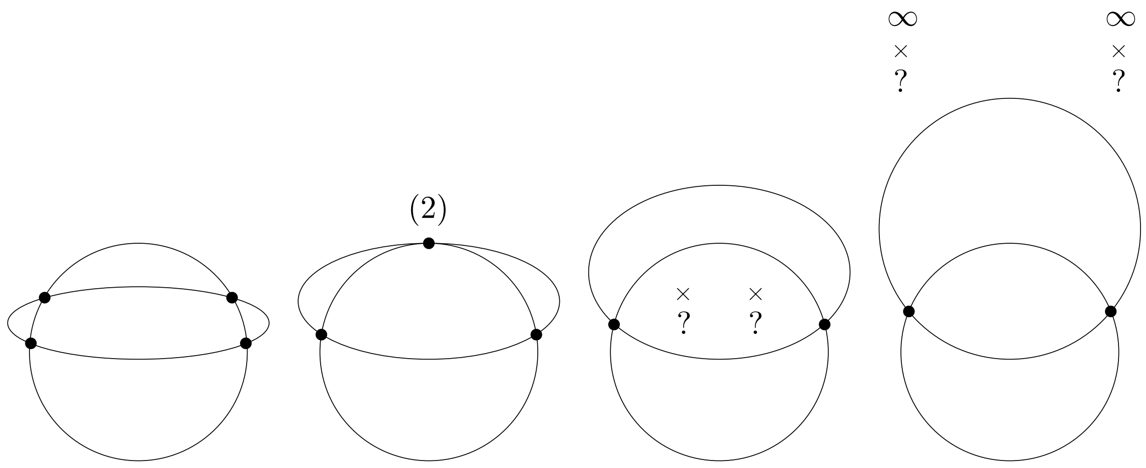
J.P. Demailly's address to the Göttingen Academy of Sciences on the occasion of the Dannie Heineman prize attribution November 15, 1991

Analytic techniques of Complex Geometry

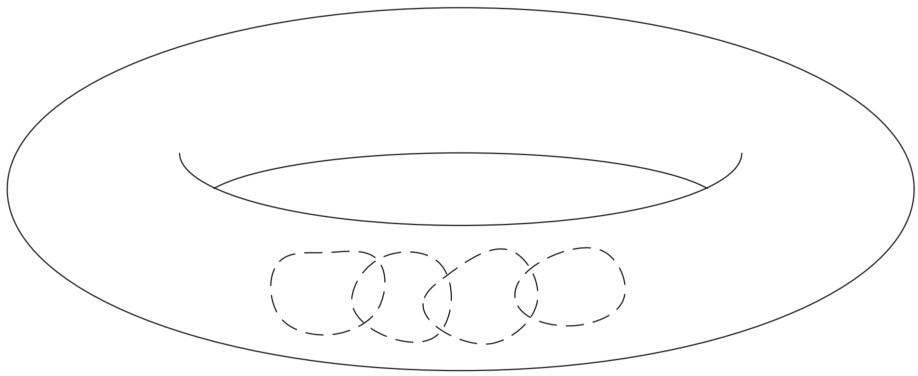
First I would like to express my deep gratitude to the Academy for the attribution of the Dannie Heineman prize rewarding my mathematical work. This is certainly the greatest honour I have ever had in my mathematical career. In this address to the Academy, I will try to outline the historical development of ideas which gradually led to the modern concepts of analytic and algebraic geometry, in particular those connected with my personal work.

Of course, the prehistory of algebraic geometry traces back to the work of the ancient Greeks, through the study of conic curves by Apollonius and the solution of geometric questions such as duplication of cubes and division of angles in equal parts. However, analytic geometry could not be developed on its own stand before the invention of cartesian coordinates by Fermat and Descartes in the first part of the 17th century. In the previous century, imaginary numbers had been introduced by Tartaglia and Cardano as a way to circumvent the nonexistence of real solutions to algebraic equations of degree 2 or more. These numbers remained mythical quantities for more than two hundred years. The clear picture everyone has in mind today that imaginary numbers are in one-to-one correspondence with points of a plane was surprisingly discovered only at the end of the 18th century by Gauss, Argand and Wessel. Roughly at the same period took place an important development of projective geometry which culminated with the work of Poncelet. Originally, projective geometry was designed to describe properties of figures depending only on incidence properties of their constituents, namely points, lines, curves, etc. For that purpose, the introduction of points at infinity appeared as a very efficient way of avoiding exceptional cases in the general statements of intersection theory, for example in Bezout's theorem that two plane algebraic curves of degrees p and q meet in $p \times q$ points exactly. Since conic curves are of degree 2, Bezout's theorem tells us for instance that two conic curves should always intersect in four points.



The figure shows that the intersection of a circle with an ellipse always has four points, provided points are counted with multiplicities and imaginary points are taken into account (they are hidden from our eyes in some third or fourth dimension). In the case of two circles, the imaginary points of intersection are located at infinity in the directions corresponding to slopes $\pm\sqrt{-1}$

So, in some sense, it became more or less clear to geometers around 1850 that a nice theory could only be made on what we now call in modern language "compact complex manifolds", that is, generalized surfaces of arbitrary dimension, on which coordinates are allowed to take imaginary values, including points at infinity.



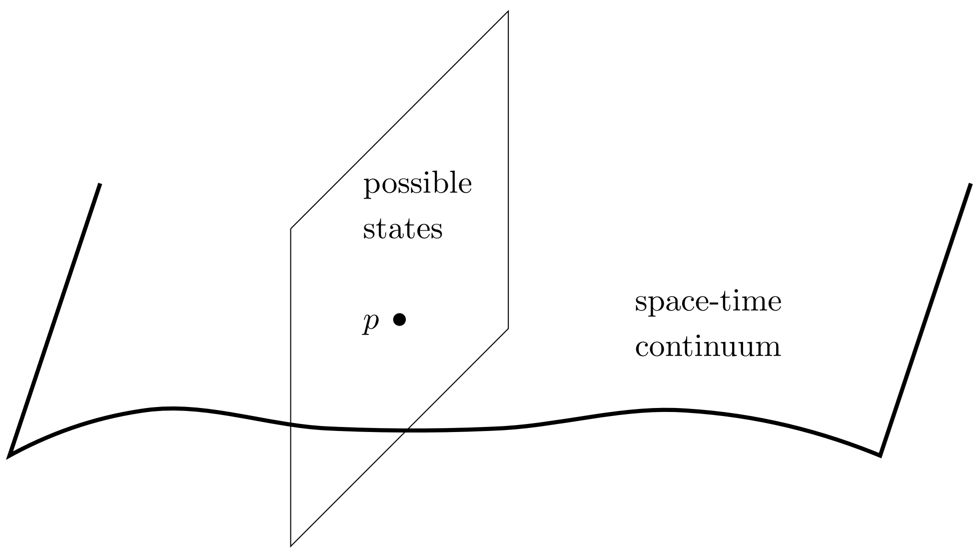
A manifold is a composite object obtained by attaching open coordinate patches together. The manifold is said to be compact if it has no boundary points at finite or infinite distance, as a sphere or a torus for instance, in contrast with a disk or a ball which do have boundary points.

As a culmination of Euler's achievements, analytic function theory was developed during the 19th century by several prominent analysts among which Legendre, Abel, Cauchy, Gauss, Jacobi. Finally Riemann realized the very rich interplay between analytic function theory and geometry. Especially, he was the first mathematician to have a clear picture of the general concept of manifolds of arbitrary dimension, although the correct definition could only be introduced much later by Hermann Weyl. Riemann was also the first to make use of the basic properties of harmonic functions in a geometric context. It was not before 1900 that Hilbert could finally establish on solid grounds Riemann's intuitive application of the Dirichlet principle. As the audience has probably noticed, a large part of the mathematicians involved in these crucial achievements lived or worked in Göttingen.

Another strong impetus in the interconnection between geometry and analysis was inspired by Physics. This impetus came in particular through Maxwell's theory of electromagnetism and the introduction of the concept of space-time continuum by Einstein at the turn of the 20th century, following previous works by Lorentz and Minkowski. According to Einstein's general relativity theory, the gravitational force can be described in terms of the curvature of the space-time continuum via the so called Einstein equations

$$R^i_{i\,\alpha\beta} = \lambda g_{\alpha\beta},$$

where g denotes the Minkowski metric and R its curvature tensor. The other forces can be also described in terms of vector or scalar fields satisfying suitable partial differential equations. In quantum field theory, the behaviour of any particle is characterized by its wave function, which is obtained by computing solutions of the Schrödinger equation. From a geometric point of view, a magnetic field can be also seen as the curvature of a geometric object called a vector bundle, representing all possible variations of the state of the particle in function of its position p in the space-time continuum.



In quantum field theory, there is a corresponding energy functional for the particle submitted to the electromagnetic field. In 1985, I developed a new theory which is now rewarded by the Dannie Heineman prize, according to which it is possible to obtain asymptotic estimates of the energy eigenstates of the particle when the electromagnetic field becomes very large. All this has striking consequences when the universe is taken to be a compact complex manifold, that is, includes imaginary points and points at infinity. In fact complex analytic functions on a manifold are solutions of the Cauchy-Riemann $\overline{\partial}$ operator and the obstructions to the solvability of the corresponding equation are computed by what mathematicians call analytic cohomology groups. These groups were introduced around 1950 and studied by many prominent mathematicians, among whom I should mention Kodaira, Cartan, Serre, Grothendieck, Prof. Friedrich Hirzebruch from the Max Planck Institut in Bonn and Prof. Hans Grauert from Göttingen; some other important contributions to analytic geometry were also made by Prof. C.L. Siegel in Göttingen. In the 1985 work already mentioned, I obtained a rather general formula relating the curvature tensor to the analytic cohomology groups, that is, to the ground state energy levels of the particle.

More recently, during a visit to Bayreuth university in 1989, I realized that a modified version of Einstein's gravitational equations could be used to solve a long standing problem of algebraic geometry concerning 3 or higher dimensional varieties of general type. The idea is to use Einstein's equations for a large pointwise mass distribution: physicists would probably imagine a black hole in that context (in their terminology, a black hole is a big star collapsed onto itself). Similar ideas have led physicists to introduce new models of our universe called Calabi-Yau manifolds, which are supposed to achieve the long-awaited unified theory of forces. Shing-Tung Yau is a famous Chinese-American mathematician who received the Fields medal in mathematics in 1978 for his solution of Calabi's conjectures on Einstein equations.

I hope that these rough indications will contribute to give an idea of the very rich interplay existing today between analysis, geometry and physical theories. I would like to conclude by saying that the role of past and present mathematicians working in Göttingen has been extremely influential in all these domains.

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