Kobayashi-Lübke inequalities for Chern classes of Hermite-Einstein vector bundles and Guggenheimer-Yau-Bogomolov-Miyaoka inequalities for Chern classes of Kähler-Einstein manifolds

Let (X,ω) be a compact Kähler manifold, $n = \dim X$, and let E be a holomorphic vector bundle over X, $r = \operatorname{rank} E$. We suppose that E is equipped with a hermitian metric, h and denote by $D_{E,h}$ the Chern connection on (E,h). The Chern curvature form is

$$\Theta_h(E) = D^2_{E,h}.$$

In a (local) orthonormal frame $(e_\alpha)_{1\le\alpha\le r}$ of E, we write

$$\Theta_h(E) = (\Theta_{\alpha\beta})_{1\le\alpha,\beta\le r}$$

where the $\Theta_{\alpha\beta}$ are complex valued $(1,1)$-forms satisfying the hermitian condition $\overline{\Theta}_{\alpha\beta} = \Theta_{\beta\alpha}$. We denote

$$\Theta_{\alpha\beta} = i \sum_{1\le\alpha,\beta\le r,\ 1\le j,k\le n} \Theta_{\alpha\beta jk} dz_j \wedge dz_k.$$

The hermitian symmetry condition can then be read $\overline{\Theta}_{\alpha\beta jk} = \Theta_{\beta\alpha kj}$. If at some point $x_0 \in X$ the coordinates (z_j) are chosen so that $(dz_j(x_0))$ is an orthonormal basis of $T^\star_{X,x_0}$, we define

$$\mathrm{Tr}_\omega \Theta_h(E) = \Big(\sum_j \Theta_{\alpha\beta jj}\Big) \in C^\infty(X, \hom(E,E)).$$

Definition. — *The hermitian vector bundle (E,h) is said to be Hermite-Einstein with respect to the Kähler metric ω if there is a constant $\lambda > 0$ such that $\mathrm{Tr}_\omega \Theta_h(E) = \lambda \mathrm{Id}_E$.*

Recall that the Chern forms $c_k(E)_h$ are defined by the formula

$$\det\big(I + t\Theta_h(E)\big) = \det(\delta_{\alpha\beta} + t\Theta_{\alpha\beta}) = 1 + t\,c_1(E)_h + \ldots + t^r c_r(E)_h.$$

This gives in particular the identities

$$c_1(E)_h = \sum_\alpha \Theta_{\alpha\alpha},$$

$$c_2(E)_h = \sum_{\alpha<\beta} \Theta_{\alpha\alpha}\wedge\Theta_{\beta\beta} - \Theta_{\alpha\beta}\wedge\Theta_{\beta\alpha} = \frac{1}{2}\sum_{\alpha,\beta} \Theta_{\alpha\alpha}\wedge\Theta_{\beta\beta} - \Theta_{\alpha\beta}\wedge\Theta_{\beta\alpha}.$$

The trace $\mathrm{Tr}_\omega \Theta_h(E)$ can be computed by the formula

$$\Theta_h(E) \wedge \frac{\omega^{n-1}}{(n-1)!} = \mathrm{Tr}_\omega \Theta_h(E)\,\frac{\omega^n}{n!}.$$

By taking the trace with respect to the indices α in E and taking the Hermite-Einstein equation into account, we find

$$c_1(E)_h \wedge \frac{\omega^{n-1}}{(n-1)!} = \lambda r \, \frac{\omega^n}{n!}.$$

This implies that the number λ in the definition of Hermite-Einstein metrics is a purely numerical invariant, namely

$$\lambda = \frac{n}{r} \int_X c_1(E) \wedge \omega^{n-1} \Big/ \int_X \omega^n.$$

KOBAYASHI-LÜBKE INEQUALITY. — *If E admits a Hermite-Einstein metric h with respect to ω, then*

$$\big[(r-1)c_1(E)_h^2 - 2r\, c_2(E)_h\big] \wedge \omega^{n-2} \le 0$$

at every point of X. Moreover, the equality holds if and only if

$$\Theta_h(E) = \frac{1}{r} c_1(E)_h \otimes \mathrm{Id}_E.$$

Observe that the equality holds pointwise already if we have the numerical equality

$$\int_X \big[(r-1)c_1(E)^2 - 2r\, c_2(E)\big] \wedge \omega^{n-2} = 0.$$

If we introduce the (formal) vector bundle $\widetilde{E} = E \otimes (\det E)^{-1/r}$ ($\widetilde{E}$ is the "normalized" vector bundle such that $\det \widetilde{E} = \mathcal{O}$), then $c_1(\widetilde{E})_h = 0$ and

$$\Theta_h(\widetilde{E}) = \Big(\Theta_h(E) - \frac{1}{r} c_1(E)_h \otimes \mathrm{Id}_E\Big) \otimes \mathrm{Id}_{(\det E)^{-1/r}}.$$

By the formula for the chern classes of $E \otimes L$, the Kobayashi-Lübke inequality can be rewritten as

$$c_2(\widetilde{E})_h \wedge \omega^{n-2} \le 0,$$

with equality if and only if $\widetilde{E}$ is unitary flat. In that case, we say that E is *projectively flat*.

Proof. By the above,

$$(r-1)c_1(E)_h^2 - 2r\, c_2(E)_h = \sum_{\alpha,\beta} -\Theta_{\alpha\alpha} \wedge \Theta_{\beta\beta} + r\, \Theta_{\alpha\beta} \wedge \Theta_{\beta\alpha}.$$

Taking the wedge product with $\omega^{n-2}/(n-2)!$ means taking the trace, i.e. the sum of coefficients of the terms $i\, dz_j \wedge d\overline{z}_j \wedge i\, dz_k \wedge d\overline{z}_k$ for all $j < k$. For this, we have to look at products of the type $(i\, dz_j \wedge d\overline{z}_j) \wedge (i\, dz_k \wedge d\overline{z}_k)$ or $(i\, dz_j \wedge d\overline{z}_k) \wedge (i\, dz_k \wedge d\overline{z}_j)$. This yields

$$\begin{aligned} &2\big[(r-1)c_1(E)_h^2 - 2r\, c_2(E)_h\big] \wedge \frac{\omega^{n-2}}{(n-2)!} \\ &\quad = \sum_{\alpha,\beta,j,k} -(\Theta_{\alpha\alpha jj}\Theta_{\beta\beta kk} - \Theta_{\alpha\alpha jk}\Theta_{\beta\beta kj}) + r(\Theta_{\alpha\beta jj}\Theta_{\beta\alpha kk} - \Theta_{\alpha\beta jk}\Theta_{\beta\alpha kj}). \end{aligned}$$

The initial factor 2 comes from the fact that the final sum is taken over all unordered indices j, k (terms with $j = k$ cancel). The Hermite-Einstein condition yields $\sum_j \Theta_{\alpha\beta jj} = \lambda\,\delta_{\alpha\beta}$, so we get

$$\sum_{\alpha,\beta,j,k} -\Theta_{\alpha\alpha jj}\Theta_{\beta\beta kk} + r\,\Theta_{\alpha\beta jj}\Theta_{\beta\alpha kk} = \sum_{\alpha,\beta} -\lambda^2\delta_{\alpha\alpha}\delta_{\beta\beta} + r\lambda^2\delta_{\alpha\beta}\delta_{\beta\alpha}$$
$$= -r^2\lambda^2 + r^2\lambda^2 = 0.$$

Hence, using the hermitian symmetry of $\Theta_{\alpha\beta jk}$, we find

$$\begin{aligned}
&2\big[(r-1)c_1(E)_h^2 - 2r\,c_2(E)_h\big] \wedge \frac{\omega^{n-2}}{(n-2)!}\\
&\qquad = \sum_{\alpha,\beta,j,k} \Theta_{\alpha\alpha jk}\overline{\Theta}_{\beta\beta jk} - r\,|\Theta_{\alpha\beta jk}|^2\\
&\qquad = -r\sum_{\alpha\neq\beta,j,k} |\Theta_{\alpha\beta jk}|^2 + \sum_{j,k}\Big(\sum_{\alpha,\beta}\Theta_{\alpha\alpha jk}\overline{\Theta}_{\beta\beta jk} - r\sum_{\alpha}|\Theta_{\alpha\alpha jk}|^2\Big)\\
&\qquad = -r\sum_{\alpha\neq\beta,j,k} |\Theta_{\alpha\beta jk}|^2 - \frac{1}{2}\sum_{\alpha,\beta,j,k}\big|\Theta_{\alpha\alpha jk} - \Theta_{\beta\beta jk}\big|^2 \leq 0.
\end{aligned}$$

This proves the expected inequality. Moreover, the equality holds if and only if we have

$$\Theta_{\alpha\beta jk} = 0 \quad \text{for} \quad \alpha \neq \beta, \qquad \Theta_{\alpha\alpha jk} = \gamma_{jk} \quad \text{for all} \quad \alpha.$$

where $\gamma = i\sum_{j,k}\gamma_{jk}dz_j \wedge d\overline{z}_k$ is a $(1,1)$-form (take e.g., $\gamma_{jk} = \Theta_{11jk}$). Hence $\Theta_h(E) = \gamma \otimes \mathrm{Id}_E$. By taking the trace with respect to E in this last equality, we get $c_1(E)_h = r\,\gamma$. Therefore the equality occurs if and only if

$$\Theta_h(E) = \frac{1}{r}c_1(E)_h \otimes \mathrm{Id}_E. \qquad \square$$

Corollary 1. — *Let (E, h) be a Hermite-Einstein vector bundle with $c_1(E) = 0$ and $c_2(E) = 0$. Then E is unitary flat for some hermitian metric $h' = h\,e^{-\varphi}$.*

Proof. By the assumption $c_1(E) = 0$, we can write $c_1(E)_h = \frac{i}{2\pi}\partial\overline{\partial}\psi$ for some global function ψ on X. The equality case of the Kobayashi-Lübke inequality yields

$$\Theta_{h\,e^{\psi/r}}(E) = \Theta_h(E) - \frac{1}{r}\frac{i}{2\pi}\partial\overline{\partial}\psi \otimes \mathrm{Id}_E = 0. \qquad \square$$

Corollary 2. — *Let X be a compact Kähler manifold with $c_1(X) = c_2(X) = 0$. Then X is a finite unramified quotient of a torus.*

Proof. By the Aubin-Calabi-Yau theorem, X admits a Ricci-flat Kähler metric ω. Since $\mathrm{Ricci}(\omega) = \mathrm{Tr}_\omega\Theta_\omega(T_X)$, we see that (T_X, ω) is a Hermite-Einstein vector bundle, and $c_1(T_X)_\omega = \mathrm{Ricci}(\omega) = 0$. By the Kobayashi-Lübke inequality, we conclude that (T_X, ω) is unitary flat, given by a unitary representation

$\pi_1(X) \to U(n)$. Let $\widetilde{X}$ be the universal covering of X and $\widetilde{\omega}$ the induced metric. Then $(T_{\widetilde{X}}, \widetilde{\omega})$ is a trivial vector bundle equipped with a flat metric. Let $(\xi_1, \ldots, \xi_n)$ be an orthonormal parallel frame of T_X. Since $\nabla \xi_j = 0$, we conclude that $d\xi_j^\star = 0$, and it is easy to infer from this that $[\xi_j, \xi_k] = 0$. The flow of each vector field $\sum \lambda_j \xi_j$ is defined for all times (this follows from the fact the length of a trajectory is proportional to the time, and $\widetilde{\omega}$ is complete). Hence we get an action of $\mathbb{C}^n$ on $\widetilde{X}$, and it follows easily that $(\widetilde{X}, \widetilde{\omega}) \simeq (\mathbb{C}^n, \mathrm{can})$. Now, $\pi_1(X)$ acts by isometries on this $\mathbb{C}^n$. The classification of subgroups of affine transformations acting freely (and with compact quotient) shows that $\pi_1(X)$ must be a semi-direct product of a finite group of isometries by a group of translations associated to a lattice $\Lambda \subset \mathbb{C}^n$. Hence there is an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \pi_1(X) \longrightarrow G \longrightarrow 0$$

where G is a finite group of isometries. It follows that there is a finite unramified covering map $\mathbb{C}^n/\Lambda \to \widetilde{X}/\pi_1(X) \simeq X$ of X by a torus. □

We now discuss the special case of the tangent bundle T_X in case (X, ω) is a compact Kähler-Einstein manifold. The Kähler-Einstein condition means that $\mathrm{Ricci}(\omega) = \lambda\omega$ for some real constant λ, i.e., $\mathrm{Tr}_\omega \Theta_\omega(T_X) = \lambda \mathrm{Id}_{T_X}$. In particular, (T_X, ω) is a Hermite-Einstein vector bundle. Here, however, the coefficients $(\Theta_{\alpha\beta jk})_{1\le\alpha,\beta,j,k\le n}$ of the curvature tensor $\Theta_\omega(T_X)$ satisfy the additional symmetry relations

$$(\star) \qquad \Theta_{\alpha\beta jk} = \Theta_{j\beta\alpha k} = \Theta_{\alpha kj\beta} = \Theta_{jk\alpha\beta}.$$

These relations follow easily from the identity $\Theta_{\alpha\beta jk} = -\partial^2\omega_{\alpha\beta}/\partial z_j \partial\overline{z}_k$ in normal coordinates, when we apply the Kähler condition $\partial\omega_{\alpha\beta}/\partial z_j = \partial\omega_{j\beta}/\partial z_\alpha$. It follows that the Chern forms satisfy a slightly stronger inequality than the general inequality valid for Hermite-Einstein bundles. In fact (T_X, ω) satisfies a similar inequality where the rank $r = n$ is replaced by $n + 1$.

Guggenheimer-Yau inequality. — *Let (T_X, ω) be a compact n-dimensional Kähler-Einstein manifold, with constant $\lambda \in \mathbb{R}$. If $\lambda = 0$, then $c_2(T_X)_\omega \wedge \omega^{n-2} \ge 0$. If $\lambda \ne 0$, we have the inequality*

$$\big[nc_1(X)^2 - (2n+2)c_2(X)\big] \cdot (\lambda c_1(X))^{n-2} \le 0,$$

and the equality also holds pointwise if we replace the Chern classes by the Chern forms $c_k(T_X)_\omega$. The equality occurs in the following cases:

(i) *If $\lambda = 0$, then (X, ω) is a finite unramified quotient of a torus.*

(ii) *If $\lambda > 0$, then $(X, \omega) \simeq (\mathbb{P}^n, \text{Fubini Study})$.*

(iii) *If $\lambda < 0$, then $(X, \omega) \simeq (\mathbb{B}_n/\Gamma, \text{Poincaré metric})$, i.e. X is a compact unramified quotient of the ball in $\mathbb{C}^n$.*

Corollary (Bogomolov-Miyaoka-Yau). — *Let X be a surface of general type with K_X ample. Then there is an inequality $c_1(X)^2 \le 3\, c_2(X)$, and the equality occurs if and only if X is a quotient of the ball $\mathbb{B}_2$.*

Miyaoka has shown that the inequality holds in fact as soon as X is a surface with general type.

Proof. As in the proof of the Kobayashi-Lübke inequality, we find

$$n\,c_1(T_X)^2_\omega-(2n+2)c_2(T_X)_\omega=\sum_{\alpha,\beta}-\Theta_{\alpha\alpha}\wedge\Theta_{\beta\beta}+(n+1)\Theta_{\alpha\beta}\wedge\Theta_{\beta\alpha}.$$

Taking the wedge product with $\omega^{n-2}/(n-2)!$, we get

$$\begin{aligned}&2\big[n\,c_1(T_X)^2_\omega-(2n+2)\,c_2(T_X)_\omega\big]\wedge\frac{\omega^{n-2}}{(n-2)!}\\&=\sum_{\alpha,\beta,j,k}-(\Theta_{\alpha\alpha jj}\Theta_{\beta\beta kk}-\Theta_{\alpha\alpha jk}\Theta_{\beta\beta kj})+(n+1)(\Theta_{\alpha\beta jj}\Theta_{\beta\alpha kk}-\Theta_{\alpha\beta jk}\Theta_{\beta\alpha kj}).\end{aligned}$$

If we had the factor $r=n$ instead of $(n+1)$ in the right hand side, the terms in jj and kk would cancel (as they did before). Hence we find

$$\begin{aligned}2\big[n\,c_1(T_X)^2_\omega-(2n+2)\,c_2(T_X)_\omega\big]\wedge\frac{\omega^{n-2}}{(n-2)!}&\\=\sum_{\alpha,\beta,j,k}\Theta_{\alpha\alpha jk}\Theta_{\beta\beta kj}+\Theta_{\alpha\beta jj}\Theta_{\beta\alpha kk}-(n+1)\Theta_{\alpha\beta jk}\Theta_{\beta\alpha kj}&\\=\sum_{\alpha,\beta,j,k}2\,\Theta_{\alpha\alpha jk}\Theta_{\beta\beta kj}-(n+1)\Theta_{\alpha\beta jk}\Theta_{\beta\alpha kj}.&\end{aligned}$$

Here, the symmetry relation $(\star)$ was used in order to obtain the equality of the summation of the first two terms. Using also the hermitian symmetry relation, our sum Σ can be rewritten as

$$\begin{aligned}\Sigma=&-\sum_{\alpha,\beta,j,k}|\Theta_{\alpha\alpha jk}-\Theta_{\beta\beta jk}|^2-(n+1)|\Theta_{\alpha\beta jk}|^2+2n\sum_{\alpha,j,k}|\Theta_{\alpha\alpha jk}|^2\\=&-\sum_{\alpha,\beta,j,k}|\Theta_{\alpha\alpha jk}-\Theta_{\beta\beta jk}|^2-(n+1)\sum_{\alpha,\beta,j,k,\,\text{pairwise}\neq}|\Theta_{\alpha\beta jk}|^2\\&-(n+1)\Big[8\sum_{\alpha\neq j<k\neq\alpha}|\Theta_{\alpha\alpha jk}|^2+4\sum_{\alpha\neq j}|\Theta_{\alpha\alpha\alpha j}|^2+4\sum_{\alpha<j}|\Theta_{\alpha\alpha jj}|^2+\sum_{\alpha}|\Theta_{\alpha\alpha\alpha\alpha}|^2\Big]\\&\quad+2n\Big[2\sum_{\alpha\neq j<k\neq\alpha}|\Theta_{\alpha\alpha jk}|^2+2\sum_{\alpha\neq j}|\Theta_{\alpha\alpha\alpha j}|^2+2\sum_{\alpha<j}|\Theta_{\alpha\alpha jj}|^2+\sum_{\alpha}|\Theta_{\alpha\alpha\alpha\alpha}|^2\Big]\\=&-\sum_{\alpha\neq\beta,j,k}|\Theta_{\alpha\alpha jk}-\Theta_{\beta\beta jk}|^2-(n+1)\sum_{\alpha,\beta,j,k,\,\text{pairwise}\neq}|\Theta_{\alpha\beta jk}|^2\\&-(4n+8)\sum_{\alpha\neq j<k\neq\alpha}|\Theta_{\alpha\alpha jk}|^2-4\sum_{\alpha\neq j}|\Theta_{\alpha\alpha\alpha j}|^2-4\sum_{\alpha<j}|\Theta_{\alpha\alpha jj}|^2\\&+(n-1)\sum_{\alpha}|\Theta_{\alpha\alpha\alpha\alpha}|^2.\end{aligned}$$

All terms are negative except the last one. We try to absorb this term in the summations involving the coefficients $\Theta_{\alpha\alpha jj}$. This gives

$$\begin{aligned}\Sigma = &- \sum_{\alpha\neq\beta, j\neq k} |\Theta_{\alpha\alpha jk} - \Theta_{\beta\beta jk}|^2 - (n+1) \sum_{\alpha,\beta,j,k,\,\text{pairwise}\neq} |\Theta_{\alpha\beta jk}|^2 \\ &- (4n+8) \sum_{\alpha\neq j<k\neq\alpha} |\Theta_{\alpha\alpha jk}|^2 - 4\sum_{\alpha\neq j} |\Theta_{\alpha\alpha\alpha j}|^2 \\ &- \sum_{\alpha\neq\beta, j} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2 - 4\sum_{\alpha<j} |\Theta_{\alpha\alpha jj}|^2 + (n-1)\sum_{\alpha} |\Theta_{\alpha\alpha\alpha\alpha}|^2.\end{aligned}$$

The last line is equal to

$$\begin{aligned}&- \sum_{\alpha\neq\beta\neq j\neq\alpha} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2 - 2\sum_{\alpha\neq\beta} |\Theta_{\alpha\alpha\beta\beta} - \Theta_{\beta\beta\beta\beta}|^2 \\ &\qquad\qquad - 4\sum_{\alpha<\beta} |\Theta_{\alpha\alpha\beta\beta}|^2 + (n-1)\sum_{\alpha} |\Theta_{\alpha\alpha\alpha\alpha}|^2 \\ = &- \sum_{\alpha\neq\beta\neq j\neq\alpha} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2 \\ &- (n-1)\sum_{\alpha} |\Theta_{\alpha\alpha\alpha\alpha}|^2 - 8\sum_{\alpha<\beta} |\Theta_{\alpha\alpha\beta\beta}|^2 + 2\sum_{\alpha\neq\beta} \Theta_{\alpha\alpha\beta\beta}\overline{\Theta}_{\beta\beta\beta\beta} + \overline{\Theta}_{\alpha\alpha\beta\beta}\Theta_{\beta\beta\beta\beta} \\ = &- \sum_{\alpha\neq\beta\neq j\neq\alpha} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2 - \sum_{\alpha\neq\beta} |\Theta_{\alpha\alpha\alpha\alpha} - 2\Theta_{\alpha\alpha\beta\beta}|^2.\end{aligned}$$

Therefore we find

$$\begin{aligned}\Sigma = &- \sum_{\alpha\neq\beta, j\neq k} |\Theta_{\alpha\alpha jk} - \Theta_{\beta\beta jk}|^2 - (n+1) \sum_{\alpha,\beta,j,k,\,\text{pairwise}\neq} |\Theta_{\alpha\beta jk}|^2 \\ &- (4n+8) \sum_{\alpha\neq j<k\neq\alpha} |\Theta_{\alpha\alpha jk}|^2 - 4\sum_{\alpha\neq j} |\Theta_{\alpha\alpha\alpha j}|^2 \\ &- \sum_{\alpha\neq\beta\neq j\neq\alpha} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2 - \sum_{\alpha\neq\beta} |\Theta_{\alpha\alpha\alpha\alpha} - 2\Theta_{\alpha\alpha\beta\beta}|^2.\end{aligned}$$

This proves the expected inequality $\Sigma \leq 0$. Moreover, we have $\Sigma = 0$ if and only if there is a scalar μ such that

$$\Theta_{\alpha\alpha\beta\beta} = \Theta_{\alpha\beta\beta\alpha} = \Theta_{\alpha\beta\alpha\beta} = \mu \ \text{ for } \ \alpha\neq\beta, \qquad \Theta_{\alpha\alpha\alpha\alpha} = 2\mu,$$

and all other coefficients $\Theta_{\alpha\beta jk}$ are zero. By taking the trace $\sum_j \Theta_{\alpha\alpha jj}$, we get $\lambda = (n+1)\mu$. We thus obtain that the hermitian form associated to the curvature tensor is

$$\begin{aligned}\langle \Theta_\omega(T_X)(\xi\otimes\eta), \xi\otimes\eta\rangle_\omega &= \frac{\lambda}{n+1}\sum_{\alpha,\beta} \xi_\alpha\eta_\beta\overline{\xi}_\alpha\overline{\eta}_\beta + \xi_\alpha\eta_\beta\overline{\xi}_\beta\overline{\eta}_\alpha \\ (\star\star) \qquad &= \frac{\lambda}{n+1}\big(|\xi|^2|\eta|^2 + |\langle\xi,\eta\rangle|^2\big)\end{aligned}$$

for all $\xi, \eta \in T_X$. When $\lambda = 0$, the curvature tensor vanishes identically and we have already seen that X is a finite unramified quotient of a torus. Assume from now on that $\lambda \neq 0$. The formula $(\star\star)$ shows that the curvature tensor is constant and coincides with the curvature tensor of $\mathbb{P}^n$ (case $\lambda > 0$), or of the ball $\mathbb{B}_n$ (case $\lambda < 0$), relatively to the canonical metrics on these spaces. By a well-known result from the theory of hermitian symmetric spaces*, it follows that (X, ω) is locally isometric to $\mathbb{P}^n$ (resp. $\mathbb{B}_n$). Since the universal covering $\widetilde{X}$ is a complete and locally symmetric hermitian manifold, we conclude that $\widetilde{X} \simeq \mathbb{P}^n$, resp. $\widetilde{X} \simeq \mathbb{B}^n$. In the case $\lambda > 0$, X is a Fano manifold, thus X is simply connected and $X = \widetilde{X}$. The proof is complete. □

To compute the curvature of $\mathbb{P}^n$ and $\mathbb{B}^n$, we use the fact that the canonical metric is

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\log(1+|z|^2) \text{ on } \mathbb{P}^n, \qquad \text{resp. } \omega = -\frac{i}{2\pi}\partial\overline{\partial}\log(1-|z|^2) \text{ on } \mathbb{B}^n,$$

with respect to the non homogeneous coordinates on $\mathbb{P}^n$. We thus get

$$\omega = \frac{i}{2\pi}\Big(\frac{dz \otimes d\overline{z}}{1+|z|^2} - \frac{|\langle dz, z\rangle|^2}{(1+|z|^2)^2}\Big), \qquad \text{resp.} \quad \omega = \frac{i}{2\pi}\Big(\frac{dz \otimes d\overline{z}}{1-|z|^2} + \frac{|\langle dz, z\rangle|^2}{(1-|z|^2)^2}\Big).$$

By computing the derivatives $\Theta_{\alpha\beta jk} = -\partial^2\omega_{\alpha\beta}/\partial z_j \partial\overline{z}_k$ at $z = 0$, we easily see that the curvature is given by $(\star\star)$ with $\lambda = \pm(n+1)$. The equality also holds at any other point by the homogeneity of $\mathbb{P}^n$ and $\mathbb{B}^n$.

Observe that the riemannian exponential map $\exp : T_{X,0} \to X$ at the origin of $\mathbb{C}^n \subset \mathbb{P}^n$ or $\mathbb{B}^n$ is unitary invariant. It follows that the holomorphic part h of the Taylor expansion of exp at 0 is unitary invariant. This invariance forces h to coincide with the identity map in the standard coordinates of $\mathbb{P}^n$ and $\mathbb{B}^n$. From this observation, it is not difficult to justify intuitively the local isometry statement used above. In fact let X be a hermitian manifold whose curvature tensor is given by $(\star\star)$, $\lambda \neq 0$. Then ω is proportional to $c_1(T_X)_\omega$ and so ω is a Kähler-Einstein metric. Since the Kähler-Einstein equation is (nonlinear) elliptic with real analytic coefficients in terms of any real analytic Kähler form, it follows that ω is real analytic, and so is the exponential map. Fix a point $x_0 \in X$ and let $h : \mathbb{C}^n \simeq T_{X,x_0} \to X$ be the holomorphic part of the Taylor expansion of exp at the origin. Then h must provide the holomorphic coordinates we are looking for, i.e. $h^\star\omega$ must coincide with the metric of $\mathbb{P}^n$ (resp. $\mathbb{B}^n$) in a neighborhood of the origin.

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* See for example, F. Tricerri et L. Vanhecke, *Variétés riemanniennes dont le tenseur de courbure est celui d'un espace symétrique riemannien irréductible*, C. R. Acad. Sc. Paris, **302** (1986), 233-235.