2 (a,b) if and only if $\alpha_1,\ldots,\alpha_n$ (resp. $u_1,\ldots,u_n$) are proportional. In our inductive proof, the restriction morphism $H^{1,1}(X,\mathbb{R}) \longrightarrow H^{1,1}(D,\mathbb{R})$ is injective for $n\ge 3$ by the hard Lefschetz theorem, hence it is enough to consider the case of $\alpha^p\wedge\beta^{n-p}$. The equality between arithmetic and geometric means occurs only when all numbers $\lambda_{j_1}\ldots\lambda_{j_p}$ are equal, so all λ_j must be equal and $\alpha = \lambda_1\beta$, as desired. More generally, one can show (exercise to the reader!) that there is an inequality

$$\begin{aligned}&\alpha_1\wedge\ldots\wedge\alpha_p\wedge\beta_1\wedge\ldots\wedge\beta_{n-p} \ge\\ &\qquad\ge (\alpha_1^p\wedge\beta_1\wedge\ldots\wedge\beta_{n-p})^{1/p}\ldots(\alpha_p^p\wedge\beta_1\wedge\ldots\wedge\beta_{n-p})^{1/p}\end{aligned}$$

for all $(1,1)$-forms $\alpha_j,\beta_k\ge 0$, and a similar inequality for products of nef cohomology classes u_j, v_k. □

We now show how the Aubin-Calabi-Yau theorem can be applied to construct singular metrics on ample (or more generally big and nef) line bundles. We first suppose that L is an ample line bundle over a projective n-fold X and that L is equipped with a smooth metric of positive curvature. We consider the Kähler metric $\omega = \frac{i}{2\pi}\Theta(L)$. Any form $\widetilde{\omega}$ in the Kähler class of ω can be written as $\widetilde{\omega} = \omega + dd^c\psi$, i.e. is the curvature form of L after multiplication of the original metric by a smooth weight function $e^{-\psi}$. By lemma 5.1, the Monge-Ampère equation

$$(\omega + dd^c\psi)^n = f \tag{10.4}$$

can be solved for ψ, whenever f is a smooth (n,n)-form with $f > 0$ and $\int_X f = L^n$. In order to produce logarithmic poles at given points $x_1, \ldots, x_N \in X$, the main idea is to let f converge to a Dirac measure at x_j ; then $\widetilde{\omega}$ will be shown to converge to a closed positive $(1,1)$-current with non zero Lelong number at x_j.

Let $(z_1, \ldots, z_n)$ be local coordinates defined on some neighborhood V_j of x_j, and let

$$\alpha_{j,\varepsilon} = dd^c\big(\chi(\log|z_j - x_j|/\varepsilon)\big) \tag{10.5}$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth convex increasing function such that $\chi(t) = t$ for $t \geq 0$ and $\chi(t) = -1/2$ for $t \leq -1$. Then $\alpha_{j,\varepsilon}$ is a smooth positive $(1,1)$-form, and $\alpha_{j,\varepsilon} = dd^c \log|z_j - x_j|$ in the complement of the ball $|z_j - x_j| \leq \varepsilon$. It follows that $\alpha_{j,\varepsilon}^n$ has support in the ball $|z_j - x_j| \leq \varepsilon$, and Stokes' formula gives

$$\int_{B(x_j,\varepsilon)} \alpha_{j,\varepsilon}^n = \int_{B(x_j,\varepsilon)} \big(dd^c \log|z_j - x_j|\big)^n = 1. \tag{10.6}$$

Hence $\alpha_{j,\varepsilon}^n$ converges weakly to the Dirac measure δ_{x_j} as ε tends to 0. For all positive numbers $\tau_j > 0$ such that $\sigma := \sum \tau_j^n < L^n = \int_X \omega^n$, Theorem 10.1 gives a solution of the Monge-Ampère equation

$$\omega_\varepsilon^n = \sum_{1 \leq j \leq N} \tau_j^n\, \alpha_{j,\varepsilon}^n + \Big(1 - \frac{\sigma}{L^n}\Big)\omega^n \quad \text{with} \ \ \omega_\varepsilon = \omega + dd^c\psi_\varepsilon, \tag{10.7}$$

since the right-hand side of the first equation is > 0 and has the correct integral value L^n over X. The solution ψ_ε is merely determined up to a constant. If γ is an arbitrary Kähler metric on X, we can normalize ψ_ε in such a way that $\int_X \psi_\varepsilon\, \gamma^n = 0$.

(10.8) Lemma. *There is a sequence ε_ν converging to zero such that ψ_{ε_ν} has a limit ψ in $L^1(X)$ and such that the sequence of $(1,1)$-forms ω_{ε_ν} converges weakly towards a closed positive current T of type $(1,1)$. Moreover, the cohomology class of T is equal to $c_1(L)$ and $T = \omega + dd^c\psi$.*

Proof. The integral $\int_X \omega_\varepsilon \wedge \gamma^{n-1} = L \cdot \{\gamma\}^{n-1}$ remains bounded, so we can find a sequence ε_ν converging to zero such that the subsequence ω_{ε_ν} converges weakly towards a closed positive current T of bidegree $(1,1)$. The cohomology class of a current is continuous with respect to the weak topology (this can be seen by Poincaré duality). The cohomology class of T is thus equal to $c_1(L)$. The function ψ_ε satisfies the equation $\frac{1}{\pi}\Delta\psi_\varepsilon = \mathrm{tr}_\gamma(\omega_\varepsilon - \omega)$ where Δ is the Laplace operator associated to γ. Our normalization of ψ_ε implies

$$\psi_\varepsilon = \pi\, G \operatorname{tr}_\gamma(\omega_\varepsilon - \omega),$$

where G is the Green operator of Δ. As G is a compact operator from the Banach space of bounded Borel measures into $L^1(X)$, we infer that some subsequence (ψ_{ε_ν}) of our initial subsequence converges to a limit ψ in $L^1(X)$. By the weak continuity of dd^c, we get $T = \lim(\omega + dd^c\psi_{\varepsilon_\nu}) = \omega + dd^c\psi$. □

Let $\Omega \subset X$ be an open coordinate patch such that L is trivial on a neighborhood of $\overline{\Omega}$, and let e^{-h} be the weight representing the initial hermitian metric on $L_{\restriction\overline{\Omega}}$. Then $dd^c h = \omega$ and $dd^c(h + \psi_\varepsilon) = \omega_\varepsilon$, so the function $\varphi_\varepsilon = h + \psi_\varepsilon$ defines a psh weight on $L_{\restriction\Omega}$, as well as its limit $\varphi = h + \psi$. By the continuity of G, we also infer from the proof of Lemma 10.8 that the family (ψ_ε) is bounded in $L^1(X)$. The usual properties of subharmonic functions then show that there is a uniform constant C such that $\varphi_\varepsilon \le C$ on $\overline{\Omega}$. We use this and equation (10.7) to prove that the limit φ has logarithmic poles at all points $x_j \in \Omega$, thanks to Bedford and Taylor's maximum principle for solutions of Monge-Ampère equations [BT76]:

(10.9) Lemma. *Let u, v be smooth (or continuous) psh functions on $\overline{\Omega}$, where Ω is a bounded open set in $\mathbb{C}^n$. If*

$$u_{\restriction\partial\Omega} \ge v_{\restriction\partial\Omega} \quad \text{and} \quad (dd^c u)^n \le (dd^c v)^n \quad \text{on} \quad \Omega,$$

then $u \ge v$ on Ω. □

In the application of Lemma 10.9, we suppose that Ω is a neighborhood of x_j and take

$$u = \tau_j\big(\chi(\log|z_j - x_j|/\varepsilon) + \log\varepsilon\big) + C_1, \qquad v = \varphi_\varepsilon,$$

where C_1 is a large constant. Then for $\varepsilon > 0$ small enough

$$u_{\restriction\partial\Omega} = \tau_j \log|z_j - x_j| + C_1, \qquad v_{\restriction\partial\Omega} \le C,$$
$$(dd^c v)^n = \omega_\varepsilon^n \ge \tau_j^n\, \alpha_{j,\varepsilon}^n = (dd^c u)^n \quad \text{on} \quad \Omega.$$

For C_1 sufficiently large, we infer $u \ge v$ on Ω, hence

$$\varphi_\varepsilon \le \tau_j \log(|z_j - x_j| + \varepsilon) + C_2 \quad \text{on} \quad \Omega.$$

(10.10) Corollary. *The psh weight $\varphi = h + \psi$ on $L_{\restriction\Omega}$ associated to the limit function $\psi = \lim \psi_{\varepsilon_\nu}$ satisfies $dd^c\varphi = T$. Moreover, φ has logarithmic poles at all points $x_j \in \Omega$ and*

$$\varphi(z) \le \tau_j \log|z_j - x_j| + O(1) \quad \text{at} \quad x_j. \qquad \square$$

(10.11) Remark. The choice of the coefficients τ_j is made according to the order s_j of jets at x_j which sections in $H^0(X, K_X + L)$ should generate. Corollary 5.13 requires $\nu(\varphi, x_j) \ge n + s_j$, hence we need only take $\tau_j = n + s_j$. Accordingly, L must satisfy the numerical condition

$$L^n > \sigma = \sum (n + s_j)^n.$$

In particular, for the question of global generation, we need only take one point with $s_1 = 0$, thus $\sigma = n^n$, and for the question of very ampleness we need either two points with $s_1 = s_2 = 0$ or one point with $s_1 = 1$, thus $\sigma = \max(2n^n, (n+1)^n) = (n+1)^n$. In fact $\sigma = 2n^n$ is enough to get very ampleness: in the the case of two infinitely near points $x_1 = x_2 = 0$ in the direction $\frac{\partial}{\partial z_n}$ (say), we can replace (10.5) and (10.7) respectively by

$$(10.5') \qquad \alpha'_\varepsilon = dd^c\chi\Big(\frac{1}{2}\log\frac{|z_1|^2 + \ldots + |z_{n-1}|^2 + |z_n|^4}{\varepsilon^2}\Big),$$

$$(10.7') \qquad \omega_\varepsilon^n = n^n(\alpha'_\varepsilon)^n + \Big(1 - \frac{2n^n}{L^n}\Big)\omega^n \quad \text{with } \omega_\varepsilon = \omega + dd^c\psi_\varepsilon.$$

In this case, we have $\int_X (\alpha'_\varepsilon)^n = 2$. Arguments similar to those used in the proof of Corollary 10.10 show that

$$\varphi(z) \le n\log\big(|z_1| + \ldots + |z_{n-1}| + |z_n|^2\big) + O(1),$$

whence $\mathcal{I}(\varphi)_0 \subset (z_1, \ldots, z_{n_1}, z_n^2)$ by the result of Exercise 5.10. Corollary 5.12 can then be used to obtain separation of infinitesimally near points. □

Case of a big nef line bundle. All our arguments were developed under the assumption that L is ample, but if L is only nef and big, we can proceed in the following way. Let A be a fixed ample line bundle with smooth curvature form $\gamma = \frac{\mathrm{i}}{2\pi}\Theta(A) > 0$. As $mL + A$ is ample for any $m \ge 1$, by Theorem 5.1 there exists a smooth hermitian metric on L depending on m, such that $\omega_m = \frac{\mathrm{i}}{2\pi}\Theta(L)_m + \frac{1}{m}\frac{\mathrm{i}}{2\pi}\Theta(A) > 0$ and

$$(10.12) \qquad \omega_m^n = \frac{(L + \frac{1}{m}A)^n}{A^n}\,\gamma^n.$$

However, a priori we cannot control the asymptotic behaviour of ω_m when m tends to infinity, so we introduce the sequence of non necessarily positive $(1,1)$-forms $\omega'_m = \frac{\mathrm{i}}{2\pi}\Theta(L)_1 + \frac{1}{m}\frac{\mathrm{i}}{2\pi}\Theta(A) \in \{\omega_m\}$, which is uniformly bounded in $C^\infty(X)$ and converges to $\Theta(L)_1$. Then we solve the Monge-Ampère equation

$$(10.13) \qquad \omega_{m,\varepsilon}^n = \sum_{1\le j\le N} \tau_j^n\,\alpha_{j,\varepsilon}^n + \Big(1 - \frac{\sigma}{(L + \frac{1}{m}A)^n}\Big)\omega_m^n$$

with $\omega_{m,\varepsilon} = \omega'_m + dd^c\psi_{m,\varepsilon}$ and some smooth function $\psi_{m,\varepsilon}$ such that $\int_X \psi_{m,\varepsilon}\gamma^n = 0$; this is again possible by Yau's theorem 5.1. The numerical condition needed on σ to solve (10.13) is obviously satisfied for all m if we suppose

$$\sigma = \sum \tau_j^n < L^n < \Big(L + \frac{1}{m}A\Big)^n.$$

The same arguments as before show that there exist a convergent subsequence $\lim_{\nu\to+\infty} \psi_{m_1,\varepsilon_\nu} = \psi$ in $L^1(X)$ and a closed positive $(1,1)$-current $T = \lim \omega_{m_1,\varepsilon_\nu} = \frac{\mathrm{i}}{2\pi}\Theta(L)_1 + dd^c\psi \in c_1(L)$ such that Corollary 10.10 is still valid; in this case, h is taken to be the weight function corresponding to $\Theta(L)_1$. Everything thus works as in the ample case.

11. Numerical Criteria for Very Ample Line Bundles

In the last section, we explained a construction of psh weights admitting singularities at all points in a given finite set. Our next goal is to develope a technique for bounding the Lelong numbers at other points. An explicit numerical criterion can then be derived from Corollary 5.13.

First suppose that L is an ample line bundle over X. The idea is to apply the self-intersection inequality 9.5 to the $(1,1)$-current $T = \lim \omega_{\varepsilon_\nu}$ produced by equation (10.7), and to integrate the inequality with respect to the Kähler form $\omega = \frac{\mathrm{i}}{2\pi}\Theta(L)$. Before doing this, we need to estimate the excess of intersection in terms of T_{ac}^n.

(11.1) Proposition. *The absolutely continuous part T_{ac} of T satisfies*

$$T_{\mathrm{ac}}^n \geq \Big(1 - \frac{\sigma}{L^n}\Big)\omega^n \qquad \textit{a.e. on } X.$$

Proof. The result is local, so we can work in an open set Ω which is relatively compact in a coordinate patch of X. Let ρ_δ be a family of smoothing kernels. By a standard lemma based on the comparison between arithmetic and geometric means (see e.g. [BT76], Proposition 5.1), the function $A \mapsto (\det A)^{1/n}$ is concave on the cone of nonnegative hermitian $n \times n$ matrices. Thanks to this concavity property we get

$$\big[\big(\omega_\varepsilon \star \rho_\delta(x)\big)^n\big]^{1/n} \geq (\omega_\varepsilon^n)^{1/n} \star \rho_\delta(x) \geq \Big(1 - \frac{\sigma}{L^n}\Big)^{1/n} (\omega^n)^{1/n} \star \rho_\delta(x),$$

thanks to equation (6.5). As ε_ν tends to 0, $\omega_{\varepsilon_\nu} \star \rho_\delta$ converges to $T \star \rho_\delta$ in the strong topology of $C^\infty(\Omega)$, thus

$$\big((T \star \rho_\delta)^n\big)^{1/n} \geq \Big(1 - \frac{\sigma}{L^n}\Big)^{1/n} (\omega^n)^{1/n} \star \rho_\delta \qquad \text{on } \Omega.$$

Now, take the limit as δ goes to 0. By the Lebesgue density theorem $T \star \rho_\delta(x)$ converges almost everywhere to $T_{\mathrm{ac}}(x)$ on Ω, so we are done. □

According to the notation used in § 9, we consider an arbitrary subset $\Xi \subset X$ and introduce the jumping values

$$b_p = \inf\big\{c > 0\,;\, \mathrm{codim}\big(E_c(T), x\big) \geq p,\ \forall x \in \Xi\big\}.$$

By Proposition 11.1 and Inequality 10.2 a), we have

$$T_{\mathrm{ac}}^j \wedge \omega^{n-j} \geq \Big(1 - \frac{\sigma}{L^n}\Big)^{j/n} \omega^n. \tag{11.2}$$

Now, suppose that the “formal vector bundle” $T_X \otimes \mathcal{O}(aL)$ is nef, i.e. that the $\mathbb{R}$-divisor $\mathcal{O}_{T_X}(1) + a\,\pi^\star L$ is nef for some constant $a \geq 0$. We can then apply Theorem 9.5 with $u = a\,\omega$ and

$$\{\Theta_p\} = (1 + b_1 a) \cdots (1 + b_p a)\{\omega^p\}\,;$$

by taking the wedge product of Θ_p with ω^{n-p}, we get

$$(1+b_1a)\ldots(1+b_pa)\int_X\omega^n \ge \sum_{k\ge 1}(\nu_{p,k}-b_1)\ldots(\nu_{p,k}-b_p)\int_X[Z_{p,k}]\wedge\omega^{n-p} + \int_X(T_{\rm ac}+b_1a\,\omega)\wedge\ldots\wedge(T_{\rm ac}+b_pa\,\omega)\wedge\omega^{n-p}.$$

Combining this inequality with (11.2) for $T_{\rm ac}^{p-j}$ yields

$$(1+b_1a)\ldots(1+b_pa)\,L^n \ge \sum_{k\ge 1}(\nu_{p,k}-b_1)\ldots(\nu_{p,k}-b_p)\,L^{n-p}\cdot Z_{p,k} + \sum_{0\le j\le p} S_j^p(b)\,a^j\Big(1-\frac{\sigma}{L^n}\Big)^{(p-j)/n}L^n,$$

where $S_j^p(b)$, $1\le j\le p$, denotes the elementary symmetric function of degree j in $b_1,\ldots,b_p$ and $S_0^p(b)=1$. As $\prod(1+b_ja)=\sum S_j^p(b)a^j$, we get

$$\sum_{k\ge 1}(\nu_{p,k}-b_1)\ldots(\nu_{p,k}-b_p)\,L^{n-p}\cdot Z_{p,k} \le \sum_{0\le j\le p} S_j^p(b)\,a^j\Big(1-\Big(1-\frac{\sigma}{L^n}\Big)^{(p-j)/n}\Big)L^n. \tag{11.3}$$

If L is only supposed to be big and nef, we follow essentially the same arguments and replace ω in all our inequalities by $\omega_m=\frac{\mathrm{i}}{2\pi}(\Theta(L)_m+\frac{1}{m}\Theta(A))$ with A ample (see section 6). Note that all (n,n)-forms ω_m^n were defined to be proportional to $\gamma^n=(\frac{\mathrm{i}}{2\pi}\Theta(A))^n$, so inequality 11.1 becomes in the limit

$$T_{\rm ac}^n \ge \Big(1-\frac{\sigma}{L^n}\Big)\frac{L^n}{A^n}\gamma^n = \Big(1-\frac{\sigma}{L^n}\Big)\frac{L^n}{(L+\frac{1}{m}A)^n}\omega_m^n.$$

The intersection inequality (11.3) is the expected generalization of Proposition 8.2 in arbitrary codimension. In this inequality, $\nu_{p,k}$ is the generic Lelong number of T along $Z_{p,k}$, and $Z_{p,k}$ runs over all p-codimensional components Y of $\bigcup_{c>b_p}E_c(T)$ intersecting Ξ ; by definition of b_j we have $\max_k \nu_{p,k}=b_{p+1}$. Hence we obtain:

(11.4) Proposition. *Let L be a big nef line bundle such that $\mathcal{O}_{T_X}(1)+a\,\pi^\star L$ is nef, and let $T\in c_1(L)$ be the positive curvature current obtained by concentrating the Monge-Ampère mass L^n into a finite sum of Dirac measures with total mass σ, plus some smooth positive density spread over X (equation (10.7)). Then the jumping values b_p of the Lelong number of T over an arbitrary subset $\Xi\subset X$ satisfy the recursive inequalities*

$$(b_{p+1}-b_1)\ldots(b_{p+1}-b_p) \le \frac{1}{\min_Y L^{n-p}\cdot Y}\sum_{0\le j\le p-1} S_j^p(b)\,a^j\sigma_{p-j}\,, \tag{11.5}$$

where $\sigma_j=\big(1-(1-\sigma/L^n)^{j/n}\big)L^n$, and where Y runs over all p-codimensional subvarieties of X intersecting Ξ.

Observe that σ_j is increasing in j; in particular $\sigma_j < \sigma_n = \sigma$ for $j \le n-1$. Moreover, the convexity of the exponential function shows that

$$t \mapsto \frac{1}{t}\big(1-(1-\sigma/L^n)^t\big)L^n$$

is decreasing, thus $\sigma_j > \sigma_p j/p$ for $j < p$; in particular $\sigma_j > \sigma j/n$ for $j \le n-1$. We are now in a position to prove the following general result.

(11.6) Main Theorem. *Let X be a projective n-fold and let L be a big nef line bundle over X. Fix points $x_j \in X$ and multiplicities $s_j \in \mathbb{N}$. Set*

$$\sigma_0 = \sum_{1\le j\le N} (n+s_j)^n \qquad \textit{resp. } \sigma_0 = 2n^n \textit{ if } N=1 \textit{ and } s_1 = 1, \qquad \textit{and}$$

$$\sigma_p = \Big(1-(1-\sigma_0/L^n)^{p/n}\Big)L^n, \qquad 1 \le p \le n-1.$$

Suppose that the formal vector bundle $T_X \otimes \mathcal{O}(aL)$ is nef for some $a \ge 0$, that $L^n > \sigma_0$, and that there is a sequence $0 = \beta_1 < \ldots < \beta_n \le 1$ with

$$L^{n-p}\cdot Y > (\beta_{p+1}-\beta_1)^{-1}\ldots(\beta_{p+1}-\beta_p)^{-1} \sum_{0\le j\le p-1} S_j^p(\beta)\, a^j \sigma_{p-j} \tag{11.7}$$

for every subvariety $Y \subset X$ of codimension $p = 1, 2, \ldots, n-1$ passing through one of the points x_j. Then there is a surjective map

$$H^0(X, K_X+L) \longrightarrow\!\!\!\!\rightarrow \bigoplus_{1\le j\le N} \mathcal{O}(K_X+L)_{x_j} \otimes \big(\mathcal{O}_{X,x_j}/\mathfrak{m}_{X,x_j}^{s_j+1}\big),$$

i.e. $H^0(X, K_X+L)$ generates simultaneously all jets of order s_j at x_j.

As the notation is rather complicated, it is certainly worth examining the particular case of surfaces and 3-folds, for the problem of getting global generation (taking $\sigma_0 = n^n$), resp. very ampleness (taking $\sigma_0 = 2n^n$). If X is a surface, we find $\sigma_0 = 4$ (resp. $\sigma_0 = 8$), and we take $\beta_1 = 0$, $\beta_2 = 1$. This gives only two conditions, namely

$$L^2 > \sigma_0, \qquad L\cdot C > \sigma_1 \tag{11.7$^{n=2}$}$$

for every curve C intersecting Ξ. These bounds are not very far from those obtained with Reider's theorem, although they are not exactly as sharp. If X is a 3-fold, we have $\sigma_0 = 27$ (resp. $\sigma_0 = 64$), and we take $\beta_1 = 0 < \beta_2 = \beta < \beta_3 = 1$. Therefore our condition is that there exists $\beta \in\,]0,1[$ such that

$$L^3 > \sigma_0, \qquad L^2\cdot S > \beta^{-1}\sigma_1, \qquad L\cdot C > (1-\beta)^{-1}(\sigma_2 + \beta\, a\sigma_1) \tag{11.7$^{n=3}$}$$

for every curve C or surface S intersecting Ξ. If we take β to be of the order of magnitude of $a^{-2/3}$, these bounds show that the influence of a on the numerical conditions for L is at worst $a^{1/3}$ in terms of the homogeneous quantities $(L^p\cdot Y)^{1/p}$. In higher dimensions, a careful choice of the β_p's shows that the influence of a on these quantities is always less than $O(a^{1-\delta_n})$, $\delta_n > 0$.

Proof of Theorem 11.6. Select $\tau_j > n + s_j$ so that $L^{n-p} \cdot Y$ still satisfies the above lower bound with the corresponding value $\sigma = \sum \tau_j^n > \sigma_0$. Then apply Theorem 11.4 with $\Xi = \{x_1, \ldots, x_N\}$. Inequality (11.5) shows inductively that $b_p < \beta_p$ for $p \geq 2$, so $b_n < 1$ and therefore x_j is an isolated point in $E_1(T)$. On the other hand, the Monge-Ampère equation gives us a weight function φ admitting a logarithmic pole at each point x_j, in such a way that $\nu(\varphi, x_j) \geq \tau_j > n + s_j$ (Corollary 10.10). However $T = dd^c\varphi$ need not be positive definite. In order to apply our vanishing theorems, we still have to make the curvature positive definite everywhere. Since L is nef and big, it has a singular metric for which the curvature current $T_0 = dd^c\varphi_0$ satisfies $\nu(T_0, x) < 1$ everywhere and $T_0 \geq \varepsilon\gamma$ for some Kähler metric (Corollary 6.8). Then $T' = (1-\delta)T + \delta T_0$ still has Lelong numbers $\nu(T', x_j) > n + s_j$ for $\delta > 0$ small, and the inequality $\nu(T', x) \leq (1-\delta)\nu(T, x) + \delta$ implies that we have $\nu(T', x) < 1$ for x near x_j, $x \neq x_j$. Corollaries 5.12 and 5.13 imply the Theorem. When $N = 1$ and $s_1 = 1$, we can use Remark 10.11 to show that the value $\sigma_0 = 2n^n$ is admissible in place of $(n+1)^n$. □

(11.8) Corollary. *Let X be a smooth algebraic surface, and let L be a big nef line bundle over X. Then on a given subset $\Xi \subset X$*

$K_X + L$		*is spanned*	*separates points*			*generates s-jets*
when	$L^2 >$	4	8	9	12	$(2+s)^2$
	$\forall C, \quad L \cdot C >$	2	6	5	4	$2 + 3s + s^2$

for all curves $C \subset X$ intersecting Ξ. In particular, if L is ample, $K_X + mL$ is always globally spanned for $m \geq 3$ and very ample for $m \geq 5$.

Proof. For s-jets, we have $\sigma_0 = (2+s)^2$, so we find the condition

$$L^2 > (2+s)^2, \qquad L \cdot C > \big(1 - (1 - (2+s)^2/L^2)^{1/2}\big)L^2.$$

The last constant decreases with L^2 and is thus at most equal to the value obtained when $L^2 = (2+s)^2 + 1$; the integral part is then precisely $2 + 3s + s^2$. □

(11.9) Remark. Ein and Lazarsfeld [EL94] have recently obtained an algebraic proof of the Main Theorem, in which they can even weaken the condition that $T_X \otimes \mathcal{O}(aL)$ is nef into the condition that $-K_X + aL$ is nef (by taking determinants, $T_X \otimes \mathcal{O}(aL)$ nef $\Rightarrow$ $-K_X + naL$ nef). However, it seems that the lower bounds they get for $L^{n-p} \cdot Y$ are substantially larger than the ones given by the analytic method. Also, the assumption on T_X is sufficient to get universal bounds for $2K_X + L$, as we will see later. □

For the applications, we introduce a convenient definition of higher jet generation, following an idea of [BSo93].

(11.10) Definition. *We say that L generates s-jets on a given subset $\Xi \subset X$ if $H^0(X, L) \longrightarrow\!\!\!\!\rightarrow \bigoplus J^{s_j}_{x_j} L$ is onto for any choice of points $x_1, \ldots, x_N \in \Xi$ and integers*

$s_1, \ldots, s_N$ with $\sum(s_j+1) = s+1$. We say that L is s-jet ample if the above property holds for $\Xi = X$.

With this terminology, L is 0-jet ample if and only if L is generated by global sections and 1-jet ample if and only if L is very ample. In order that K_X+L generates s-jets on Ξ, the constant to be used in Theorem 11.6 is $\sigma_0 = \max\sum(n+s_j)^n$ over all decompositions $s+1 = \sum(s_j+1)$. In fact, if we set $t_j = s_j+1$, the following lemma gives $\sigma_0 = (n+s)^n$, that is, the maximum is reached when only one point occurs.

(11.11) Lemma. *Let $t_1, \ldots, t_N \in [1, +\infty[$. Then*

$$\sum_{1\le j\le N} (n-1+t_j)^n \le \Big(n-1+\sum_{1\le j\le N} t_j\Big)^n.$$

Proof. The right hand side is a polynomial with nonnegative coefficients and the coefficient of a monomial t_j^k involving exactly one variable is the same as in the left hand side (however, the constant term is smaller). Thus the difference is increasing in all variables and we need only consider the case $t_1 = \ldots = t_N = 1$. This case follows from the obvious inequality

$$n^n N = n^n + \binom{n}{1} n^{n-1}(N-1) \le (n+N-1)^n. \qquad \square$$

(11.12) Corollary. *If all intersection numbers $L^p \cdot Y$ satisfy the inequalities in Theorem 11.6 with $\sigma_0 = (n+s)^n$, then $K_X + L$ is s-jet ample.*

In order to find universal conditions for $K_X + L$ to be very ample, our main theorem would require a universal value a depending only on $n = \dim_{\mathbb{C}} X$ such that $T_X \otimes \mathcal{O}(aL)$ is always nef. However, this is clearly impossible as the example of curves already shows: if X is a curve of genus g and L has degree 1, then $T_X \otimes \mathcal{O}(aL)$ is nef if and only if $a \ge 2g-2$. As we will see in § 13, an explicit value a depending only on the intersection numbers L^n and $L^{n-1} \cdot K_X$ exists, but this value is very large. Here, these difficulties can be avoided by means of the following simple lemma.

(11.13) Lemma. *Let F be a very ample line bundle over X. Then the vector bundle $T_X \otimes \mathcal{O}(K_X + nF)$ is nef and generated by global sections.*

Proof. By the assumption that F is very ample, the 1-jet bundle J^1F is generated by its sections. Consider the exact sequence

$$0 \longrightarrow T_X^\star \otimes F \longrightarrow J^1F \longrightarrow F \longrightarrow 0$$

where $\mathrm{rank}(J^1F) = n+1$ and $\det(J^1F) = K_X + (n+1)F$. The n-th exterior power $\Lambda^n(J^1F) = (J^1F)^\star \otimes \det(J^1F)$ is also generated by sections. Hence, by dualizing the sequence, we get a surjective morphism

$$\Lambda^n(J^1F) \longrightarrow (T_X^\star \otimes F)^\star \otimes \det(J^1F) = T_X \otimes \mathcal{O}(K_X + nF).$$

Therefore $T_X \otimes \mathcal{O}(K_X + nF)$ is generated by sections and, in particular, it is nef. □

Before going further, it is convenient to introduce the following quantitative measurement of ampleness:

(11.14) Definition. *Let F be a nef line bundle and let $\Xi \subset X$ be an arbitrary subset. We define*

$$\mu(F, \Xi) = \min_{1 \le p \le n} \min_{\dim Y = p,\, Y \cap \Xi \neq \emptyset} (F^p \cdot Y)^{1/p}$$

where Y runs over all p-dimensional subvarieties intersecting Ξ. The main properties of this invariant are:

- *Linearity with respect to F :* $\forall k \ge 0, \quad \mu(kF, \Xi) = k\, \mu(F, \Xi)$;
- *Nakai-Moishezon criterion*: *F is ample if and only if $\mu(F, X) > 0$.*

The next idea consists in the following iteration trick: Lemma 11.13 suggests that a universal lower bound for the nefness of $T_X \otimes \mathcal{O}(aL')$ can be achieved with $L' = K_X + L$ if L is sufficiently ample. Then it follows from the Main Theorem 11.6 that $K_X + L' = 2K_X + L$ is very ample under suitable numerical conditions. Lemma 11.13 applied with $F = 2K_X + L$ shows that the bundle $T_X \otimes \mathcal{O}\big((2n+1)K_X + nL\big)$ is nef, and thus $T_X \otimes \mathcal{O}\big((2n+1)L''\big)$ is nef with $L'' = K_X + \frac{1}{2}L \le L'$. Hence we see that the Main Theorem can be iterated. The special value $a = 2n + 1$ will play an important role.

(11.15) Lemma. *Let L' be an ample line bundle over X. Suppose that the vector bundle $T_X \otimes \mathcal{O}\big((2n+1)L'\big)$ is nef. Then, for any $s \ge 1$, $K_X + L'$ is s-jet ample as soon as $\mu(L', X) \ge 3(n+s)^n$. (By Remark 10.11, this is still true with $3(n+s)^n$ replaced by $6n^n$ if $s = 1$.)*

Proof. We apply the Main Theorem 11.6 with $\sigma_0 = (n+s)^n$ and $a = 2n+1$. Thanks to the inequality $\sigma_j \le \sigma_0$ and to the identity

$$\sum_{0 \le j \le p-1} S_j^p(\beta) = (1 + \beta_1 a) \ldots (1 + \beta_p a),$$

(recall that $\beta_1 = 0$), we find lower bounds $L^{n-p} \cdot Y > M_p$ with

$$M_p \le \frac{(1 + \beta_1 a) \ldots (1 + \beta_p a)\, \sigma_0}{(\beta_{p+1} - \beta_1) \ldots (\beta_{p+1} - \beta_p)}.$$

If we choose an increasing sequence $0 = \beta_1 < \ldots < \beta_n = 1$ such that β_p/β_{p+1} is increasing, we get $\beta_j/\beta_{p+1} < \beta_{j+n-p-1}/\beta_n = \beta_{j+n-p-1}$, hence

$$\prod_{1 \le j \le p} (\beta_{p+1} - \beta_j) = \beta_{p+1}^p \prod_{1 \le j \le p} \Big(1 - \frac{\beta_j}{\beta_{p+1}}\Big) \ge \beta_{p+1}^p \prod_{1 \le j \le p} (1 - \beta_{j+n-p-1}),$$

$$M_p \le \prod_{1 \le j \le n-1} \frac{(1 + \beta_j a)}{(1 - \beta_j)} \beta_{p+1}^{-p} \sigma_0.$$

A suitable choice is $\beta_p = n^{-n(n-p)/(p-1)}$. We then find

$$M_p \le C_n n^{n(n-p-1)}(n+s)^n \le C_n(n+s)^{n(n-p)}$$

where

$$C_n = \prod_{1\le p\le n-1} \frac{\big(1+(2n+1)n^{-n(n-p)/(p-1)}\big)}{\big(1-n^{-n(n-p)/(p-1)}\big)}.$$

Numerical calculations left to the reader show that that $C_n < 3$ for all $n \ge 2$ and that $C_n = 3 - 4\log n/n + O(1/n)$ as $n \to +\infty$. Lemma 11.15 follows. □

(11.16) Lemma. *Let F be a line bundle which generates s-jets at every point. Then $F^p \cdot Y \ge s^p$ for every p-dimensional subvariety $Y \subset X$.*

Proof. Fix an arbitrary point $x \in Y$. Then consider the singular metric on F given by

$$||\xi||^2 = \frac{|\xi|^2}{\sum |u_j(z)|^2},$$

where $(u_1, \ldots, u_N)$ is a basis of $H^0(X, F \otimes \mathfrak{m}_x^s)$. By our assumption, these sections have an isolated common zero of order s at x. Hence F possesses a singular metric such that the weight $\varphi = \frac{1}{2}\log\sum|u_j|^2$ is psh and has an isolated logarithmic pole of Lelong number s at x. By the comparison inequality (3.6) with $\psi(z) = \log|z-x|$, we get

$$F^p \cdot Y \ge \int_{B(x,\varepsilon)} [Y] \wedge (dd^c\varphi)^p \ge s^p\nu([Y],\psi) = s^p\nu(Y,x) \ge s^p. \qquad \square$$

(11.17) Theorem. *Let $s \ge 1$ and $m \ge 2$ be arbitrary integers and let L be an ample line bundle. If L satisfies the numerical condition*

$$(m-1)\,\mu(L,X) + s \ge 6(n+s)^n,$$

then $2K_X + mL$ is s-jet ample. If $s=1$, the result still holds with $6(n+s)^n$ replaced by $12n^n$, in particular $2K_X + 12n^nL$ is always very ample.

Proof. We denote here simply $\mu(L,X) = \mu(L)$, for $\Xi = X$ everywhere in the proof. As L is ample, there exists an integer q (possibly very large) such that

$$(11.18) \qquad \begin{cases} K_X + qL & \text{is ample,}\\ T_X \otimes \mathcal{O}\big((2n+1)(K_X+qL)\big) & \text{is nef,}\\ \mu(K_X+qL) \ge 3(n+s)^n. & \end{cases}$$

By lemma 11.15 applied to $L' = K_X + qL$, we find that $F = K_X + L' = 2K_X + qL$ is very ample and generates s-jets. In particular $K_X + \frac{q}{2}L$ is an ample $\mathbb{Q}$-divisor, and for any p-dimensional subvariety $Y \subset X$ we have

$$\begin{aligned}\big(K_X + (q-1)L\big)^p \cdot Y &= \Big(\frac{1}{2}F + (q/2-1)L\Big)^p \cdot Y\\ &= \sum_{0\le k\le p}\binom{p}{k} 2^{k-p}(q/2-1)^k F^{p-k}\cdot L^k \cdot Y.\end{aligned}$$

By the convexity inequality 10.2 b) and Lemma 11.16 we get

$$F^{p-k} \cdot L^k \cdot Y \geq (F^p \cdot Y)^{1-k/p} (L^p \cdot Y)^{k/p} \geq s^{p-k}\big(\mu(L)\big)^k.$$

Hence $\big(K_X + (q-1)L\big)^p \cdot Y \geq \big((q/2 - 1)\,\mu(L) + s/2\big)^p$ and

$$\mu\big(K_X + (q-1)L, X\big) \geq \frac{1}{2}\big((q-2)\,\mu(L) + s\big).$$

Moreover, Lemma 11.13 applied to F shows that

$$T_X \otimes \mathcal{O}(K_X + nF) = T_X \otimes \mathcal{O}\big((2n+1)K_X + nqL\big)$$

is nef. As $nq/(2n+1) \leq q/2 \leq q-1$ for $q \geq 2$, we find that all properties (11.18) except perhaps the last one remain valid with $q-1$ in place of q:

$$(11.19) \qquad \begin{cases} K_X + (q-1)L & \text{is ample,} \\ T_X \otimes \mathcal{O}\big((2n+1)(K_X + (q-1)L)\big) & \text{is nef,} \\ \mu\big(K_X + (q-1)L, X\big) \geq \frac{1}{2}\big((q-2)\,\mu(L) + s\big). \end{cases}$$

By induction we conclude that (11.19) is still true for the smallest integer $q-1 = m$ such that

$$\frac{1}{2}\big((q-2)\,\mu(L) + s\big) = \frac{1}{2}\big((m-1)\,\mu(L) + s\big) \geq 3(n+s)^n.$$

For this value of m, Lemma 11.15 implies that $2K_X + mL$ generates s-jets. □

(11.20) Remark. The condition $(m-1)\,\mu(L,X) + s \geq 6(n+s)^n$ is never satisfied for $m = 1$. However, Lemma 8.6 applied with $F = K_X + L$ and $\mu = 2$ allows us to obtain also a sufficient condition in order that $2K_X + L$ generates s-jets. It is sufficient that $2(K_X + L)$ generates jets of order $s'_j = 2(n + s_j) + 1$ at any of the points x_j whenever $\sum(s_j + 1) = s + 1$. For $n \geq 2$ we get

$$\sum (n + s'_j)^n = \sum (2s_j + 3n + 1)^n \leq (3n + 3 + 2s)^n$$

after a short computation. The proof of Theorem 11.17 then yields the sufficient condition $\mu(L,X) \geq 6(3n + 3 + 2s)^n$. □

(11.21) Remark. If G is a nef line bundle, the Main Theorem 11.6 is still valid for the line bundle $K_X + L + G$, with the same lower bounds in the numerical conditions for L; indeed, the proof rests on the existence of suitable singular hermitian metrics with positive definite curvature on L, and adding G preserves all properties of these metrics. It follows that Theorem 11.17 and Remark 11.20 can be applied as well to the line bundle $2K_X + mL + G$, under the same numerical conditions. □

12. Holomorphic Morse Inequalities

Let X be a compact Kähler manifold, E a holomorphic vector bundle of rank r and L a line bundle over X. If L is equipped with a smooth metric of curvature form $\Theta(L)$, we define the q-index set of L to be the open subset

$$X(q,L)=\left\{x\in X\ ;\ \mathrm{i}\Theta(L)_x \text{ has } \begin{matrix} q & \text{negative eigenvalues} \\ n-q & \text{positive eigenvalues}\end{matrix}\right\} \tag{12.1}$$

for $0\le q\le n$. Hence X admits a partition $X=\Delta\amalg\coprod_q X(q,L)$ where $\Delta=\{x\in X\,;\,\det(\Theta(L)_x)=0\}$ is the degeneracy set. We also introduce

$$X(\le q,L)=\bigcup_{0\le j\le q} X(j,L). \tag{12.1$'$}$$

It is shown in [Dem85b] that the cohomology groups $H^q\big(X,E\otimes\mathcal{O}(kL)\big)$ satisfy the following asymptotic *weak Morse inequalities* as $k\to+\infty$

$$h^q\big(X,E\otimes\mathcal{O}(kL)\big)\le r\frac{k^n}{n!}\int_{X(q,L)}(-1)^q\Big(\frac{\mathrm{i}}{2\pi}\Theta(L)\Big)^n+o(k^n). \tag{12.2}$$

A sharper form is given by the *strong Morse inequalities*

$$\begin{aligned}\sum_{0\le j\le q}&(-1)^{q-j}h^j\big(X,E\otimes\mathcal{O}(kL)\big)\\ &\le r\frac{k^n}{n!}\int_{X(\le q,L)}(-1)^q\Big(\frac{\mathrm{i}}{2\pi}\Theta(L)\Big)^n+o(k^n).\end{aligned} \tag{12.2$'$}$$

These inequalities are a useful complement to the Riemann-Roch formula when information is needed about individual cohomology groups, and not just about the Euler-Poincaré characteristic.

One difficulty in the application of these inequalities is that the curvature integral is in general quite uneasy to compute, since it is neither a topological nor an algebraic invariant. However, the Morse inequalities can be reformulated in a more algebraic setting in which only algebraic invariants are involved. We give here two such reformulations.

(12.3) Theorem. *Let $L=F-G$ be a holomorphic line bundle over a compact Kähler manifold X, where F and G are numerically effective line bundles. Then for every $q=0,1,\ldots,n=\dim X$, there is an asymptotic strong Morse inequality*

$$\sum_{0\le j\le q}(-1)^{q-j}h^j(X,kL)\le\frac{k^n}{n!}\sum_{0\le j\le q}(-1)^{q-j}\binom{n}{j}F^{n-j}\cdot G^j+o(k^n).$$

Proof. By adding ε times a Kähler metric ω to the curvature forms of F and G, $\varepsilon>0$ one can write $\frac{\mathrm{i}}{2\pi}\Theta(L)=\theta_\varepsilon(F)-\theta_\varepsilon(G)$ where $\theta_\varepsilon(F)=\frac{\mathrm{i}}{2\pi}\Theta(F)+\varepsilon\omega$ and $\theta_\varepsilon(G)=\frac{\mathrm{i}}{2\pi}\Theta(G)+\varepsilon\omega$ are positive definite. Let $\lambda_1\ge\ldots\ge\lambda_n>0$ be the eigenvalues of $\theta_\varepsilon(G)$ with respect to $\theta_\varepsilon(F)$. Then the eigenvalues of $\frac{\mathrm{i}}{2\pi}\Theta(L)$ with respect to $\theta_\varepsilon(F)$

are the real numbers $1-\lambda_j$ and the set $X(\le q, L)$ is the set $\{\lambda_{q+1}<1\}$ of points $x\in X$ such that $\lambda_{q+1}(x)<1$. The strong Morse inequalities yield

$$\sum_{0\le j\le q}(-1)^{q-j}h^j(X,kL)\le\frac{k^n}{n!}\int_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\le j\le n}(1-\lambda_j)\theta_\varepsilon(F)^n+o(k^n).$$

On the other hand we have

$$\binom{n}{j}\theta_\varepsilon(F)^{n-j}\wedge\theta_\varepsilon(G)^j=\sigma_n^j(\lambda)\,\theta_\varepsilon(F)^n,$$

where $\sigma_n^j(\lambda)$ is the j-th elementary symmetric function in $\lambda_1,\ldots,\lambda_n$, hence

$$\sum_{0\le j\le q}(-1)^{q-j}\binom{n}{j}F^{n-j}\cdot G^j=\lim_{\varepsilon\to 0}\int_X\sum_{0\le j\le q}(-1)^{q-j}\sigma_n^j(\lambda)\,\theta_\varepsilon(F)^n.$$

Thus, to prove the Lemma, we only have to check that

$$\sum_{0\le j\le q}(-1)^{q-j}\sigma_n^j(\lambda)-\mathbf{1}_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\le j\le n}(1-\lambda_j)\ge 0$$

for all $\lambda_1\ge\ldots\ge\lambda_n\ge 0$, where $\mathbf{1}_{\{\ldots\}}$ denotes the characteristic function of a set. This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda)=\sigma_{n-1}^j(\lambda)+\sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). □

In the case $q=1$, we get an especially interesting lower bound (this bound has been observed and used by S. Trapani [Tra95] in a similar context).

(12.4) Consequence. $h^0(X,kL)-h^1(X,kL)\ge\frac{k^n}{n!}(F^n-nF^{n-1}\cdot G)-o(k^n)$. *Therefore some multiple kL has a section as soon as $F^n-nF^{n-1}\cdot G>0$.*

(12.5) Remark. The weaker inequality

$$h^0(X,kL)\ge\frac{k^n}{n!}(F^n-nF^{n-1}\cdot G)-o(k^n)$$

is easy to prove if X is projective algebraic. Indeed, by adding a small ample $\mathbb{Q}$-divisor to F and G, we may assume that F, G are ample. Let m_0G be very ample and let k' be the smallest integer $\ge k/m_0$. Then $h^0(X,kL)\ge h^0(X,kF-k'm_0G)$. We select k' smooth members G_j, $1\le j\le k'$ in the linear system $|m_0G|$ and use the exact sequence

$$0\to H^0(X,kF-\sum G_j)\to H^0(X,kF)\to\bigoplus H^0(G_j,kF_{|G_j}).$$

Kodaira's vanishing theorem yields $H^q(X,kF)=0$ and $H^q(G_j,kF_{|G_j})=0$ for $q\ge 1$ and $k\ge k_0$. By the exact sequence combined with Riemann-Roch, we get

$$\begin{aligned} h^0(X,kL) &\ge h^0(X,kF-\sum G_j)\\ &\ge \frac{k^n}{n!}F^n - O(k^{n-1}) - \sum\Big(\frac{k^{n-1}}{(n-1)!}F^{n-1}\cdot G_j - O(k^{n-2})\Big)\\ &\ge \frac{k^n}{n!}\Big(F^n - n\frac{k'm_0}{k}F^{n-1}\cdot G\Big) - O(k^{n-1})\\ &\ge \frac{k^n}{n!}\Big(F^n - n\,F^{n-1}\cdot G\Big) - O(k^{n-1}). \end{aligned}$$

(This simple proof is due to F. Catanese.) □

(12.6) Corollary. *Suppose that F and G are nef and that F is big. Some multiple of $mF-G$ has a section as soon as*

$$m > n\,\frac{F^{n-1}\cdot G}{F^n}.$$

In the last condition, the factor n is sharp: this is easily seen by taking $X=\mathbb{P}_1^n$ and $F=\mathcal{O}(a,\ldots,a)$ and $G=\mathcal{O}(b_1,\ldots,b_n)$ over $\mathbb{P}_1^n$; the condition of the Corollary is then $m>\sum b_j/a$, whereas $k(mF-G)$ has a section if and only if $m\ge\sup b_j/a$; this shows that we cannot replace n by $n(1-\varepsilon)$. □

We now discuss another application of Morse inequalities in the case where $c_1(L)\in N_{\rm psef}$. Then the regularization theorem 9.1 allows us to measure the distance of L to the nef cone $N_{\rm nef}$. In that case, a use of singular metrics combined with 9.1 produces smooth metrics on L for which an explicit bound of the negative part of the curvature is known. It follows that (12.2) gives an explicit upper bound of the cohomology groups of $E\otimes\mathcal{O}(kL)$ in terms of a polynomial in the first Chern class $c_1(L)$ (related techniques have already been used in [Sug87] in a slightly different context).

(12.7) Theorem. *Suppose that there is a nef cohomology class $\{u\}$ in $H^{1,1}(X)$ such that $c_1\big(\mathcal{O}_{T_X}(1)\big)+\pi^\star\{u\}$ is nef over the hyperplane bundle $P(T_X^\star)$. Suppose moreover that L is equipped with a singular metric such that $T=\frac{\mathrm{i}}{2\pi}\Theta(L)\ge 0$. For $p=1,2,\ldots,n,n+1$ set*

$$b_p=\inf\{c>0\,;\,\operatorname{codim}E_c(T)\ge p\},$$

with $b_{n+1}=\max_{x\in X}\nu(T,x)$. Then for any holomorphic vector bundle E of rank r over X we have

$$h^q\big(X,E\otimes\mathcal{O}(kL)\big)\le A_q r\,k^n+o(k^n)$$

where A_q is the cup product

$$A_q=\frac{1}{q!\,(n-q)!}\big(b_{n-q+1}\{u\}\big)^q\cdot\big(c_1(L)+b_{n-q+1}\{u\}\big)^{n-q}$$

in $H^{2n}(X,\mathbb{R})$, identified with a positive number.

(12.8) Remark. When X is projective algebraic and $\kappa(L)=n$, the proof of 6.6 f) shows that $mL\simeq\mathcal{O}(A+D)$ with A ample and D effective, for some $m\ge 1$.

Then we can choose a singular metric on L such that $T = \frac{\mathrm{i}}{2\pi}\Theta(L) = \omega + m^{-1}[D]$, where $\omega = m^{-1}\frac{\mathrm{i}}{2\pi}\Theta(A)$ is a Kähler metric. As $\nu(T,x) = m^{-1}\nu(D,x)$ at each point, the constants b_j of theorem 12.7 are obtained by counting the multiplicities of the singular points of D; for example, if D only has isolated singularities, then $b_1 = 0$, $b_2 = \ldots = b_n = 1/m$. Observe moreover that the nefness assumption on $\mathcal{O}_{T_X}(1)$ is satisfied with $\{u\} = c_1(G)$ if G is a nef $\mathbb{Q}$-divisor such that $\mathcal{O}(T_X) \otimes \mathcal{O}(G)$ is nef, e.g. if $\mathcal{O}(S^m T_X) \otimes \mathcal{O}(mG)$ is spanned by sections for some $m \geq 1$. □

Proof of theorem 12.7. By definition, we have $0 = b_1 \leq b_2 \leq \ldots \leq b_n \leq b_{n+1}$, and for $c \in \,]b_p, b_{p+1}]$, $E_c(T)$ has codimension $\geq p$ with some component(s) of codimension p exactly. Let ω be a fixed Kähler metric on X. By adding $\varepsilon\omega$ to u if necessary, we may assume that $u \geq 0$ and that $\mathcal{O}_{T_X}(1)$ has a smooth hermitian metric such that $c\big(\mathcal{O}_{T_X}(1)\big) + \pi^\star u \geq 0$.

Under this assumption, the approximation theorem 9.1 shows that the metric of L can be approximated by a sequence of smooth metrics such that the associated curvature forms T_j satisfy the uniform lower bound

$$T_j \geq -\lambda_j(x)\, u(x) - \varepsilon_j\, \omega(x) \tag{12.9}$$

where $\lim\!\downarrow_{j\to+\infty} \varepsilon_j = 0$ and $(\lambda_j)_{j>0}$ is a decreasing sequence of continuous functions on X such that $\lim_{j\to+\infty} \lambda_j(x) = \nu(T,x)$ at each point.

The estimate (12.2) cannot be used directly with $T = \frac{\mathrm{i}}{2\pi}\Theta(L)$ because wedge products of currents do not make sense in general. Therefore, we replace $\frac{\mathrm{i}}{2\pi}\Theta(L)$ by its approximations T_j and try to find an upper bound for the limit.

(12.10) Lemma. *Let $U_j = X(q, T_j)$ be the q-index set associated to T_j and let c be a positive number. On the open set $\Omega_{c,j} = \{x \in X\,;\, \lambda_j(x) < c\}$ we have*

$$(-1)^q \mathbf{1}_{U_j} T_j^n \leq \frac{n!}{q!\,(n-q)!}\big(c\,u + \varepsilon_j\,\omega\big)^q \wedge \big(T_j + c\,u + \varepsilon_j\,\omega\big)^{n-q}.$$

Proof. Write $v = c\,u + \varepsilon_j\,\omega > 0$ and let $\alpha_{1,j} \leq \ldots \leq \alpha_{n,j}$ be the eigenvalues of T_j with respect to v at each point. Then $T_j^n = \alpha_{1,j} \ldots \alpha_{n,j}\, v^n$ and

$$v^q \wedge (T_j + v)^{n-q} = \frac{q!\,(n-q)!}{n!} \sum_{1 \leq i_1 < \ldots < i_{n-q} \leq n} (1 + \alpha_{i_1,j}) \ldots (1 + \alpha_{i_{n-q},j})\, v^n.$$

On $\Omega_{c,j}$ we get $T_j \geq -v$ by inequality (12.9), thus $\alpha_{i,j} \geq -1$; moreover, we have $\alpha_1 \leq \ldots \leq \alpha_q < 0$ and $0 < \alpha_{q+1} \leq \ldots \leq \alpha_n$ on U_j. On $\Omega_{c,j}$ we thus find

$$0 \leq (-1)^q \mathbf{1}_{U_j} \alpha_{1,j} \ldots \alpha_{n,j} \leq \mathbf{1}_{U_j} \alpha_{q+1,j} \ldots \alpha_{n,j} \leq (1 + \alpha_{q+1,j}) \ldots (1 + \alpha_{n,j}),$$

therefore $(-1)^q \mathbf{1}_{U_j} T_j^n \leq \big(n!/q!\,(n-q)!\big)\, v^q \wedge (T_j + v)^{n-q}$. □

Proof of theorem 12.7 (end). Set $\Lambda = \max_X \lambda_1(x)$. By Lemma 12.10 applied with an arbitrary $c > \Lambda$ we have

$$(-1)^q \mathbf{1}_{U_j} T_j^n \leq \frac{n!}{q!(n-q)!}(\Lambda\, u + \varepsilon_1 \omega)^q \wedge (T_j + \Lambda\, u + \varepsilon_1 \omega)^{n-q} \quad \text{on } X.$$

Then estimate (12.2) and Lemma 12.10 again imply

$$\begin{aligned} h^q\big(X, E\otimes\mathcal{O}(kL)\big) &\le r\frac{k^n}{n!}\int_X (-1)^q \mathbf{1}_{U_j} T_j^n + o(k^n)\\ &\le \frac{r\,k^n}{q!\,(n-q)!}\Big\{\int_{\Omega_{c,j}} (c\,u+\varepsilon_j\,\omega)^q\wedge(T_j+c\,u+\varepsilon_j\,\omega)^{n-q}\\ &\qquad + \int_{X\smallsetminus\Omega_{c,j}} (\Lambda\,u+\varepsilon_1\omega)^q\wedge(T_j+\Lambda\,u+\varepsilon_1\omega)^{n-q}\Big\} + o(k^n). \end{aligned} \tag{12.11}$$

Since $\lambda_j(x)$ decreases to $\nu(T,x)$ as $j\to+\infty$, the set $X\smallsetminus\Omega_{c,j}$ decreases to $E_c(T)$. Now, $T_j+\Lambda\,u+\varepsilon_1\omega$ is a closed positive $(1,1)$-form belonging to a fixed cohomology class, so the mass of any wedge power $(T_j+\Lambda\,u+\varepsilon_1\omega)^p$ with respect to ω is constant. By weak compactness, there is a subsequence (j_ν) such that $(T_{j_\nu}+\Lambda\,u+\varepsilon_1\omega)^p$ converges weakly to a closed positive current Θ_p of bidegree (p,p), for each $p=1,\ldots,n$. For $c>b_{p+1}$, we have $\operatorname{codim} E_c(T)\ge p+1$, hence $\mathbf{1}_{E_c(T)}\Theta_p=0$. It follows that the integral over $X\smallsetminus\Omega_{c,j}$ in (12.11) converges to 0 when $c>b_{n-q+1}$. For the same reason the integral over $\Omega_{c,j}$ converges to the same limit as its value over X: observe that $(T_j+c\,u+\varepsilon_j\,\omega)^{n-q}$ can be expressed in terms of powers of u,ω and of the positive forms $(T_j+\Lambda\,u+\varepsilon_1\omega)^p$ with $p\le n-q$; thus the limit is a linear combination with smooth coefficients of the currents Θ_p, which carry no mass on $E_c(T)$. In the limit, we obtain

$$h^q\big(X,E\otimes\mathcal{O}(kL)\big)\le\frac{r\,k^n}{q!\,(n-q)!}(c\{u\})^q\cdot\big(c_1(L)+c\{u\}\big)^{n-q}+o(k^n),$$

and since this is true for every $c>b_{n-q+1}$, Theorem 12.7 follows. □

13. Effective Version of Matsusaka's Big Theorem

An important problem of algebraic geometry is to find effective bounds m_0 such that multiples mL of an ample line bundle become very ample for $m\ge m_0$. From a theoretical point of view, this problem has been solved by Matsusaka [Mat72] and Kollár-Matsusaka [KoM83]. Their result is that there is a bound $m_0=m_0(n,L^n,L^{n-1}\cdot K_X)$ depending only on the dimension and on the first two coefficients L^n and $L^{n-1}\cdot K_X$ in the Hilbert polynomial of L. Unfortunately, the original proof does not tell much on the actual dependence of m_0 in terms of these coefficients. The goal of this section is to find effective bounds for such an integer m_0, along the lines of [Siu93]. However, one of the technical lemmas used in [Siu93] to deal with dualizing sheaves can be sharpened. Using this sharpening of the lemma, Siu's bound will be here substantially improved. We first start with the simpler problem of obtaining merely a nontrivial section in mL. The idea, more generally, is to obtain a criterion for the ampleness of $mL-B$ when B is nef. In this way, one is able to subtract from mL any multiple of K_X which happens to get added by the application of Nadel's vanishing theorem (for this, replace B by B plus a multiple of $K_X+(n+1)L$).

(13.1) Proposition. *Let L be an ample line bundle over a projective n-fold X and let B be a nef line bundle over X. Then K_X+mL-B has a nonzero section for some integer m such that*

$$m \le n\,\frac{L^{n-1}\cdot B}{L^n} + n + 1.$$

Proof. Let m_0 be the smallest integer $> n\,\frac{L^{n-1}\cdot B}{L^n}$. Then $m_0L - B$ can be equipped with a singular hermitian metric of positive definite curvature. Let φ be the weight of this metric. By Nadel's vanishing theorem, we have

$$H^q(X, \mathcal{O}(K_X + mL - B) \otimes \mathcal{I}(\varphi)) = 0 \qquad \text{for } q \ge 1,$$

thus $P(m) = h^0(X, \mathcal{O}(K_X + mL - B) \otimes \mathcal{I}(\varphi))$ is a polynomial for $m \ge m_0$. Since P is a polynomial of degree n and is not identically zero, there must be an integer $m \in [m_0, m_0 + n]$ which is not a root. Hence there is a nontrivial section in

$$H^0(X, \mathcal{O}(K_X + mL - B)) \supset H^0(X, \mathcal{O}(K_X + mL - B) \otimes \mathcal{I}(\varphi))$$

for some $m \in [m_0, m_0 + n]$, as desired. □

(13.2) Corollary. *If L is ample and B is nef, then $mL - B$ has a nonzero section for some integer*

$$m \le n\Big(\frac{L^{n-1}\cdot B + L^{n-1}\cdot K_X}{L^n} + n + 1\Big).$$

Proof. By Fujita's result 8.3 a), $K_X + (n+1)L$ is nef. We can thus replace B by $B + K_X + (n+1)L$ in the result of Prop. 13.1. Corollary 13.2 follows. □

(13.3) Remark. We do not know if the above Corollary is sharp, but it is certainly not far from being so. Indeed, for $B = 0$, the initial constant n cannot be replaced by anything smaller than $n/2$: take X to be a product of curves C_j of large genus g_j and $B = 0$; our bound for $L = \mathcal{O}(a_1[p_1]) \otimes \ldots \otimes \mathcal{O}(a_n[p_n])$ to have $|mL| \ne \emptyset$ becomes $m \le \sum(2g_j - 2)/a_j + n(n+1)$, which fails to be sharp only by a factor 2 when $a_1 = \ldots = a_n = 1$ and $g_1 \gg g_2 \gg \ldots \gg g_n \to +\infty$. On the other hand, the additive constant $n+1$ is already best possible when $B = 0$ and $X = \mathbb{P}^n$. □

So far, the method is not really sensitive to singularities (the Morse inequalities are indeed still true in the singular case as is easily seen by using desingularizations of the ambient variety). The same is true with Nadel's vanishing theorem, provided that K_X is replaced by the L^2 dualizing sheaf ω_X (according to the notation introduced in Remark 5.17, $\omega_X = K_X(0)$ is the sheaf of holomorphic n-forms u on $X_{\rm reg}$ such that $\mathrm{i}^{n^2} u \wedge \overline{u}$ is integrable in a neighborhood of the singular set). Then Prop. 13.1 can be generalized as

(13.4) Proposition. *Let L be an ample line bundle over a projective n-fold X and let B be a nef line bundle over X. For every p-dimensional* (*reduced*) *algebraic subvariety Y of X, there is an integer*

$$m \le p\frac{L^{p-1}\cdot B\cdot Y}{L^p\cdot Y} + p + 1$$

such that the sheaf $\omega_Y \otimes \mathcal{O}_Y(mL - B)$ has a nonzero section. □

To proceed further, we need the following useful "upper estimate" about L^2 dualizing sheaves (this is one of the crucial steps in Siu's approach; unfortunately, it has the effect of producing rather large final bounds when the dimension increases).

(13.5) Proposition. *Let H be a very ample line bundle on a projective algebraic manifold X, and let $Y \subset X$ be a p-dimensional irreducible algebraic subvariety. If $\delta = H^p \cdot Y$ is the degree of Y with respect to H, the sheaf*

$$\mathcal{H}om\big(\omega_Y, \mathcal{O}_Y((\delta - p - 2)H)\big)$$

has a nontrivial section.

Observe that if Y is a smooth hypersurface of degree δ in $(X, H) = (\mathbb{P}^{p+1}, \mathcal{O}(1))$, then $\omega_Y = \mathcal{O}_Y(\delta - p - 2)$ and the estimate is optimal. On the other hand, if Y is a smooth complete intersection of multidegree $(\delta_1, \ldots, \delta_r)$ in $\mathbb{P}^{p+r}$, then $\delta = \delta_1 \ldots \delta_r$ whilst $\omega_Y = \mathcal{O}_Y(\delta_1 + \ldots + \delta_r - p - r - 1)$; in this case, Prop. (13.5) is thus very far from being sharp.

Proof. Let $X \subset \mathbb{P}^N$ be the embedding given by H, so that $H = \mathcal{O}_X(1)$. There is a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{p+1}$ whose restriction $\pi : Y \to \mathbb{P}^{p+1}$ to Y is a finite and regular birational map of Y onto an algebraic hypersurface Y' of degree δ in $\mathbb{P}^{p+1}$. Let $s \in H^0(\mathbb{P}^{p+1}, \mathcal{O}(\delta))$ be the polynomial of degree δ defining Y'. We claim that for any small Stein open set $W \subset \mathbb{P}^{p+1}$ and any L^2 holomorphic p-form u on $Y' \cap W$, there is a L^2 holomorphic $(p+1)$-form $\widetilde{u}$ on W with values in $\mathcal{O}(\delta)$ such that $\widetilde{u}_{\restriction Y' \cap W} = u \wedge ds$. In fact, this is precisely the conclusion of the Ohsawa-Takegoshi extension theorem [OT87], [Ohs88] (see also [Man93] for a more general version); one can also invoke more standard local algebra arguments (see Hartshorne [Har77], Th. III-7.11). As $K_{\mathbb{P}^{p+1}} = \mathcal{O}(-p-2)$, the form $\widetilde{u}$ can be seen as a section of $\mathcal{O}(\delta - p - 2)$ on W, thus the sheaf morphism $u \mapsto u \wedge ds$ extends into a global section of $\mathcal{H}om\big(\omega_{Y'}, \mathcal{O}_{Y'}(\delta - p - 2)\big)$. The pull-back by $\pi^\star$ yields a section of $\mathcal{H}om\big(\pi^\star\omega_{Y'}, \mathcal{O}_Y((\delta - p - 2)H)\big)$. Since π is finite and generically $1:1$, it is easy to see that $\pi^\star\omega_{Y'} = \omega_Y$. The Proposition follows. $\square$

By an appropriate induction process based on the above results, we can now improve Siu's effective version of the Big Matsusaka Theorem [Siu93]. Our version depends on a constant λ_n such that $m(K_X + (n+2)L) + G$ is very ample for $m \geq \lambda_n$ and every nef line bundle G. Corollary (8.5) shows that $\lambda_n \leq \binom{3n+1}{n} - 2n$, and a similar argument involving the recent results of Angehrn-Siu [AS94] implies $\lambda_n \leq n^3 - n^2 - n - 1$ for $n \geq 2$. Of course, it is expected that $\lambda_n = 1$ in view of the Fujita conjecture.

(13.6) Effective version of the Big Matsusaka Theorem. *Let L and B be nef line bundles on a projective n-fold X. Assume that L is ample and let H be the very ample line bundle $H = \lambda_n(K_X + (n+2)L)$. Then $mL - B$ is very ample for*

$$m \geq (2n)^{(3^{n-1}-1)/2} \frac{(L^{n-1} \cdot (B+H))^{(3^{n-1}+1)/2} (L^{n-1} \cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(L^n)^{3^{n-2}(n/2-1/4)+1/4}}.$$

In particular mL is very ample for

$$m \geq C_n \; (L^n)^{3^{n-2}} \left(n + 2 + \frac{L^{n-1} \cdot K_X}{L^n}\right)^{3^{n-2}(n/2+3/4)+1/4}$$

with $C_n = (2n)^{(3^{n-1}-1)/2}(\lambda_n)^{3^{n-2}(n/2+3/4)+1/4}$.

Proof. We use Prop. (13.4) and Prop. (13.5) to construct inductively a sequence of (non necessarily irreducible) algebraic subvarieties $X = Y_n \supset Y_{n-1} \supset \ldots \supset Y_2 \supset Y_1$ such that $Y_p = \bigcup_j Y_{p,j}$ is p-dimensional, and Y_{p-1} is obtained for each $p \geq 2$ as the union of zero sets of sections

$$\sigma_{p,j} \in H^0(Y_{p,j}, \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B))$$

with suitable integers $m_{p,j} \geq 1$. We proceed by induction on decreasing values of the dimension p, and find inductively upper bounds m_p for the integers $m_{p,j}$.

By Cor. (13.2), an integer m_n for $m_nL - B$ to have a section σ_n can be found with

$$m_n \leq n\,\frac{L^{n-1} \cdot (B + K_X + (n+1)L)}{L^n} \leq n\,\frac{L^{n-1} \cdot (B + H)}{L^n}.$$

Now suppose that the sections σ_n, …, $\sigma_{p+1,j}$ have been constructed. Then we get inductively a p-cycle $\widetilde{Y}_p = \sum \mu_{p,j} Y_{p,j}$ defined by $\widetilde{Y}_p =$ sum of zero divisors of sections $\sigma_{p+1,j}$ in $\widetilde{Y}_{p+1,j}$, where the mutiplicity $\mu_{p,j}$ on $Y_{p,j} \subset Y_{p+1,k}$ is obtained by multiplying the corresponding multiplicity $\mu_{p+1,k}$ with the vanishing order of $\sigma_{p+1,k}$ along $Y_{p,j}$. As cohomology classes, we find

$$\widetilde{Y}_p \equiv \sum (m_{p+1,k}L - B) \cdot (\mu_{p+1,k} Y_{p+1,k}) \leq m_{p+1} L \cdot \widetilde{Y}_{p+1}.$$

Inductively, we thus have the numerical inequality

$$\widetilde{Y}_p \leq m_{p+1} \ldots m_n L^{n-p}.$$

Now, for each component $Y_{p,j}$, Prop. (13.4) shows that there exists a section of $\omega_{Y_{p,j}} \otimes \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ for some integer

$$m_{p,j} \leq p \frac{L^{p-1} \cdot B \cdot Y_{p,j}}{L^p \cdot Y_{p,j}} + p + 1 \leq p m_{p+1} \ldots m_n \, L^{n-1} \cdot B + p + 1.$$

Here, we have used the obvious lower bound $L^{p-1} \cdot Y_{p,j} \geq 1$ (this is of course a rather weak point in the argument). The degree of $Y_{p,j}$ with respect to H admits the upper bound

$$\delta_{p,j} := H^p \cdot Y_{p,j} \leq m_{p+1} \ldots m_n H^p \cdot L^{n-p}.$$

We use the Hovanski-Teissier concavity inequality (10.2 b)

$$(L^{n-p} \cdot H^p)^{\frac{1}{p}} (L^n)^{1-\frac{1}{p}} \leq L^{n-1} \cdot H$$

to express our bounds in terms of the intersection numbers L^n and $L^{n-1} \cdot H$ only. We then get

$$\delta_{p,j} \le m_{p+1} \ldots m_n \frac{(L^{n-1} \cdot H)^p}{(L^n)^{p-1}}.$$

By Prop. (13.5), there is a nontrivial section in

$$\mathcal{H}om\big(\omega_{Y_{p,j}}, \mathcal{O}_{Y_{p,j}}((\delta_{p,j} - p - 2)H)\big).$$

Combining this section with the section in $\omega_{Y_{p,j}} \otimes \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ already constructed, we get a section of $\mathcal{O}_{Y_{p,j}}(m_{p,j}L - B + (\delta_{p,j} - p - 2)H)$ on $Y_{p,j}$. Since we do not want H to appear at this point, we replace B with $B + (\delta_{p,j} - p - 2)H$ and thus get a section $\sigma_{p,j}$ of $\mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ with some integer $m_{p,j}$ such that

$$\begin{aligned} m_{p,j} &\le pm_{p+1} \ldots m_n \, L^{n-1} \cdot (B + (\delta_{p,j} - p - 2)H) + p + 1 \\ &\le p\, m_{p+1} \ldots m_n \, \delta_{p,j} \, L^{n-1} \cdot (B + H) \\ &\le p\, (m_{p+1} \ldots m_n)^2 \frac{(L^{n-1} \cdot H)^p}{(L^n)^{p-1}} \, L^{n-1} \cdot (B + H). \end{aligned}$$

Therefore, by putting $M = n\, L^{n-1} \cdot (B + H)$, we get the recursion relation

$$m_p \le M \frac{(L^{n-1} \cdot H)^p}{(L^n)^{p-1}} (m_{p+1} \ldots m_n)^2 \qquad \text{for } 2 \le p \le n-1,$$

with initial value $m_n \le M/L^n$. If we let $(\overline{m}_p)$ be the sequence obtained by the same recursion formula with equalities instead of inequalities, we get $m_p \le \overline{m}_p$ with $\overline{m}_{n-1} = M^3 (L^{n-1} \cdot H)^{n-1} / (L^n)^n$ and

$$\overline{m}_p = \frac{L^n}{L^{n-1} \cdot H} \, \overline{m}_{p+1}^2 \overline{m}_{p+1}$$

for $2 \le p \le n-2$. We then find inductively

$$m_p \le \overline{m}_p = M^{3^{n-p}} \frac{(L^{n-1} \cdot H)^{3^{n-p-1}(n-3/2)+1/2}}{(L^n)^{3^{n-p-1}(n-1/2)+1/2}}.$$

We next show that $m_0 L - B$ is nef for

$$m_0 = \max\big(m_2\,,\, m_3, \ldots, m_n\,,\, m_2 \ldots m_n \, L^{n-1} \cdot B\big).$$

In fact, let $C \subset X$ be an arbitrary irreducible curve. Either $C = Y_{1,j}$ for some j or there exists an integer $p = 2, \ldots, n$ such that C is contained in Y_p but not in Y_{p-1}. If $C \subset Y_{p,j} \smallsetminus Y_{p-1}$, then $\sigma_{p,j}$ does not vanish identically on C. Hence $(m_{p,j}L - B)_{\restriction C}$ has nonnegative degree and

$$(m_0 L - B) \cdot C \ge (m_{p,j} L - B) \cdot C \ge 0.$$

On the other hand, if $C = Y_{1,j}$, then

$$(m_0 L - B) \cdot C \ge m_0 - B \cdot \widetilde{Y}_1 \ge m_0 - m_2 \ldots m_n \, L^{n-1} \cdot B \ge 0.$$

By the definition of λ_n (and by Cor. (8.5) showing that such a constant exists), $H + G$ is very ample for every nef line bundle G, in particular $H + m_0 L - B$ is very ample. We thus replace again B with $B + H$. This has the effect of replacing M with $M = n\,\big(L^{n-1} \cdot (B + 2H)\big)$ and m_0 with

$$m_0 = \max\big(m_n\,,\ m_{n-1},\ldots,m_2\,,\ m_2\ldots m_n\,L^{n-1}\cdot(B+H)\big).$$

The last term is the largest one, and from the estimate on $\overline{m}_p$, we get

$$\begin{aligned} m_0 &\le M^{(3^{n-1}-1)/2}\frac{(L^{n-1}\cdot H)^{(3^{n-2}-1)(n-3/2)/2+(n-2)/2}(L^{n-1}\cdot(B+H))}{(L^n)^{(3^{n-2}-1)(n-1/2)/2+(n-2)/2+1}} \\ &\le (2n)^{(3^{n-1}-1)/2}\frac{(L^{n-1}\cdot(B+H))^{(3^{n-1}+1)/2}(L^{n-1}\cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(L^n)^{3^{n-2}(n/2-1/4)+1/4}} \end{aligned}$$

□

(13.7) Remark. In the surface case $n=2$, one can take $\lambda_n=1$ and our bound yields mL very ample for

$$m\ge 4\,\frac{(L\cdot(K_X+4L))^2}{L^2}.$$

If one looks more carefully at the proof, the initial constant 4 can be replaced by 2. In fact, it has been shown recently by Fernández del Busto that mL is very ample for

$$m>\frac{1}{2}\left[\frac{(L\cdot(K_X+4L)+1)^2}{L^2}+3\right],$$

and an example of G. Xiao shows that this bound is essentially optimal (see [FdB94]).

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