TRANSCENDENTAL PROOF OF A GENERALIZED KAWAMATA-VIEHWEG VANISHING THEOREM

Jean-Pierre DEMAILLY

Université de Grenoble I,
Institut Fourier, BP 74,
Laboratoire associé au C.N.R.S. n° 188,
F-38402 Saint-Martin d'Hères

Dedicated to Professor Leon Ehrenpreis on his sixtieth birthday

Abstract.— *Let $\mathcal{L}$ be a holomorphic line bundle over a projective algebraic manifold X . It is shown that the differential geometric technique of Bochner-Kodaira-Nakano and the L^2 estimates for $\overline{\partial}$ yield a very elementary proof of the Kawamata-Viehweg theorem [7,13,15]: if $\mathcal{L}$ is numerically effective, then $H^q(X, \mathcal{L}^{-1}) = 0$ for $q < s$, where s is the largest integer such that $c_1(\mathcal{L})^s \neq 0$. More generally, our method implies a vanishing result when $\mathcal{L}$ is tensorized with an effective $\mathbf{Q}$–divisor which may have non normal crossings, under a natural integrability hypothesis for the divisor.*

1. Statement of results.

Recall that a line bundle $\mathcal{L}$ over a projective algebraic manifold X is said to be *numerically effective* (nef) if $c_1(\mathcal{L})_{\restriction\Gamma} \geq 0$ for every curve Γ in X . Then, it is known [10] that $c_1(\mathcal{L})^d_{\restriction Y} \geq 0$ for any subvariety $Y \subset X$ of dimension d . On the other hand, the well-known Nakai-Moishezon criterion says that a line bundle $\mathcal{H}$ is ample if and only if $c_1(\mathcal{H})^d_{\restriction Y} > 0$ for any subvariety Y of dimension d . If $\mathcal{H}$ is a given ample line bundle, it follows easily that a line bundle $\mathcal{L}$ is nef if and only if $\mathcal{L}^m \otimes \mathcal{H}$ is ample for every integer $m \geq 0$.

Let $\mathcal{F}$ be a line bundle over X and $Z(V_m)$ the set of common zeros of all sections in $V_m = H^0(X, \mathcal{F}^m)$. The *Kodaira dimension* $\kappa(\mathcal{F})$ is the maximum when m runs over all positive integers of the generic rank of the analytic morphism $\Phi_m : X \setminus Z(V_m) \longrightarrow P(V_m^\star)$ which maps a point $x \in X \setminus Z(V_m)$ to the hyperplane of sections $\sigma \in V_m$ such that $\sigma(x) = 0$. As usual, we set $\kappa(\mathcal{F}) = -\infty$ if $V_m = \{0\}$ for all m . The following definition (related to the concept of log-terminal singularities) gives a way of measuring how singular is a divisor.

Definition.— *We say that a divisor $D = \sum \alpha_j D_j$ with rational coefficients $\alpha_j \in \mathbb{Q}$ is integrable at a point $x_0 \in X$ if the function $\prod |g_j|^{-2\alpha_j}$ equal to the product of local generators g_j of the ideal of D_j at x_0 is integrable on a neighborhood of x_0 .*

When D has normal crossings, the g_j's can be taken to be coordinates at x_0; thus D is integrable if and only if $\alpha_j < 1$ for all j . When D is effective and has arbitrary singularities, a sufficient (but rather restrictive) condition for integrability of D at x_0 is that the multiplicity (or Lelong number)

$$m(D, x_0) = \sum \alpha_j \, m(D_j, x_0)$$

be < 1; indeed, if $\alpha_j = p_j/q \geq 0$ and if $\prod g_j(z)^{p_j}$ is written as a Weierstrass polynomial $\prod_{1 \leq k \leq p}(z_n - r_k(z'))$ of degree $p = \sum p_j \, m(D_j, x_0)$ in z_n , the inequality between arithmetic and geometric mean shows that

$$\prod |g_j|^{-2p_j/q} = \Big|\prod \big(z_n - r_k(z')\big)\Big|^{-2/q} \leq p^{-1} \sum |z_n - r_k(z')|^{-2p/q}$$

is integrable as soon as $q > p$.

If neither D has normal crossings nor $m(D, x_0) < 1$, the integrability condition can be checked by means of a sequence of blowing-ups which lift D into a divisor with normal crossings (this is always possible by Hironaka [11]). Let $\varphi : \widetilde{X} \longrightarrow X$ be the blowing-up of X along a submanifold Y , $\operatorname{codim} Y = r$, and $\widetilde{Y} \subset \widetilde{X}$ the exceptional divisor. On a coordinate neighborhood U of a point $x_0 \in X$, we have

$$\int_U \prod |g_j(z)|^{-2\alpha_j} d\lambda(z) = \int_{\varphi^{-1}(U)} \prod \big|g_j(\varphi(\zeta))\big|^{-2\alpha_j} |J\varphi(\zeta)|^2 \, d\lambda(\zeta)$$

where $d\lambda$ is the Lebesgue measure. The divisor of zeros of the Jacobian $J\varphi$ is $(r-1)\widetilde{Y}$, thus D is integrable on U if and only if $D^{(1)} = \varphi^\star D - (r-1)\widetilde{Y}$ is integrable on $\varphi^{-1}(U) \subset \widetilde{X}$. At the end we get a divisor $D^{(N)}$ with normal crossings and D is integrable if and only if all coefficients of $D^{(N)}$ are < 1 . A consequence of this is that integrability is an open condition : if E is an arbitrary effective divisor and if D is integrable at x_0 (resp. on X), then $D + p^{-1}E$ is again integrable at x_0 (resp. on X) for p large enough. Observe also that the set of points B where D is non integrable is an analytic subset of X (the image in X of the union of all components of $D^{(N)}$ which have a coefficient ≥ 1).

Main theorem.— *Let $\mathcal{F}$ be a line bundle over a projective manifold X . Assume that some positive power $\mathcal{F}^k$ can be written $\mathcal{F}^k = \mathcal{L} \otimes \mathcal{O}(D)$ with $\mathcal{L}$ nef and with an effective divisor D such that $k^{-1}D$ is integrable on $X \setminus B$. Let s be the largest integer such that $c_1(\mathcal{L})^s \neq 0$. Then*

$$H^q(X, \mathcal{F}^{-1}) = 0 \quad \text{for} \quad q < \min\{\max\{s, \kappa(\mathcal{F})\}, \operatorname{codim} B\} .$$

The special case $D = 0$ and $\mathcal{F} = \mathcal{L}$ gives in particular $H^q(X, \mathcal{L}^{-1}) = 0$ for $\mathcal{L}$ nef and $q < s$: this is the original Kawamata-Viehweg vanishing theorem ([13,15]). The above theorem has also been proved by Kawamata [13] in some important cases and it can be deduced formally from the method of Esnault-Viehweg [7], although their main result (theorem 2.13) is not stated in such a wide generality (they assume D to be a normal crossing divisor and $\kappa(\mathcal{F}) = n$). The proof given here is an expanded and improved version of the one given in our note [5]. It is elementary in the sense that it avoids some rather sophisticated tools of algebraic geometry that were used in the previous proofs. Our proof involves only one non trivial ingredient — the L^2-estimates for $\overline{\partial}$ deduced from the Bochner-Kodaira-Nakano curvature inequality [1] — combined with standard slicing arguments and a simple trick of Kawamata [13] which reduces the situation to the case when $\mathcal{L}$ is ample.

Thus, it now appears that the original arguments of Akizuki-Nakano [1] were in fact good enough to prove the Kawamata-Viehweg theorem and the above strong refinement! In the same vein, we shall give at the end a new differential-geometric proof of the Bogomolov-Sommese vanishing theorem. Since our approach rests heavily on methods of partial differential equations to which Leon Ehrenpreis contributed so much, it is a pleasure for us to dedicate this work to him on the occasion of his sixtieth birthday.

2. L^2 estimates for $\overline{\partial}$ on line bundles.

Let X be a complex manifold admitting a complete Kähler metric ω and $\mathcal{G}$ a holomorphic line bundle over X equipped with a hermitian metric of class C^∞. If $d\sigma$ denotes the Kähler volume form and $\lambda_1(x) \geq \ldots \geq \lambda_n(x)$ are the eigenvalues of the curvature form $c(\mathcal{G})$ at any point $x \in X$, the well-known Bochner-Kodaira-Nakano inequality gives

$$(\star) \qquad ||\overline{\partial} u||^2 + ||\overline{\partial}^\star u||^2 \geq \int_X (\lambda_1 + \cdots + \lambda_q - \lambda_{p+1} - \cdots - \lambda_n)|u|^2 d\sigma$$

for every (p,q)–form u of class C^∞ with values in $\mathcal{G}$ and compact support .

From this we readily infer the Kodaira-Akizuki-Nakano vanishing theorem [1]. Indeed, let $\mathcal{G}$ be an ample line bundle on a compact manifold X ; then there exists a C^∞ metric with positive curvature on $\mathcal{G}$. If we take $\omega = c(\mathcal{G})$, then $\lambda_j(x) = 1$ for all j and x , hence

$$||\overline{\partial} u||^2 + ||\overline{\partial}^\star u||^2 \geq (p + q - n)||u||^2 \ .$$

For $p + q > n$, it follows that every (p,q)–form that is harmonic (*i.e.* such that $\overline{\partial} u = \overline{\partial}^\star u = 0$) must vanish. Hodge theory implies the desired result:

$$\mathcal{G} \ \text{ ample} \Longrightarrow H^q(X, \Omega_X^p \otimes \mathcal{G}) = 0 \quad \text{ for } \ p + q > n \ .$$

By Serre duality, we also infer that $H^q(X, \Omega_X^p \otimes \mathcal{G}^{-1}) = 0$ for $p + q < n$. A natural problem is to relax the ampleness or positivity hypothesis. Historically, the first results proved in this direction are contained in the following:

Proposition.— *Assume X is a compact connected Kähler manifold and that $c(\mathcal{G})$ is semi-positive everywhere.*

(i) (Grauert-Riemenschneider [9]) *If $c(\mathcal{G})$ has rank at least equal to s at some point $x_0 \in X$ then*

$$H^q(X, \mathcal{G}^{-1}) = 0 \quad \text{ for } \ q < s \ .$$

(ii) (Girbau [8]) *If $c(\mathcal{G})$ has rank at least equal to s at every point, then*

$$H^q(X, \Omega_X^p \otimes \mathcal{G}^{-1}) = 0 \quad \text{ for } \ p + q < s \ .$$

Sketch of proof.— First observe that the ordered sequence of eigenvalues of $c(\mathcal{G}^{-1}) = -c(\mathcal{G})$ is $-\lambda_n \leq \ldots \leq -\lambda_1 \leq 0$.

(i) For $(0,q)$–forms with values in $\mathcal{G}^{-1}$, inequality $(\star)$ gives

$$||\overline{\partial}u||^2 + ||\overline{\partial}^\star u||^2 \geq \int_X (\lambda_1 + \cdots + \lambda_{n-q})|u|^2 d\sigma \ .$$

As $\lambda_{n-s+1}(x_0) > 0$, we see that any harmonic $(0,q)$–form u with $q < s$ must vanish identically on a neighborhood of x_0 . Hence u vanishes identically on X (by Aronszajn' lemma on solutions of elliptic equations).

(ii) In this case, we replace the Kähler metric ω by $\omega_\varepsilon = \varepsilon\omega + c(\mathcal{G})$ with $\varepsilon > 0$ very small. Then all eigenvalues $\lambda_{j,\varepsilon}$ with respect to ω_ε are in $[0,1[$ and $\lambda_{n-s+1,\varepsilon}, \ldots, \lambda_{n,\varepsilon} > 0$ become very close to 1 , say $\geq 1-\delta$. For a (p,q)–form with values in $\mathcal{G}^{-1}$, inequality $(\star)$ gives

$$\begin{aligned} ||\overline{\partial}u||^2 + ||\overline{\partial}^\star u||^2 &\geq \int_X (\lambda_1 + \cdots + \lambda_{n-q} - \lambda_{n-p+1} - \cdots - \lambda_n)|u|^2 d\sigma \\ &\geq \big((s-q)(1-\delta) - p\big)||u||^2 \ , \end{aligned}$$

and the Girbau vanishing result follows. □

The vanishing result in (ii) is no longer true under the weaker hypothesis of (i), even if $c(\mathcal{G})$ is assumed to be of rank n on a dense open set : in fact, an example of C.P. Ramanujam [14] shows that for X equal to the blow-up of $\mathbb{P}^n$ at one point and $\mathcal{G}$ equal to the pull-back of $\mathcal{O}(1)$ over X , then $H^1(X, \Omega^1_X \otimes \mathcal{G}^{-1}) \neq 0$. The failure of the proof when $p > 0$ is due to the fact that the sum $\lambda_{n-p+1} + \cdots + \lambda_n$ may become larger than $\lambda_1 + \cdots + \lambda_{n-q}$ at points where the rank of $c(\mathcal{G})$ drops below its generic value s . The easiest way of avoiding this difficulty is to restrict ourselves to the case $H^q(X, \mathcal{G}^{-1})$ with $p = 0$, or, by Serre duality, to the dual case $H^q(X, K_X \otimes \mathcal{G})$ with $p = n$.

Another draw-back of the above proposition is that semi-positivity cannot be translated easily in purely algebraic terms : it is clear that $c(\mathcal{G}) \geq 0$ implies $\mathcal{G}$ nef, but the converse implication is false. Our desire is to formulate vanishing theorems in such a way that the hypotheses involve only numerical conditions on the Chern class or easily understood conditions on divisors of sections. Then the difficulty in the analytic proof is to find hermitian metrics with the required (semi)-positivity properties. As we will see below, this can be carried out in a simple way if we allow *singular metrics.* If $\mathcal{G}_{\restriction\Omega} \simeq \Omega \times \mathbb{C}$ is a local trivialization of $\mathcal{G}$, the metric is defined by a positive function on Ω that will be denoted $e^{-\varphi}$. The only requirement we make on φ is that $\varphi \in L^1_{\rm loc}(\Omega)$. Then the curvature form $c(\mathcal{G}) = i\partial\overline{\partial}\varphi$, computed in the sense of distributions, is a real $(1,1)$–current. The main analytic tool that will be used is the following consequence of Hörmander's L^2–estimates for $\overline{\partial}$ [12].

Theorem.— *Let (X,ω) be a complete Kähler manifold and $\mathcal{G}$ a line bundle over X with a singular metric such that $c(\mathcal{G}) \geq \varepsilon\omega$ in the sense of currents. Then for every (n,q)–form v with values in $\mathcal{G}$ such that $\overline{\partial}v = 0$ and $\int_X |v|^2 d\sigma < +\infty$, there exists a $(n,q-1)$–form u such that*

$$\overline{\partial}u = v \quad \text{and} \quad \int_X |u|^2 d\sigma \leq (q\varepsilon)^{-1} \int_X |v|^2 d\sigma \ .$$

Proof.— If the metric is smooth, $(\star)$ implies $||\overline{\partial}u||^2 + ||\overline{\partial}^\star u||^2 \geq q\varepsilon||u||^2$ for all smooth (n,q)–forms, and the result follows by the usual Hilbert space method. The hypothesis on the curvature implies that the weight functions φ are plurisubharmonic. The general case follows from the smoothing procedure for plurisubharmonic functions on Kähler manifolds developed in [4] (théorèmes 5.1 et 9.1). Since the arguments are rather technical, we give here a simple proof when X is algebraic. In this case, there is an affine open subset $\Omega \subset X$ such that $\mathcal{G}_{\restriction\Omega}$ is trivial. Then the hermitian metric is given by a global plurisubharmonic function φ on Ω . As Ω is Stein, one can find an increasing sequence of Stein open sets $\Omega_j \subset\subset \Omega$ and plurisubharmonic functions $\varphi_j \in C^\infty(\Omega_j)$ decreasing to φ , such that $i\partial\overline{\partial}\varphi_j \geq \varepsilon_j\omega$ with $\lim \varepsilon_j = \varepsilon$ (embed Ω in some $\mathbb{C}^N$, use a holomorphic retraction ρ of a neighborhood of Ω onto Ω and approximate $\varphi\circ\rho$ by convolution). By the standard Hörmander L^2–estimates, there is a sequence of $(n, q-1)$–forms u_j on Ω_j such that $\overline{\partial}u_j = v$ and

$$\int_{\Omega_j} |u_j|^2 e^{-\varphi_j} d\sigma \leq (q\varepsilon_j)^{-1} \int_{\Omega_j} |v|^2 e^{-\varphi_j} d\sigma \leq (q\varepsilon_j)^{-1} \int_{\Omega} |v|^2 e^{-\varphi} d\sigma \ .$$

Then the sequence u_j is uniformly bounded in L^2 norm on compact subsets of Ω (φ_j is uniformly locally bounded from above), so there is a weak limit u satisfying $\overline{\partial}u = v$ on Ω and $\int_X |u|^2_{\mathcal{G}} d\sigma < \infty$ (observe that $X \setminus \Omega$ has measure zero). This estimate implies that u has coefficients in $L^2_{\rm loc}$. Then, by easy estimates (see e.g. [4], lemme 6.9), the relation $\overline{\partial}u = v$ extends to X . □

Corollary.— *Let $\mathcal{G}$ be a line bundle with a singular hermitian metric. Assume that $c(\mathcal{G}) \geq \varepsilon\omega$ and that the metric (i.e. the weight $e^{-\varphi}$) is integrable near all but finitely many points of X . Then*

$$H^q(X, K_X \otimes \mathcal{G}) = 0 \ \text{ for } \ q > 0 \ , \qquad H^q(X, \mathcal{G}^{-1}) = 0 \ \text{ for } \ q < n \ .$$

Proof.— By Serre duality, we only have to consider the case of $H^q(X, K_X \otimes \mathcal{G})$. Let $x_1, \ldots, x_m$ be the points where the metric is not integrable and v a smooth (n, q)–form with values in $\mathcal{G}$ such that $\overline{\partial}v = 0$. Let u_j be a smooth solution of $\overline{\partial}u_j = v$ in a neighborhood of V_j of x_j and ψ_j a cut-off function with support in V_j such that $\psi_j(x_j) = 1$. Then $v' = v - \sum \overline{\partial}(\psi_j u_j)$ satisfies $\overline{\partial}v' = 0$ and $\int_X |v'|^2_{\mathcal{G}} d\sigma < +\infty$ because the metric is integrable except at the x_j's and v' vanishes in a neighborhood of x_j . By the above theorem, there is a L^2 solution u' of $\overline{\partial}u' = v'$, hence $u = u' + \sum \psi_j u_j$ is a solution of $\overline{\partial}u = v$. It is well-known that the existence of a $L^2_{\rm loc}$ solution implies the existence of a smooth one, whence the corollary. □

3. Proof of the main theorem.

The proof consists of three steps. The first two are easy algebraic reductions and the third one is a direct application of the above corollary. We set $n = \dim_{\mathbb{C}} X$.

• **First step :** *reduction to the case when* $\min\{\max\{s, \kappa(\mathcal{F})\}, \operatorname{codim} B\} = n$.

This step is achieved by induction on n and a use of suitable hyperplane sections. Suppose $n > \min\{\max\{s, \kappa(\mathcal{F})\}, \operatorname{codim} B\}$ and choose a very ample line bundle $\mathcal{H}$ on X . Take a generic section of $\mathcal{H}$ admitting a non singular zero divisor A and consider the exact sequence

$$0 \longrightarrow \mathcal{F}^{-1} \otimes \mathcal{O}(-A) \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^{-1}_{\restriction A} \longrightarrow 0 \ ,$$

together with the associated long exact sequence of cohomology. Then all groups $H^q(X,\mathcal{F}^{-1}\otimes\mathcal{O}(-A))$ vanish for $q<n$ if $\mathcal{H}=\mathcal{O}(A)$ is ample enough (if not, replace $\mathcal{H}$ by some power). The vanishing of $H^q(X,\mathcal{F}^{-1})$ is then equivalent to the vanishing of $H^q(A,\mathcal{F}^{-1}_{\restriction A})$, so we have to choose A in such a way that the hypotheses still hold for $\mathcal{F}_{\restriction A}$ and $\mathcal{F}^k_{\restriction A}=\mathcal{L}_{\restriction A}\otimes\mathcal{O}(D_{\restriction A})$. The Fubini theorem applied to the slices of functions $\prod|g_j|^{-2\alpha_j}$ associated to $k^{-1}D$ shows that $D_{\restriction A}$ is integrable on $A\backslash(B\cap A)$ for generic A; in addition, we select A such that $\mathrm{codim}_A B\cap A=\mathrm{codim}_X B$. Now if we choose A transversal to the fiber of $\Phi_m: X\setminus Z(V_m)\longrightarrow P(V_m^\star)$, at a point where Φ_m has maximal rank $\kappa(\mathcal{F})$, we see that $\Phi_{m\restriction A}$ has the same rank as Φ_m if $\kappa(\mathcal{F})<n$ and has rank $n-1$ otherwise. Hence $\kappa(\mathcal{F}_{\restriction A})\geq\min\{\kappa(\mathcal{F}),n-1\}$. Finally $\mathcal{L}_{\restriction A}$ is nef and :

Lemma 1.— *For $r=\min\{s,n-1\}$ we have* $c_1(\mathcal{L}_{\restriction A})^r\neq 0$.

Indeed the line bundle $\mathcal{L}^m\otimes\mathcal{H}$ is ample, so $\big(c_1(\mathcal{L})+(1/m)c_1(\mathcal{H})\big)^r$ is the cohomology class of a positive rational cycle $Z_m\geq 0$ of codimension r , obtained by intersecting r generic divisors of some power of $\mathcal{L}^m\otimes\mathcal{H}$. By extracting a weak limit as m tends to infinity, we see that $c_1(\mathcal{L})^r$ is represented by a closed positive current T on X , and $T\neq 0$ by hypothesis. The class $\{A\}=c_1(\mathcal{H})$ is represented by a smooth positive $(1,1)$–form, thus if i_A is the injection of A in X we have

$$(i_A)_\star\big(c_1(\mathcal{L})^r_{\restriction A}\big)=c_1(\mathcal{L})^r\wedge\{A\}=\{T\}\wedge c_1(\mathcal{H})\neq 0\ ,$$

because $T\wedge c(\mathcal{H})$ is a non zero positive current. □

Now the induction hypothesis shows that

$$H^q(A,\mathcal{F}_{\restriction A})=0\quad\text{ for }\ q<\min\{\max\{r,\kappa(\mathcal{F}_{\restriction A}\},\mathrm{codim}_A B\cap A\}\ .$$

By the above choice of the divisor A , the upper bound on q is larger than $\min\{\max\{s,\kappa(\mathcal{F})\},\mathrm{codim}\,B\}$. Next, we need the following standard lemma on numerically effective line bundles.

Lemma 2.— *If $\mathcal{L}$ is nef, then for every line bundle $\mathcal{G}$ we have*

$$h^q(X,\mathcal{L}^m\otimes\mathcal{G})\leq Cm^{n-1}\quad \textit{for}\ \ q\geq 1\ ,$$
$$h^0(X,\mathcal{L}^m\otimes\mathcal{G})=c_1(\mathcal{L})^n\,m^n/n!+\mathrm{O}(m^{n-1})\ .$$

The first is a consequence of the exact sequence

$$H^{q-1}(A,\mathcal{L}^m\otimes\mathcal{G}\otimes\mathcal{O}(A)_{\restriction A})\longrightarrow H^q(X,\mathcal{L}^m\otimes\mathcal{G})\longrightarrow H^q(X,\mathcal{L}^m\otimes\mathcal{G}\otimes\mathcal{O}(A))\ ;$$

choose A so ample that $K_X^{-1}\otimes\mathcal{G}\otimes\mathcal{O}(A)$ is ample; then the last group vanishes by the Akizuki-Kodaira-Nakano vanishing theorem and the first one has rank $\leq Cm^{n-1}$ since $\dim A=n-1$. The second line results from the fact that the Euler-Poincaré characteristic $\chi(X,\mathcal{L}^m\otimes\mathcal{G})$ is a polynomial in m of leading term $c_1(\mathcal{L})^n\,m^n/n!$. □

Remark.— Lemma 2 shows that the Kodaira dimension $\kappa(\mathcal{L})$ is equal to n if and only if $c_1(\mathcal{L})^n>0$. In the case $s<n$, we always have $\kappa(\mathcal{L})\leq s$, for there exists a subvariety $Y\subset X$ of dimension $d=\kappa(\mathcal{L})$ such that $\kappa(\mathcal{L}_{\restriction Y})=d$ and lemma 2 gives $c_1(\mathcal{L})^d_{\restriction Y}>0$. However, it may happen that $\kappa(\mathcal{L})<s$: take $\mathcal{L}\longrightarrow X=X_1\times X_2$ equal to the tensor product of an ample line bundle on X_1 and of a hermitian flat line bundle on X_2 with non trivial powers; then $\kappa(\mathcal{L})=-\infty$ and $s=\dim X_1$.

The same example shows that the Kodaira dimension may increase by restriction to a subvariety (take $Y = X_1 \times \{\text{point}\}$). $\square$

• **Second step :** *reduction to the case when $\mathcal{L}$ is ample.*— By the first step, we may assume $\operatorname{codim} B = n$ and $\max\{s, \kappa(\mathcal{F})\} = n$. The injection $\mathcal{L} \hookrightarrow \mathcal{L} \otimes \mathcal{O}(D) = \mathcal{F}^k$ implies $\kappa(\mathcal{F}) \geq \kappa(\mathcal{L})$, hence the case $s = n$ implies also $\kappa(\mathcal{F}) = n$ by the above remark. We may therefore assume $\kappa(\mathcal{F}) = n$, whence $\limsup\ m^{-n}\, h^0(X, \mathcal{F}^m) > 0$. Let $\mathcal{H} = \mathcal{O}(A)$ be an ample line bundle. The exact sequence

$$0 \longrightarrow H^0(X, \mathcal{F}^m \otimes \mathcal{O}(-A)) \longrightarrow H^0(X, \mathcal{F}^m) \longrightarrow H^0(A, \mathcal{F}^m_{\restriction A})$$

where $h^0(A, \mathcal{F}^m_{\restriction A}) \leq Cm^{n-1}$ shows that $\mathcal{F}^m \otimes \mathcal{H}^{-1}$ has non trivial sections for m large enough. If E is the divisor of such a section, we get $\mathcal{F}^m = \mathcal{H} \otimes \mathcal{O}(E)$. After raising $\mathcal{F}^k = \mathcal{L} \otimes \mathcal{O}(D)$ to the power p and multiplying by $\mathcal{F}^m$, we obtain :

$$\mathcal{F}^{kp+m} = \mathcal{L}^p \otimes \mathcal{H} \otimes \mathcal{O}(pD + E) = \mathcal{L}' \otimes \mathcal{O}(D')\ ,$$

with $\mathcal{L}' = \mathcal{L}^p \otimes \mathcal{H}$ and $D' = pD + E$. Moreover $\mathcal{L}'$ is ample and for p large enough $(kp+m)^{-1}D' \leq k^{-1}D + (kp+m)^{-1}E$ is integrable.

• **Third step :** *Proof of the main theorem when $\mathcal{L}$ is ample.*— We choose on $\mathcal{L}$ a smooth hermitian metric with positive curvature and set $\omega = c(\mathcal{L})$. On the other hand, $\mathcal{O}(D)$ can be equipped with a natural singular metric : if s is a local section of $\mathcal{O}(D)$, viewed as a meromorphic function with poles along D , we define $|s|^2 = s\overline{s} \in [0, +\infty]$. If $g = \prod g_j^{p_j}$ is a local generator of the ideal of $D = \sum p_j D_j$, then g^{-1} is a non vanishing section of $\mathcal{O}(D)$ which may be used to get a trivialization of $\mathcal{O}(D)$. The plurisubharmonic weight corresponding to the metric is $e^{-\varphi}$ with $\varphi = \log |g|^2$. By the Lelong-Poincaré equation, the curvature of this metric is $\frac{1}{2\pi} c(\mathcal{O}(D)) = [D] =$ current of integration over D; in particular $c(\mathcal{O}(D)) \geq 0$ in the sense of currents. The induced metric on $\mathcal{F}$ has curvature

$$c(\mathcal{F}) = k^{-1}\big(c(\mathcal{L}) + c(\mathcal{O}(D))\big) = k^{-1}\omega + k^{-1}2\pi[D] \geq k^{-1}\omega\ .$$

Caution : in spite of the fact that our singular metric has strongly positive curvature, it may happen that $\mathcal{F}$ is not ample nor even nef when D itself is not nef (e.g. when D is an exceptional divisor).

Now the weight associated to $\mathcal{F}$ is equal to $\exp(-k^{-1}(\varphi + \psi))$ where ψ is the (smooth) weight of $\mathcal{L}$, so it is integrable if and only if $\exp(-k^{-1}\varphi) = |g|^{-2/k}$ is integrable. This assumption holds on $X \setminus B$ by hypothesis, and here B is finite ($\operatorname{codim} B = n$). The corollary in §2 implies $H^q(X, \mathcal{F}^{-1}) = 0$ for $q < n$. $\square$

Remark.— If the linear system $|D|$ is not reduced to the divisor D , the integrability requirement can be weakened . Let $D^{(1)} = D,\, D^{(2)},\, \ldots,\, D^{(N)}$ be the zero divisors of sections in a basis $s = (s^{(1)}, \ldots, s^{(N)})$ of $H^0(X, \mathcal{O}(D))$. Then we take the quotient metric on $\mathcal{O}(D)$ induced by the generically surjective homomorphism $s : \mathcal{O}^N \longrightarrow \mathcal{O}(D)$ and the trivial metric on $\mathcal{O}^N$. If e is a local holomorphic frame of $\mathcal{O}(D)$, we may write $s^{(\ell)} = g^{(\ell)}e$ where $g^{(\ell)}$ is a local generator of $\mathcal{I}_{D^{(\ell)}}$ and the quotient metric is given by the weight $e^{-\varphi}$ with $\varphi = \log(|g^{(1)}|^2 + \cdots + |g^{(N)}|^2)$. The integrability condition is now weakened into the integrability of $\big(|g^{(1)}|^2 + \cdots + |g^{(N)}|^2\big)^{-2/k}$ instead of $|g^{(1)}|^{-2/k}$.

4. New proof of the Bogomolov-Sommese vanishing theorem.

By similar ideas, we can obtain a new simple proof of this theorem, without using Deligne's degeneracy criterion for the Hodge spectral sequence.

Theorem ([2], [15]).— *Let (X,ω) be a compact Kähler manifold and let $\mathcal{G}$ be a line bundle over X . Then*

$$H^0(X,\Omega_X^p\otimes\mathcal{G}^{-1})=0\quad\text{ for }\ p<\kappa(\mathcal{G})\ .$$

Proof.— Let $\Phi_m : X\setminus Z\longrightarrow\mathbb{P}^N$ be a morphism of generic rank $d=\kappa(\mathcal{G})$ associated to $V_m=H^0(X,\mathcal{G}^m)$. Then $\mathcal{G}^m\simeq\Phi_m^\star\mathcal{O}(1)$ and we endow $\mathcal{G}$ with the induced metric, which is singular along Z . If we set $V_m=\langle s^{(1)},\ldots,s^{(N)}\rangle$ and use the same notations as in the last remark, the singular metric on $\mathcal{G}$ (resp. $\mathcal{G}^{-1}$) is defined by the weight $e^{-\varphi}$ (resp. by e^{φ}) with

$$\varphi=\frac{1}{m}\log(|g^{(1)}|^2+\cdots+|g^{(N)}|^2)\ .$$

It is clear that the curvature form $c(\mathcal{G})=i\partial\overline{\partial}\varphi=\Phi_m^\star c(\mathcal{O}(1))$ is a positive current, equal to a smooth form of generic rank $\kappa(\mathcal{G})$ on $X\setminus Z$. If u is a holomorphic p–form with values in $\mathcal{G}^{-1}$, the curvature inequality $(\star)$ gives :

$$0=\int_X|\overline{\partial}u|^2d\sigma\geq\int_{X\setminus Z}(\lambda_{p+1}+\cdots+\lambda_n)|u|^2d\sigma$$

where $\lambda_1\geq\ldots\geq\lambda_n\geq 0$ are the eigenvalues of $c(\mathcal{G})$. This can be checked by using a sequence of smooth weights φ_j converging to φ with $\lambda_i(\varphi_j)\geq-$ constant converging to $\lambda_i=\lambda_i(\varphi)$ and e^{φ_j} converging uniformly to e^{φ} (cf. [4]). As $\lambda_d>0$ almost everywhere on $X\setminus Z$, we conclude that $u=0$ for $p<d$. □

References

[1] Y. Akizuki and S. Nakano.— *Note on Kodaira-Spencer's proof of Lefschetz theorems*, Proc. Jap. Acad., **30** (1954), 266–272.

[2] F. Bogomolov.— *Unstable vector bundles and curves on surfaces*, Proc. Intern. Congress of Math., Helsinki (1978), 517–524.

[3] P. Deligne.— *Théorie de Hodge II et III*, Publ. Math. IHES, **40** (1972), 5–57 et **44** (1975), 5–77.

[4] J.-P. Demailly.— *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au dessus d'une variété kählérienne complète*, Ann. Sci. Ec. Norm. Sup., **15** (1982), 457–511.

[5] J.-P. Demailly.— *Une généralisation du théorème d'annulation de Kawamata-Viehweg*, C. R. Acad. Sci. Paris Sér. I Math., **309** (1989), 123–126.

[6] H. Esnault et E. Viehweg.— *Revêtements cycliques*, Algebraic Threefolds, Varenna 1981, Lecture Notes in Math. n° 947, Springer-Verlag, Berlin (1982), 241–250.

[7] H. Esnault et E. Viehweg.— *Logarithmic De Rham complexes and vanishing theorems*, Invent. Math., **86** (1986), 161–194.

[8] J. Girbau.— *Sur le théorème de Le Potier d'annulation de la cohomologie*, C. R. Acad. Sci. Paris, série A, **283** (1976), 355–358.

[9] H. Grauert and O. Riemenschneider.— *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math., **11** (1970), 263–292.

[10] R. Hartshorne.— *Ample subvarieties of algebraic varieties*, Lecture Notes in Math. n° 156, Springer-Verlag, Berlin (1970).

[11] H. Hironaka.— *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math., **79** (1964), 109–326.

[12] L. Hörmander.— *An introduction to Complex Analysis in several variables*, 2nd edition, North-Holland Math. library, Vol 7, Amsterdam, 1973.

[13] Y. Kawamata.— *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann., **261** (1982), 43–46.

[14] C.P. Ramanujam.— *Remarks on the Kodaira vanishing theorem*, J. Indian Math. Soc., **36** (1972), 41–50 and **38** (1974), 121–124.

[15] E. Viehweg.— *Vanishing theorems*, J. Reine Angew. Math., **335** (1982), 1–8.